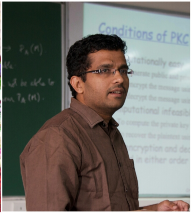
Combinatorial Bounds for Conflict-free Coloring on Open Neighborhoods

Sriram Bhyravarapu Subrahmanyam Kalyanasundaram



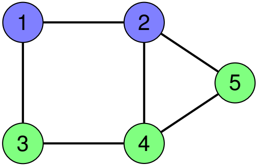
Department of Computer Science and Engineering
Indian Institute of Technology Hyderabad

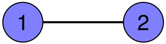
Conflict-free coloring problem - Open neighborhood

CFON problem

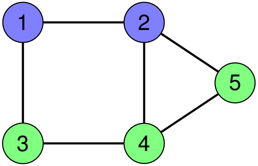
- Given a graph $G = (V, E)$, a conflict-free coloring w.r.t. open neighborhoods is an assignment of colors to $V(G)$ such that
  - Every vertex in G has a uniquely colored vertex in its open neighborhood.
  - Open Neighborhood of a vertex v is $N(v) = \{w : \{v, w\} \in E(G))\}$.
- The minimum number of colors required for such a coloring is denoted by $\chi_{ON}(G)$.

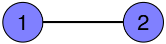
CFON Coloring Problem

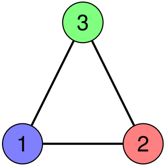


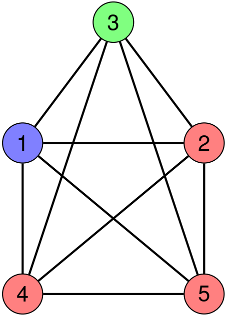


CFON Coloring Problem

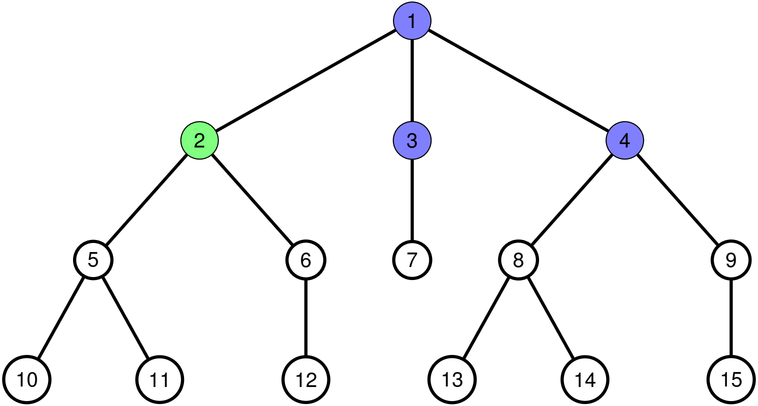






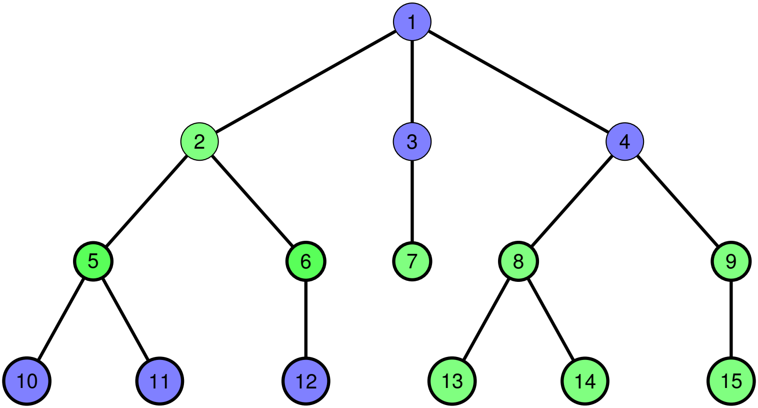


CFON Coloring a Tree



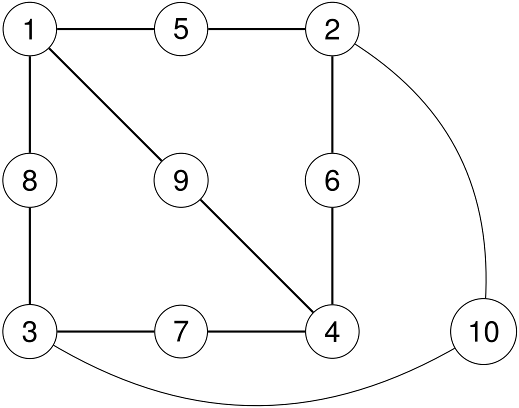
- We start with the first 2 levels.
- Every other vertex is assigned the color different from its grandparent's color.

CFON Coloring a Tree



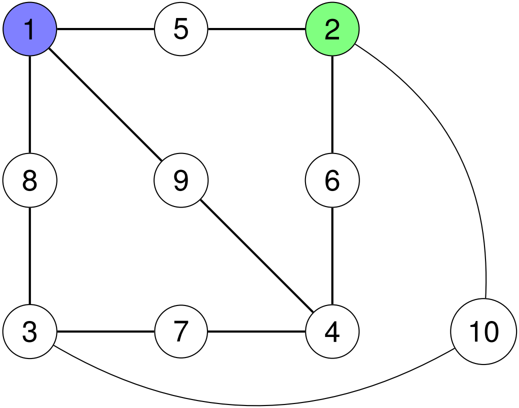
- We start with the first 2 levels.
- Every other vertex is assigned the color different from its grandparent's color.

K_n^* : Subdivision graph of the Clique



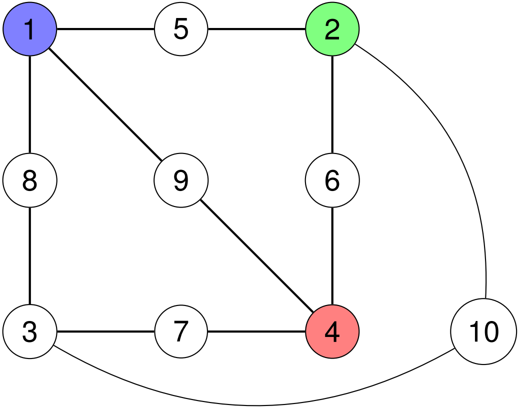
$\chi_{ON}(K_n^*) = n$ but K_n^* is bipartite.

K_n^* : Subdivision graph of the Clique



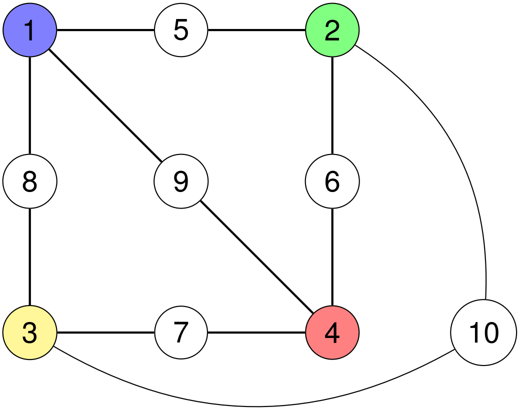
$\chi_{ON}(K_n^*) = n$ but K_n^* is bipartite.

K_n^* : Subdivision graph of the Clique



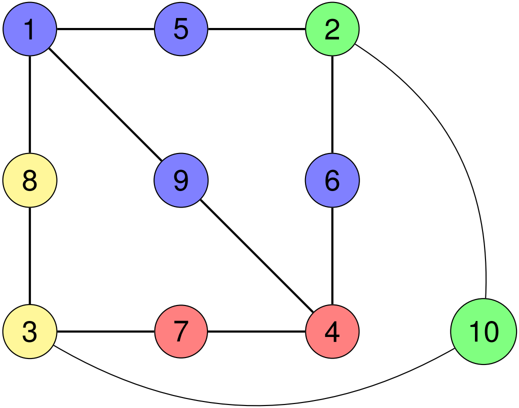
$\chi_{ON}(K_n^*) = n$ but K_n^* is bipartite.

K_n^* : Subdivision graph of the Clique



$\chi_{ON}(K_n^*) = n$ but K_n^* is bipartite.

K_n^* : Subdivision graph of the Clique



$\chi_{ON}(K_n^*) = n$ but K_n^* is bipartite.

Motivation & History

- Introduced by Even, Lotker, Ron and Smorodinsky in 2002, motivated by the Frequency Assignment Problem.
- The problem has been studied with respect to both the open neighborhoods and the closed neighborhoods.
- Pach and Tardos (2009) showed that $\chi_{ON}(G) = O(\sqrt{n})$ and $\chi_{CN}(G) = O(\log^2 n)$.
- Geometric intersection graphs like disk, square, rectangle, interval graphs, etc have attracted special interest.
- Most of the variants are NP-complete.

Results

Parameter	Known Result	Our Result	Lower Bound
Planar	8[1]	6	4
Outerplanar	6	4	3
Pathwidth (pw)	$2pw + 1$[2]	$\lfloor \frac{5}{3}(pw+1) \rfloor$	pw
FVS (f)	$f + 3$	$f + 2$	$f + 2$
Distance to Cluster (dc)	$2dc + 3$[3]	$dc + 3$	dc
Neighborhood Diversity	$\chi_{ON}(H) + cl(G) + 1$[4]	$\chi_{ON}(H) + \frac{cl(G)}{2} + 2$	-

[1] Abel, Alvare, Demaine, Fekete, Gour, Hesterberg, Keldenich, and Scheffer, "Conflict-Free Coloring of Graphs", *SIAM Journal on Discrete Mathematics, 2018*.

[2] Bodlaender, Kolay and Pieterse, "Parameterized Complexity of Conflict-free Graph Coloring", *Workshop on Algorithms and Data Structures (WADS)*, 2019.

[3] Reddy, "Parameterized algorithms for conflict-free colorings of graphs", *Theor. Comput. Sci.*, 2018.

[4] Gargano and Rescigno, "Complexity of Conflict-free Colorings of Graphs", *Theor. Comput. Sci.*, 2015.

Results

Parameter	Known Result	Our Result	Lower Bound
Planar	8[1]	6	4
Outerplanar	6	4	3
Pathwidth (pw)	$2pw + 1$[2]	$\lfloor \frac{5}{3}(pw + 1) \rfloor$	pw
FVS (f)	$f + 3$	$f + 2$	$f + 2$
Distance to Cluster (dc)	$2dc + 3$[3]	$dc + 3$	dc
Neighborhood Diversity	$\chi_{ON}(H) + cl(G) + 1$[4]	$\chi_{ON}(H) + \frac{cl(G)}{2} + 2$	-

[1] Abel, Alvare, Demaine, Fekete, Gour, Hesterberg, Keldenich, and Scheffer, "Conflict-Free Coloring of Graphs", *SIAM Journal on Discrete Mathematics, 2018*.

[2] Bodlaender, Kolay and Pieterse, "Parameterized Complexity of Conflict-free Graph Coloring", *Workshop on Algorithms and Data Structures (WADS)*, 2019.

[3] Reddy, "Parameterized algorithms for conflict-free colorings of graphs", *Theor. Comput. Sci.*, 2018.

[4] Gargano and Rescigno, "Complexity of Conflict-free Colorings of Graphs", *Theor. Comput. Sci.*, 2015.

Results

Parameter	Known Result	Our Result	Lower Bound
Planar	8[1]	6	4
Outerplanar	6	4	3
Pathwidth (pw)	$2pw + 1$[2]	$\lfloor \frac{5}{3}(pw + 1) \rfloor$	pw
FVS (f)	$f + 3$	$f + 2$	$f + 2$
Distance to Cluster (dc)	$2dc + 3$[3]	$dc + 3$	dc
Neighborhood Diversity	$\chi_{ON}(H) + cl(G) + 1$[4]	$\chi_{ON}(H) + \frac{cl(G)}{2} + 2$	-

[1] Abel, Alvare, Demaine, Fekete, Gour, Hesterberg, Keldenich, and Scheffer, “Conflict-Free Coloring of Graphs”, *SIAM Journal on Discrete Mathematics, 2018*.

[2] Bodlaender, Kolay and Pieterse, “Parameterized Complexity of Conflict-free Graph Coloring”, *Workshop on Algorithms and Data Structures (WADS)*, 2019.

[3] Reddy, “Parameterized algorithms for conflict-free colorings of graphs”, *Theor. Comput. Sci.*, 2018.

[4] Gargano and Rescigno, “Complexity of Conflict-free Colorings of Graphs”, *Theor. Comput. Sci.*, 2015.

Results

Parameter	Known Result	Our Result	Lower Bound
Planar	8[1]	6	4
Outerplanar	6	4	3
Pathwidth (pw)	$2pw + 1$[2]	$\lfloor \frac{5}{3}(pw + 1) \rfloor$	pw
FVS (f)	$f + 3$	$f + 2$	$f + 2$
Distance to Cluster (dc)	$2dc + 3$[3]	$dc + 3$	dc
Neighborhood Diversity	$\chi_{ON}(H) + cl(G) + 1$[4]	$\chi_{ON}(H) + \frac{cl(G)}{2} + 2$	-

[1] Abel, Alvare, Demaine, Fekete, Gour, Hesterberg, Keldenich, and Scheffer, "Conflict-Free Coloring of Graphs", *SIAM Journal on Discrete Mathematics, 2018*.

[2] Bodlaender, Kolay and Pieterse, "Parameterized Complexity of Conflict-free Graph Coloring", *Workshop on Algorithms and Data Structures (WADS)*, 2019.

[3] Reddy, "Parameterized algorithms for conflict-free colorings of graphs", *Theor. Comput. Sci.*, 2018.

[4] Gargano and Rescigno, "Complexity of Conflict-free Colorings of Graphs", *Theor. Comput. Sci.*, 2015.

Results

Parameter	Known Result	Our Result	Lower Bound
Planar	8[1]	6	4
Outerplanar	6	4	3
Pathwidth (pw)	$2pw + 1$[2]	$\lfloor \frac{5}{3}(pw + 1) \rfloor$	pw
FVS (f)	$f + 3$	$f + 2$	$f + 2$
Distance to Cluster (dc)	$2dc + 3$[3]	$dc + 3$	dc
Neighborhood Diversity	$\chi_{ON}(H) + cl(G) + 1$[4]	$\chi_{ON}(H) + \frac{cl(G)}{2} + 2$	-

[1] Abel, Alvare, Demaine, Fekete, Gour, Hesterberg, Keldenich, and Scheffer, "Conflict-Free Coloring of Graphs", *SIAM Journal on Discrete Mathematics, 2018*.

[2] Bodlaender, Kolay and Pieterse, "Parameterized Complexity of Conflict-free Graph Coloring", *Workshop on Algorithms and Data Structures (WADS)*, 2019.

[3] Reddy, "Parameterized algorithms for conflict-free colorings of graphs", *Theor. Comput. Sci.*, 2018.

[4] Gargano and Rescigno, "Complexity of Conflict-free Colorings of Graphs", *Theor. Comput. Sci.*, 2015.

Results

Parameter	Known Result	Our Result	Lower Bound
Planar	8[1]	6	4
Outerplanar	6	4	3
Pathwidth (pw)	$2pw + 1$[2]	$\lfloor \frac{5}{3}(pw + 1) \rfloor$	pw
FVS (f)	$f + 3$	$f + 2$	$f + 2$
Distance to Cluster (dc)	$2dc + 3$[3]	$dc + 3$	dc
Neighborhood Diversity	$\chi_{ON}(H) + cl(G) + 1$[4]	$\chi_{ON}(H) + \frac{cl(G)}{2} + 2$	-

[1] Abel, Alvare, Demaine, Fekete, Gour, Hesterberg, Keldenich, and Scheffer, "Conflict-Free Coloring of Graphs", *SIAM Journal on Discrete Mathematics, 2018*.

[2] Bodlaender, Kolay and Pieterse, "Parameterized Complexity of Conflict-free Graph Coloring", *Workshop on Algorithms and Data Structures (WADS)*, 2019.

[3] Reddy, "Parameterized algorithms for conflict-free colorings of graphs", *Theor. Comput. Sci.*, 2018.

[4] Gargano and Rescigno, "Complexity of Conflict-free Colorings of Graphs", *Theor. Comput. Sci.*, 2015.

Planar graphs

Theorem

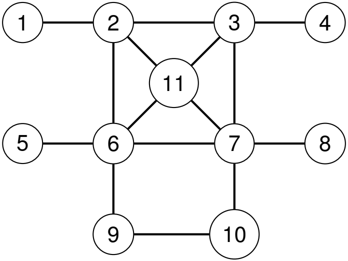
If G is a planar graph, then $\chi_{ON}(G) \leq 6$.

Assumption: G is connected and $|V(G)| \geq 2$.

Maximal distance-3 set, V_0

- Initially $V_0 = \{v\}$, for some $v \in V(G)$.
- Add u to V_0, if
  - $\exists w \in V_0$, such that $dist(u, w) = 3$.
  - $dist(u, w') \geq 3$, for all $w' \in V_0$.

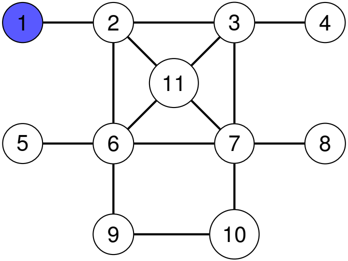
Maximal distance-3 set



Partition of $V(G)$

- $V_0 \rightarrow$ maximal distance-3 set.
- $V_1 = N(V_0)$, where $N(V_0)$ represents neighborhood of V_0.
- $V_2 = V(G) \setminus (V_0 \cup V_1)$.
  $V_2 \rightarrow$ Set of vertices that are at a distance 2 from V_0.

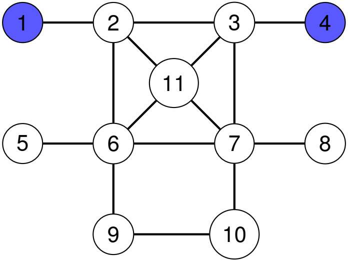
Maximal distance-3 set



Partition of $V(G)$

- $V_0 \rightarrow$ maximal distance-3 set.
- $V_1 = N(V_0)$, where $N(V_0)$ represents neighborhood of V_0.
- $V_2 = V(G) \setminus (V_0 \cup V_1)$.
  $V_2 \rightarrow$ Set of vertices that are at a distance 2 from V_0.

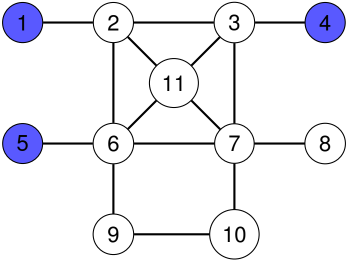
Maximal distance-3 set



Partition of $V(G)$

- $V_0 \rightarrow$ maximal distance-3 set.
- $V_1 = N(V_0)$, where $N(V_0)$ represents neighborhood of V_0.
- $V_2 = V(G) \setminus (V_0 \cup V_1)$.
  $V_2 \rightarrow$ Set of vertices that are at a distance 2 from V_0.

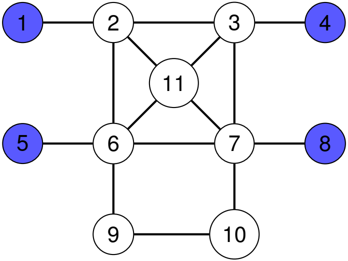
Maximal distance-3 set



Partition of $V(G)$

- $V_0 \rightarrow$ maximal distance-3 set.
- $V_1 = N(V_0)$, where $N(V_0)$ represents neighborhood of V_0.
- $V_2 = V(G) \setminus (V_0 \cup V_1)$.
  $V_2 \rightarrow$ Set of vertices that are at a distance 2 from V_0.

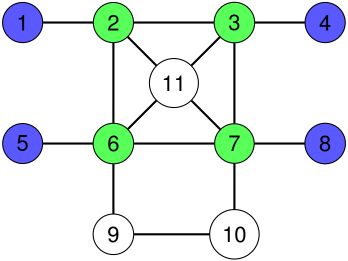
Maximal distance-3 set



Partition of $V(G)$

- $V_0 \rightarrow$ maximal distance-3 set.
- $V_1 = N(V_0)$, where $N(V_0)$ represents neighborhood of V_0.
- $V_2 = V(G) \setminus (V_0 \cup V_1)$.
  $V_2 \rightarrow$ Set of vertices that are at a distance 2 from V_0.

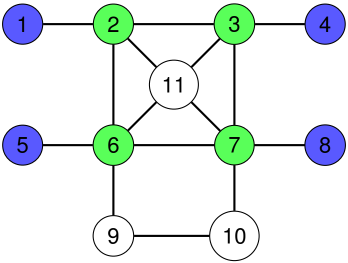
Maximal distance-3 set



Partition of $V(G)$

- $V_0 \rightarrow$ maximal distance-3 set.
- $V_1 = N(V_0)$, where $N(V_0)$ represents neighborhood of V_0.
- $V_2 = V(G) \setminus (V_0 \cup V_1)$.
  $V_2 \rightarrow$ Set of vertices that are at a distance 2 from V_0.

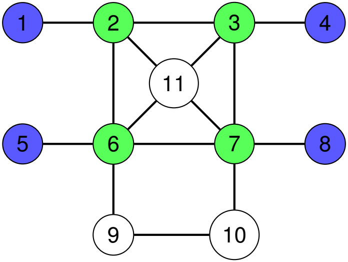
Maximal distance-3 set - Observations



Observations:

- V_0 is an independent set.
- No edges between V_0 and V_2.
- Every vertex in $V_0 \cup V_2$ has a neighbor in V_1.
- Every vertex in V_1 has exactly 1 neighbor in V_0.

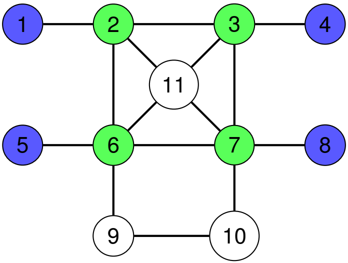
Maximal distance-3 set - Observations



Observations:

- V_0 is an independent set.
- No edges between V_0 and V_2.
- Every vertex in $V_0 \cup V_2$ has a neighbor in V_1.
- Every vertex in V_1 has exactly 1 neighbor in V_0.

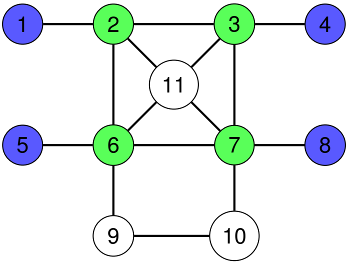
Maximal distance-3 set - Observations



Observations:

- V_0 is an independent set.
- No edges between V_0 and V_2.
- Every vertex in $V_0 \cup V_2$ has a neighbor in V_1.
- Every vertex in V_1 has exactly 1 neighbor in V_0.

Maximal distance-3 set - Observations



Observations:

- V_0 is an independent set.
- No edges between V_0 and V_2.
- Every vertex in $V_0 \cup V_2$ has a neighbor in V_1.
- Every vertex in V_1 has exactly 1 neighbor in V_0.

Maximal distance-3 set - Observations

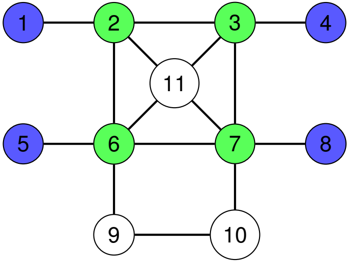
Observations:

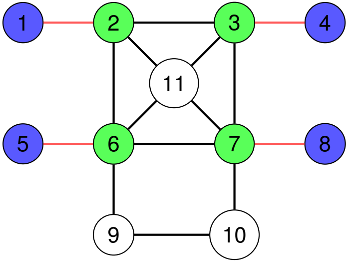
- V_0 is an independent set.
- No edges between V_0 and V_2.
- Every vertex in $V_0 \cup V_2$ has a neighbor in V_1.
- Every vertex in V_1 has exactly 1 neighbor in V_0.

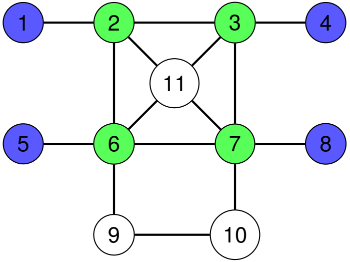
Color V_0:

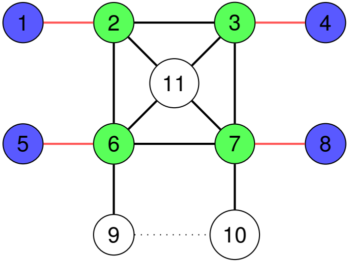
- For all $v \in V_0$, assign $C(v) = 1$.
- For each vertex in V_1, there is a uniquely colored neighbor.

Color V_1

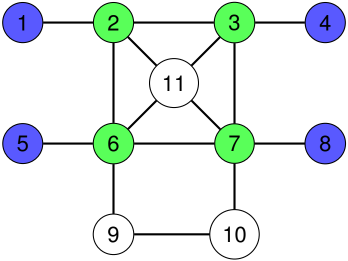


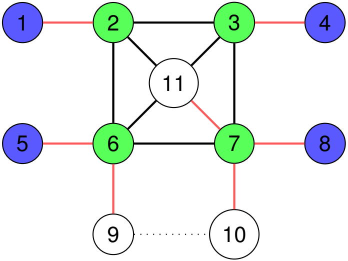



1
2
3
4
11
5
6
7
8
9
10

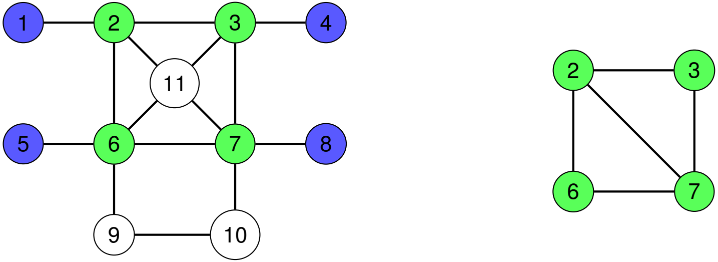

1
2
3
4
11
5
6
7
8
9
10

Color V_1

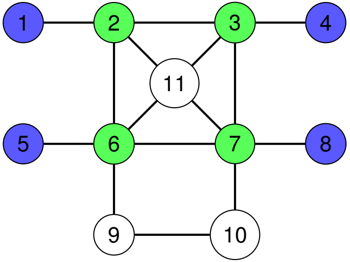


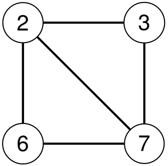


Color V_1

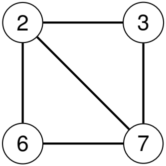


Color V_1

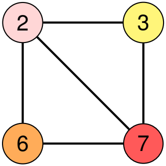




Color V_1

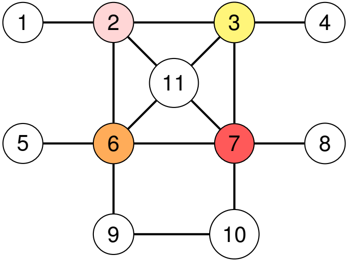


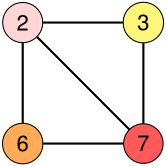
- The contracted vertex in V_1 will be the uniquely colored neighbor for each vertex in $V_0 \cup V_2$.
- All vertices of G have a uniquely colored neighbor. However, vertices in V_2 are uncolored.
- For all $v \in V_2$, assign $C(v) = 6$.



- The contracted vertex in V_1 will be the uniquely colored neighbor for each vertex in $V_0 \cup V_2$.
- All vertices of G have a uniquely colored neighbor. However, vertices in V_2 are uncolored.
- For all $v \in V_2$, assign $C(v) = 6$.

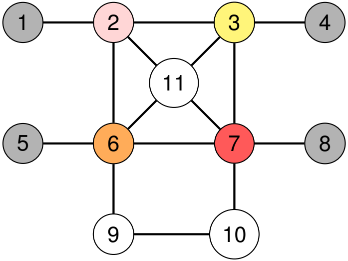
Color V_1

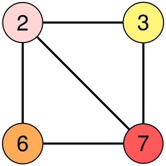




- The contracted vertex in V_1 will be the uniquely colored neighbor for each vertex in $V_0 \cup V_2$.
- All vertices of G have a uniquely colored neighbor. However, vertices in V_2 are uncolored.
- For all $v \in V_2$, assign $C(v) = 6$.

Color V_1





- The contracted vertex in V_1 will be the uniquely colored neighbor for each vertex in $V_0 \cup V_2$.
- All vertices of G have a uniquely colored neighbor. However, vertices in V_2 are uncolored.
- For all $v \in V_2$, assign $C(v) = 6$.

CFON coloring of Planar graphs

Theorem

If G is a planar graph, then $\chi_{ON}(G) \leq 6$.

Partial conflict-free coloring: *CFON**

If G is a planar graph, then $\chi^*_{ON}(G) \leq 5$.

CFON coloring of Planar graphs

Theorem

If G is a planar graph, then $\chi_{ON}(G) \leq 6$.

Partial conflict-free coloring: *CFON**

If G is a planar graph, then $\chi^*_{ON}(G) \leq 5$.

Pathwidth

- Bodlaender, Kolay and Pieterse [WADS 2019] showed that $\chi_{ON}(G) \leq 2tw + 1$, where *tw* is the treewidth of graph *G*.
- We show that $\chi_{ON}(G) \leq \lfloor \frac{5}{3}(pw + 1) \rfloor$, where *pw* is the pathwidth of *G*.

Sketch

- Every vertex v in a bag X_i gets a distinct color $C(v)$. Assign the unique color $U(v)$ when its sees its first neighbor.
- Assign a color which is not seen in the bag, which gives $\forall v \in X_i, |C(v) \cup U(v)| \leq 2|X_i|$.
  For a bag of *tw* vertices, we need $\leq 2 \cdot tw$ colors.
- We use some sophisticated rules, that reuses the colors to get $|C(v) \cup U(v)| \leq \frac{5}{3}|X_i|$.

Pathwidth

- Bodlaender, Kolay and Pieterse [WADS 2019] showed that $\chi_{ON}(G) \leq 2tw + 1$, where *tw* is the treewidth of graph *G*.
- We show that $\chi_{ON}(G) \leq \lfloor \frac{5}{3}(pw + 1) \rfloor$, where *pw* is the pathwidth of *G*.

Sketch

- Every vertex *v* in a bag X_i gets a distinct color $C(v)$. Assign the unique color $U(v)$ when its sees its first neighbor.
- Assign a color which is not seen in the bag, which gives $\forall v \in X_i, |C(v) \cup U(v)| \leq 2|X_i|$.
  For a bag of *tw* vertices, we need $\leq 2 \cdot tw$ colors.
- We use some sophisticated rules, that reuses the colors to get $|C(v) \cup U(v)| \leq \frac{5}{3}|X_i|$.

Pathwidth

- Bodlaender, Kolay and Pieterse [WADS 2019] showed that $\chi_{ON}(G) \leq 2tw + 1$, where *tw* is the treewidth of graph *G*.
- We show that $\chi_{ON}(G) \leq \lfloor \frac{5}{3}(pw + 1) \rfloor$, where *pw* is the pathwidth of *G*.

Sketch

- Every vertex v in a bag X_i gets a distinct color $C(v)$. Assign the unique color $U(v)$ when its sees its first neighbor.
- Assign a color which is not seen in the bag, which gives $\forall v \in X_i, |C(v) \cup U(v)| \leq 2|X_i|$.
  For a bag of *tw* vertices, we need $\leq 2 \cdot tw$ colors.
- We use some sophisticated rules, that reuses the colors to get $|C(v) \cup U(v)| \leq \frac{5}{3}|X_i|$.

FVS

- Bodlaender, Kolay and Pieterse [WADS 2019] showed that $\chi_{ON}(G) \leq |F| + 3$, where F is a feedback vertex set of G.
- We show that $\chi_{ON}(G) \leq |F| + 2$.

Sketch

- Color trees in $G[V \setminus F]$ using 2 colors.
- We use $|F|$ distinct colors to color F.
- If $G[F]$ is connected, we have a CFON coloring of G.
- Else, we identify a “free color”. This is the non-trivial part of the algorithm.

The bounds for the distance to cluster parameter use the same framework.

FVS

- Bodlaender, Kolay and Pieterse [WADS 2019] showed that $\chi_{ON}(G) \leq |F| + 3$, where F is a feedback vertex set of G.
- We show that $\chi_{ON}(G) \leq |F| + 2$.

Sketch

- Color trees in $G[V \setminus F]$ using 2 colors.
- We use $|F|$ distinct colors to color F.
- If $G[F]$ is connected, we have a CFON coloring of G.
- Else, we identify a “free color”. This is the non-trivial part of the algorithm.

The bounds for the distance to cluster parameter use the same framework.

Summary of our results

Parameter	Known Result	Our Result	Lower Bound
Planar	8	6	4
Outerplanar	6	4	3
Pathwidth (pw)	$2pw + 1$	$\lfloor \frac{5}{3}(pw + 1) \rfloor$	pw
FVS (f)	$f + 3$	$f + 2$	$f + 2$
Distance to Cluster (dc)	$2dc + 3$	$dc + 3$	dc
Neighborhood Diversity	$\chi_{ON}(H) + cl(G) + 1$	$\chi_{ON}(H) + \frac{cl(G)}{2} + 2$	-

Our techniques also yield results for CFCN coloring.

Parameter	Known Result	Our Result
Distance to Cluster (dc)	$dc + 2$	$\max\{3, dc + 1\}$
Neighborhood Diversity	$\chi_{CN}(H) + ind(G) + 1$	$\chi_{CN}(H) + \frac{ind(G)}{3} + 3$

Future Work

- Get tight results for some parameters.
  - For the distance to cluster (*dc*) parameter, we now have matching upper and lower bounds of $dc + 1$, [B., Kalyanasundaram 20+]
- Can we show that $\chi_{ON}(G) \leq \frac{5}{3}(tw + 1)$?
- Look at the list coloring variant of the problem.
- Complexity status for various graph classes is unknown, like block graphs, permutation graphs, interval graphs, etc.

References

[1] Zachary Abel, Victor Alvarez, Erik D. Demaine, Sándor P. Fekete, Aman Gour, Adam Hesterberg, Phillip Keldenich, and Christian Scheffer, "Conflict-Free Coloring of Graphs", *SIAM Journal on Discrete Mathematics*, 2018.

[2] Hans L. Bodlaender, Sudeshna Kolay and Astrid Pieterse, "Parameterized Complexity of Conflict-free Graph Coloring", *Workshop on Algorithms and Data Structures (WADS)*, 2019.

[3] I. Vinod Reddy, "Parameterized algorithms for conflict-free colorings of graphs", *Theor. Comput. Sci.*, 2018.

[4] Luisa Gargano and Adele A. Rescigno, "Complexity of Conflict-free Colorings of Graphs", *Theor. Comput. Sci.*, 2015.

THANK YOU

Questions?