On Structural Parameterizations of the Matching Cut Problem

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Abstract. In an undirected graph, a matching cut is a partition of vertices into two sets such that the edges across the sets induce a matching. The matching cut problem is the problem of deciding whether a given graph has a matching cut. The matching cut problem can be expressed using a monadic second-order logic (MSOL) formula and hence is solvable in linear time for graphs with bounded tree-width. However, this approach leads to a running time of $f(\phi, t)n^{O(1)}$, where ϕ is the length of the MSOL formula, t is the tree-width of the graph and n is the number of vertices of the graph.
In [Theoretical Computer Science, 2016], Kratsch and Le asked to give a single exponential algorithm for the matching cut problem with tree-width alone as the parameter. We answer this question by giving a $2^{O(t)}n^{O(1)}$ time algorithm. We also show the tractability of the matching cut problem when parameterized by neighborhood diversity and other structural parameters.

Keywords: Matching Cut, Decomposable Graphs, Parameterized Algorithm

1 Introduction

Consider an undirected graph $G = (V, E)$ such that $|V| = n$. An *edge cut* is a minimal edge set $S \subseteq E$ such that the removal of S from the graph increase the number of components in the graph. A *matching* is an edge set such that no two edges in the set have a common end point. A *matching cut* is an edge cut which is also a matching. The *matching cut* problem is the decision problem of determining whether a given graph G has a matching cut.

The matching cut problem was first introduced by Graham in [1], in the name of *decomposable graphs*. Farley and Proskurowski [2] pointed out the applications of the matching cut problem in computer networks – in studying the networks which are immune to failures of non-adjacent links. Patrignani and Pizzonia [3] pointed out the applications of the matching cut problem in graph drawing. They refer to a method of graph drawing, where one starts with a degenerate drawing where all the vertices and edges are at the same point. At each step, the vertices

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in the drawing are partitioned and progressively the drawing approaches the original graph. In this regard, the cut involving the non-adjacent edges (matching cut) yields a more efficient and effective performance.

The matching cut problem is NP-Complete for the following graph classes:

- Graphs with maximum degree 4 (Chvátal [4], Patrignani and Pizzonia [3]).
- Bipartite graphs with one partite set has maximum degree 3 and the other partite set has maximum degree 4 (Le and Randerath [5]).
- Planar graphs with maximum degree 4 and planar graphs with girth 5 (Bonsma [6]).
- $K_{1,4}$-free graphs with maximum degree 4 (inferred from the reduction in [4]).

The matching cut problem has polynomial time algorithms for the following graph classes:

- Graphs with maximum degree 3 (Chvátal [4]).
- Line graphs (Moshi [7]).
- Graphs without chordless cycles of length 5 or more (Moshi [7]).
- Series parallel graphs (Patrignani and Pizzonia [3]).
- Claw-free graphs, cographs, graphs with bounded tree-width and graphs with bounded clique-width (Bonsma [6]).
- Graphs with diameter 2 (Borowiecki and Jesse-Józefczyk [8]).
- $(K_{1,4}, K_{1,4} + e)$-free graphs (Kratsch and Le [9]).

When the graph G has degree at least 2, the matching cut problem in G is equivalent to the problem of deciding whether the line graph of G has a stable cut set. A *stable cut set* is a set $S \subseteq V$ of independent vertices, such that the removal of S from the graph G increases the number of components of G. Algorithmic aspects of stable cut set of line graphs have been studied in [5, 10–12].

Recently, Kratsch and Le [9] presented a $2^{n/2}n^{O(1)}$ time algorithm for the matching cut problem using branching techniques. They also showed that the matching cut problem is tractable for graphs with bounded vertex cover.

The matching cut problem can be expressed using a monadic second-order logic (MSOL) formula [6] and is hence solvable in linear time for graphs with bounded tree-width. This approach leads to an algorithm with running time $f(\phi, t)n^{O(1)}$, where ϕ is the length of the MSOL formula and t is the tree-width of the graph. However, for most graphs, the function $f(\phi, t)$ is a tower of exponentials of height ϕ. That raises the following question, asked in [9]: Can we have an algorithm where f is a single exponential function?

In this paper, we answer the above question by giving a $2^{O(t)}n^{O(1)}$ algorithm for the matching cut problem, where t is the tree-width of the graph. We also show that the matching cut problem is tractable for graphs with bounded neighborhood diversity and other structural parameters.

2 Preliminaries

A parameterized problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a fixed and finite alphabet. For $(x, k) \in \Sigma^* \times \mathbb{N}$, k is referred to as the parameter. A parameterized problem L is *fixed parameter tractable (FPT)* if there is an algorithm

A, a computable non-decreasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ and a constant c such that, given $(x, k) \in \Sigma^* \times \mathbb{N}$ the algorithm A correctly decides whether $(x, k) \in L$ in time bounded by $f(k).|x|^c$.

Sometimes, we write $f(n) = O^*(g(n))$ if $f(n) = O(g(n)\text{poly}(n))$, where poly$(n)$ is a polynomial in n. Two vertices u, v are called *neighbors* if $\{u, v\} \in E$, we say v is a *neighbor* of u and vice versa. The set of all neighbors of u (*open neighborhood*) is denoted by $N(u)$. The *closed neighborhood* of u, is denoted by $N[u]$, is defined as $N[u] = N(u) \cup \{u\}$. For a vertex set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. For a vertex set $S \subseteq V$, $G\backslash S$ denotes the graph $G[V\backslash S]$. When there is no ambiguity, we use the simpler notations $S\backslash x$ to denote $S\backslash\{x\}$ and $S \cup x$ to denote $S \cup \{x\}$.

3 Graphs with Bounded Tree-width

A *tree decomposition* of G is a pair $(T, \{X_i, i \in I\})$, where for $i \in I$, $X_i \subseteq V$ (usually called bags) and T is a tree with elements of I as the nodes such that:

1. For each vertex $v \in V$, there is an $i \in I$ such that $v \in X_i$.
2. For each edge $\{u, v\} \in E$, there is an $i \in I$ such that $\{u, v\} \subseteq X_i$.
3. For each vertex $v \in V$, $T[\{i \in I | v \in X_i\}]$ is connected.

The width of the tree decomposition is $\max_{i \in I}(|X_i| - 1)$. The tree-width of G is the minimum width taken over all tree decompositions of G and we denote it as t. For more details on tree-width, we refer the reader to [13]. Kloks [14] introduced *nice tree decomposition*, which is a tree decomposition where every node $i \in I$ is one of the following types:

1. Leaf node: For a leaf node i, $X_i = \emptyset$.
2. Introduce Node: An introduce node i has exactly one child j and there is a vertex $v \in V\backslash X_j$ such that $X_i = X_j \cup \{v\}$.
3. Forget Node: A forget node i has exactly one child j and there is a vertex $v \in V\backslash X_i$ such that $X_j = X_i \cup \{v\}$.
4. Join Node: A join node i has exactly two children j_1 and j_2 such that $X_i = X_{j_1} = X_{j_2}$.

Every graph G has a nice tree decomposition with $|I| = O(n)$ nodes and width equal to the tree-width of G. Moreover, such a decomposition can be found in linear time if the tree-width is bounded [14].

Now we present an $O^*(2^{O(t)})$ time algorithm for the matching cut problem. The algorithm we present is based on dynamic programming technique on the nice tree decomposition.

The matching cut problem is a graph partitioning problem, where we need to partition the vertices into two sets A and B such that the edges across the sets induce a matching. And we denote such a matching cut by (A, B). We use the following notation in the algorithm.

– i: A node in the tree decomposition.

– X_i: The set of vertices associated with bag at node i.
– $G[X_i]$: Subgraph induced by X_i.
– T_i: The sub-tree rooted at node i of the tree decomposition. This includes node i and all its descendants.
– $G[T_i]$: Subgraph induced by the vertices in node i and all its descendants.

Let $\Psi = (A_1, A_2, A_3, B_1, B_2, B_3)$ be a partition of X_i, we say that the partition Ψ is **legal** at node i if it satisfies the following conditions ($\star$):

1. Every vertex of A_1 (respectively B_1) has exactly one neighbor in B_1 (resp. A_1) and no neighbors in $B_2 \cup B_3$ (resp. $A_2 \cup A_3$).
2. Every vertex of $A_2 \cup A_3$ (resp. $B_2 \cup B_3$) has no neighbors in any of the B_i's (resp. A_i's).

We say that a legal partition ψ is **valid** for the node i if there exists a matching cut (A, B) of $G[T_i]$ such that the following conditions ($\star\star$) hold:

1. The A_i's are contained in A and the B_i's are contained in B.
2. Every vertex of A_1 (resp. B_1) has a matching cut neighbor in B_1 (resp. A_1).
3. Every vertex of $A_2 \cup B_2$ has a matching cut neighbor in $G[T_i] \setminus X_i$.
4. The vertices of $A_3 \cup B_3$ are not part of the cut-edges, i.e. every vertex of A_3 (resp. B_3) has no neighbor in B (resp. A).

A matching cut is empty if there are no edges in cut. We say that a valid partition Ψ of X_i is *locally empty* in $G[T_i]$, if every matching cut of $G[T_i]$ extending ψ (i.e. satisfying $\star\star$) is empty. Note that a necessary condition for Ψ to be locally empty is: $A_1 \cup A_2 \cup B_1 \cup B_2 = \emptyset$.

We define $M_i[\Psi]$ to be $+1$ if Ψ is valid for the node X_i and not locally empty, 0 if it is valid and locally empty, and -1 otherwise. Now, we explain how to compute $M_i[\Psi]$ for each partition Ψ at the nodes of the nice tree decomposition.

Leaf node: For a leaf node i, $X_i = \emptyset$. We have $\Psi = (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$ and $M_i[\Psi] = 0$. This step can be executed in constant time.

Introduce node: Let j be the only child of the node i. Suppose, $v \in X_i$ is the new node present in X_i, $v \notin X_j$. Let $\Psi = (A_1, A_2, A_3, B_1, B_2, B_3)$ be a partition of X_i. If Ψ is not legal, we straightaway set $M_i[\Psi]$ to -1. Otherwise, we use the below procedure to compute $M_i[\Psi]$ for $v \in A_i$, and analogously for $v \in B_i$.

Case 1: $v \in A_1$, then $M_i[\Psi] = +1$, if there exists a unique $x \in B_1$, such that, $(v, x) \in E$ and $M_j[\Psi'] \geq 0$ for $\Psi' = (A_1 \backslash v, A_2, A_3, B_1 \backslash x, B_2, B_3 \cup x)$. Otherwise $M_i[\Psi] = -1$. Note that, $M_i[\Psi]$ can not be 0, as $v \in A_1$ brings an edge into the cut if it is valid.
Case 2: $v \in A_2$, this case is not valid as v does not have any neighbor in $V(T_i) \backslash X_i$ (it is the property of the nice tree decomposition).
Case 3 $v \in A_3$, $M_i[\Psi] = M_j[\Psi']$ where $\Psi' = (A_1, A_2, A_3 \backslash v, B_1, B_2, B_3)$.

The total number of possible Ψ's for X_i is 6^{t+1}. For each Ψ, the above cases can be executed in polynomial time. Hence the total time complexity at the introduce node is $O^*(6^t)$.

Forget node: Let j be the only child of the node i. Suppose, $v \in X_j$ is the node missing in X_i, $v \notin X_i$. Let $\Psi = (A_1, A_2, A_3, B_1, B_2, B_3)$ be a partition of X_i. If Ψ is not legal, we straightaway set $M_i[\Psi]$ to -1.

Otherwise, $M_i[\Psi] = \max_{k=1}^{k=6}\{\delta_k\}$, where δ_k is computed as follows: If Ψ is valid, it should be possible to add v to one of the six sets to get a valid partition at node j.

Case 1: v is in the first set at the node j. If there is a unique $x \in B_2$ such that $(v, x) \in E$ then $\delta_1 = M_j[\Psi']$ where $\Psi' = (A_1 \cup v, A_2, A_3, B_1 \cup x, B_2 \backslash x, B_3)$. If no such x exists, then δ_1 is set to -1.
Case 2: v is in the second set at the node j.
Let $\Psi' = (A_1, A_2 \cup v, A_3, B_1, B_2, B_3)$ and $\delta_2 = M_j[\Psi']$.
Case 3: v is in the third set at the node j.
Let $\Psi' = (A_1, A_2, A_3 \cup v, B_1, B_2, B_3)$ and $\delta_3 = M_j[\Psi']$.

The values δ_4, δ_5 and δ_6 are computed analogously. The total number of possible Ψ's for X_i is 6^t. For each Ψ, the above cases can be executed in polynomial time. Hence the total time complexity at the forget node is $O^*(6^t)$.

Join node: Let j_1 and j_2 be the children of the node i. $X_i = X_{j_1} = X_{j_2}$ and $V(T_{j_1}) \cap V(T_{j_2}) = X_i$. There are no edges between $V(T_{j_1}) \backslash X_i$ and $V(T_{j_2}) \backslash X_i$. Let $\Psi = (A_1, A_2, A_3, B_1, B_2, B_3)$ be a partition of X_i. For $X \subseteq A_2$ and $Y \subseteq B_2$ let $\Psi_1 = (A_1, X, A_3 \cup \{A_2 \backslash X\}, B_1, Y, B_3 \cup \{B_2 \backslash Y\})$ and $\Psi_2 = (A_1, A_2 \backslash X, A_3 \cup X, B_1, B_2 \backslash Y, B_3 \cup Y)$.

$$M_i[\Psi] = \begin{cases} +1, \text{If } \exists X \subseteq A_2 \text{ and } Y \subseteq B_2 \text{ such that } M_{j_1}[\Psi_1] + M_{j_2}[\Psi_2] \geq 1; \\ 0, \quad \text{If } \Psi \text{ is locally empty, (i.e } M_{j_1}[\Psi] = 0 \text{ and } M_{j_2}[\Psi] = 0); \\ -1, \text{Otherwise} \end{cases}$$

The total number of possible Ψ's for X_i is 6^{t+1}. For each Ψ, we need to check 2^{t+1} different Ψ_1 and Ψ_2. The total time complexity at the join node is $O^*(12^t)$.

At each node i, let $\Delta_i = \max_{\Psi}\{M_i[\Psi]\}$. If $\Delta_i = +1$, then $G[T_i]$ has a valid non-empty matching cut. If r is the root of the nice tree decomposition, the graph G has a matching cut if $\Delta_r = +1$. By induction and the correctness of $M_i[\Psi]$ values, we can conclude the correctness of the algorithm. The total time complexity of the algorithm is $O^*(12^t) = O^*(2^{O(t)})$.

Theorem 1 *There is an algorithm with running time $O^*(2^{O(t)})$ that solves the matching cut problem, where t is the tree-width of the graph.*

4 Graphs with Bounded Neighborhood Diversity

Lampis [15] introduced a structural parameter called *neighborhood diversity* which is defined as follows:

Definition 1 (Neighborhood Diversity [15]). *In an undirected graph G, two vertices u and v have the same type if and only if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$.*

The graph G has neighborhood diversity d if there exists a partition of $V(G)$ into d sets $P_1, P_2, \ldots, P_d$ such that all the vertices in each set have the same type. Such a partition is called a type partition. Moreover, it can be computed in linear time.

Note that, each P_i forms either a clique or an independent set in G.

If a graph has vertex cover number q, then the neighborhood diversity of the graph is at most $2^q + q$ [15]. Hence, graphs with bounded vertex cover number also have bounded neighborhood diversity. However, the converse is not true since complete graphs have neighborhood diversity 1. Some NP-hard problems are shown to be tractable on graphs with bounded neighborhood diversity (see e.g., [16]). Here, we show that the matching cut problem is tractable for graphs with bounded neighborhood diversity. We describe an algorithm with time complexity $O^*(2^{2d})$, where d is the neighborhood diversity of the graph.

We start with a graph G, and its type partitioning with d partitions, i.e neighborhood diversity of G is d. We label the vertices of G (using the type partitioning) such that vertices having the same label should be entirely on one side of the cut. We assume that the graph is connected and so is the type partitioning graph. Let $P_1, P_2, \ldots, P_d$ be the sets of the type partition. We say P_i is an I-set if P_i induces an independent set. Similarly, we say P_i is a C-set if P_i induces a clique. The size of a set P_i is the number of vertices in the set P_i.

Observe that a clique K_c with $c \geq 3$ and $K_{r,s}$ with $r \geq 2$ and $s \geq 3$ do not have a matching cut. It means that all the vertices of these graphs should be entirely on one side of the cut. Consider a partition P_i, vertices of P_i are labeled according to the following rules in order:

- If P_i is a C-set with size ≥ 2, vertices in the set P_i and all the vertices in its neighboring sets get the same label.
- If P_i is an I-set with size ≥ 3 and is adjacent to an I-set with size ≥ 2, then the vertices in both the sets get the same label.
- If P_i is an I-set with size ≥ 3 and is adjacent to two or more sets of size ≥ 1, then vertices in all these sets get the same label.
- If P_i is an I-set with size ≥ 3 and has only one adjacent set of size 1, then G has a matching cut.
- If P_i is an I-set with size 2 and is adjacent to an I-set of size 2 and a set of size 1, then vertices in all these sets get the same label.
- If P_i is an I-set with size 2 and is adjacent to only one I-set of size 2, in these two sets, each vertex will get different label.
- If P_i is an I-set with size 2 and is adjacent to two sets of size 1, in these three sets, each vertex will get different label.
- If P_i is an I-set with size 2 and is adjacent to a set of size 1, then G has a matching cut.
- All the remaining sets of size 1 will get different labels.

If we apply the above rules, either we conclude that G has a matching cut, or for each set we use at most 2 labels, hence we can state the following:

Lemma 2 *The number of labels required is at most* $2d$.

The vertices of each label should entirely be in the same set of the matching cut. Hence there are 2^{2d} possible label combinations. Thus we have the following:

Theorem 3 *There is an algorithm with running time* $O^*(2^{2d})$ *that solves the matching cut problem, where d is the neighbourhood diversity of the graph.*

5 Other Structural Parameters

For graphs with bounded feedback vertex number, the tree-width is also bounded. As the matching cut problem is in FPT for tree-width, it is also in FPT for feedback vertex number. Kratsch and Le [9] showed that the matching cut problem is in FPT for the size of the vertex cover. We use the techniques used in [9] to show that the matching cut problem is in FPT for the parameters *twin cover* and the *distance to split graphs.*

Lemma 4 (stated as Lemma 3 in [9]) *Let* I *be an independent set and let* $U = V \backslash I$. *Given a partition* (X, Y) *of* U, *it can be decided in* $O(n^2)$ *time if the graph has a matching cut* (A, B) *such that* $X \subseteq A$ *and* $Y \subseteq B$.

Two non-adjacent (adjacent) vertices having the same open (closed) neighborhood are called *twins.* A *twin cover* is a vertex set S such that for each edge $\{u, v\} \in E$, either $u \in S$ or $v \in S$ or u and v are twins. Note that, for a twin cover $S \subseteq V$, $G[V \backslash S]$ is a collection of disjoint cliques.

Lemma 5 *Let* $S \subseteq V$ *be a twin cover of* G. *Given a partition* (X, Y) *of* S, *it can be decided in* $O(n^2)$ *time if the graph has a matching cut* (A, B) *such that* $X \subseteq A$ *and* $Y \subseteq B$.

Proof. Clearly, $V \backslash S$ induces a collection of disjoint cliques. Consider a maximal clique C on two or more vertices in $V \backslash S$. Let u, v be any two vertices of the clique C. Clearly, u and v are twins. If u and v has a common neighbor in both X and Y, then the graph has no matching cut such that $X \subseteq A$ and $Y \subseteq B$. Hence, without loss of generality we can assume that u and v have common neighbors only in X. Let $X' = X \cup V(C)$. Clearly, $V \backslash (S \cup V(C))$ is an independent set. Using Lemma 4, we can decide in $O(n^2)$ time if the graph has a matching cut (A, B) such that $X' \subseteq A$ and $Y \subseteq B$. □

Let S be a twin cover of the graph. By guessing a partition (X, Y) of S, we can check in $O(n^2)$ time if G has a matching cut (A, B) such that $X \subseteq A$ and $Y \subseteq B$. Hence we can state the following theorem.

Theorem 6 *There is an algorithm with running time* $O^*(2^{|S|})$ *to solve matching cut problem, where* S *is the twin cover of the graph.*

Lemma 7 *Let* G *be a graph with vertex set* V, *if* $S \subseteq V$ *be such that* $G[V \backslash S]$ *is a split graph. Given a partition* (X, Y) *of* S, *it can be decided in* $O(n^2)$ *time whether the graph* G *has a matching cut* (A, B) *such that* $X \subseteq A$ *and* $Y \subseteq B$.

Proof. Let $V \backslash S = C \cup I$ be the vertex set of the split graph, where C is a clique and I is an independent set. If $|C| = 1$ or $|C| \geq 3$, then let $X' = X \cup V(C)$ and $Y' = Y \cup V(C)$. Clearly, $V \backslash (S \cup V(C))$ is an independent set. Hence, G has matching cut (A, B) such that $X \subseteq A$ and $Y \subseteq B$ if and only if G has a matching cut such that either $X' \subseteq A$ and $Y \subseteq B$ or $X \subseteq A$ and $Y' \subseteq B$. Both these instances can be solved in $O(n^2)$ time using Lemma 4. If $|C| = 2$, depending on whether the vertices of C go to X or Y, we solve four instances of Lemma 4 to check whether graph has a matching cut (A, B) such that $X \subseteq A$ and $Y \subseteq B$. Therefore the time complexity is $O(n^2)$. □

Similar to Theorem 6, we can state the following theorem.

Theorem 8 *There is an algorithm with running time $O^*(2^{|S|})$ to solve the matching cut problem, where $S \subseteq V$ such that $G[V \backslash S]$ is a split graph.*

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