A short note on conflict-free coloring on closed neighborhoods of bounded degree graphs

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Abstract

The *closed neighborhood conflict-free chromatic number* of a graph G, denoted by $\chi_{CN}(G)$, is the minimum number of colors required to color the vertices of G such that for every vertex, there is a color that appears exactly once in its closed neighborhood. Pach and Tardos [Combin. Probab. Comput. 2009] showed that $\chi_{CN}(G) = O(\log^{2+\varepsilon} \Delta)$, for any $\varepsilon > 0$, where Δ is the maximum degree. In [Combin. Probab. Comput. 2014], Glebov, Szabó and Tardos showed existence of graphs G with $\chi_{CN}(G) = \Omega(\log^2 \Delta)$. In this paper, we bridge the gap between the two bounds by showing that $\chi_{CN}(G) = O(\log^2 \Delta)$.

Conflict-free coloring was introduced [ELRS04] in 2003 motivated by problems arising from situations in wireless communication. Over the past two decades, conflict-free coloring has been extensively studied [Smo13].

Definition 1 (Conflict-free chromatic number of hypergraphs)**.** *The* conflict-free chromatic number *of a hypergraph $H = (V, E)$ is the minimum number of colors required to color the points in V such that every $e \in E$ contains a point whose color is distinct from that of every other point in e.*

Conflict-free coloring has also been studied in the context of hypergraphs created out of simple graphs. Two such variants are *conflict-free coloring on closed neighborhoods* and *conflict-free coloring on open neighborhoods*. In this note, we focus on the former variant. For any vertex v of an undirected graph G, let $N_G(v) := \{u \in V(G) : \{u, v\} \in E(G)\}$ denote the *open neighborhood* of v in G. Let $N_G[v] := N_G(v) \cup \{v\}$ denote the *closed neighborhood* of v in G.

Definition 2 (Closed neighborhood conflict-free chromatic number)**.** *Given an undirected graph $G = (V, E)$, a conflict-free coloring on closed neighborhoods (CFCN coloring) is an assignment of colors $C : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that for every $v \in V(G)$, there exists an $i \in \{1, 2, \ldots, k\}$ such that $|N[v] \cap C^{-1}(i)| = 1$. The smallest k required for such a coloring is called the CFCN chromatic number of G, denoted $\chi_{CN}(G)$.*

In other words, given a graph G, let H be the hypergraph with $V(H) = V(G)$ and $E(H) = \{N_G[v] : v \in V(G)\}$. Then, $\chi_{CN}(G)$ is equal to the conflict-free chromatic number of this hypergraph H. Pach and Tardos [PT09] showed that for a graph G with maximum degree Δ, $\chi_{CN}(G) = O(\log^{2+\varepsilon} \Delta)$ for any $\varepsilon > 0$. We show the following improved bound.

Theorem 3. *Let G be a graph with maximum degree Δ. Then $\chi_{CN}(G) = O(\log^2 \Delta)$.*

In 2014, Glebov, Szabó and Tardos [GST14] showed the existence of graphs G on n vertices such that $\chi_{CN}(G) = \Omega(\log^2 n)$. Since $\Delta < n$, our bound in Theorem 3 is tight up to constants.

Before we proceed to the proof, we explain some notations. All logarithms we consider here are to the base 2. Given a graph G and a set $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced on the vertex set S. For any two vertices $u, v \in V(G)$, we use $dist_G(u, v)$ to denote the number of edges in a shortest path between u and v in G. We set $dist_G(u, v) = \infty$ when there is no path between u and v in G.

Definition 4 (Maximal Distance-3^+ Set)**.** *For a graph G, a* maximal distance-3^+ set *is a set $A \subseteq V(G)$ that satisfies the following:*

1. *For every two distinct $u, v \in A$, $dist_G(u, v) \geq 3$.*
2. *For every $v \in V(G) \setminus A$, $\exists u \in A$ such that $dist_G(u, v) < 3$.*

Let A be a maximal distance-3^+ set in G. Let $B = \{v \in V(G)\setminus A : v \text{ has a neighbor in } A\}$ and let $C = V(G) \setminus (A \cup B)$. We note the following:

Observation 5. *Every vertex in B has exactly one neighbor in A.*

Observation 6. *Every vertex in C has at least one neighbor in B.*

Our proof uses the following theorem on conflict-free coloring on hypergraphs due to Pach and Tardos [PT09].

Theorem 7 (Theorem 1.2 in [PT09])**.** *For any positive integers t and Γ, the conflict-free chromatic number of any hypergraph in which each edge is of size at least $2t - 1$ and each edge intersects at most Γ others is $O(t\Gamma^{1/t} \log \Gamma)$. There is a randomized polynomial time algorithm to find such a coloring.*

Proof of Theorem 3. We perform the following iterative process starting with $G_0 = G$.
Iterative coloring process: Let A_i be a maximal distance-3^+ set in G_i. Let $B_i := \{v \in V(G_i) \setminus A_i : v \text{ has a neighbor in } A_i\}$ and $C_i := V(G_i) \setminus (A_i \cup B_i)$. Assign a color c_i to all the vertices in A_i. Observation 5 combined with the fact that A_i is an independent set in G_i imply that for every vertex $v \in A_i \cup B_i$, $N_G[v]$ contains exactly one vertex with the color c_i. Repeat the above process with $G_{i+1} = G[C_i]$.

The iterative process is repeated till one of the following two conditions is satisfied: (i) G_i is the empty graph, or (ii) $i = k = 4 \log \Delta$.

If the process terminated with $i < 4 \log \Delta$, then we have CFCN-colored G with $O(\log \Delta)$ colors. Suppose it terminated with $i = k = 4 \log \Delta$. We know that every vertex in $V(G) \setminus C_k$ has some color appearing exactly once in its closed neighborhood under the present coloring. In order to complete the proof, we need to extend this 'nice' property to the vertices of C_k as well. If C_k is the empty set, then the proof is complete. Asssume C_k is non-empty. Let H be a hypergraph constructed from G as explained here. Let

$V(H) = \cup_{i=0}^{k} B_i$ and $E(H) = \{e_v : v \in C_k\}$, where $e_v = \{N_G(v) \cap V(H)\}$. We note that each vertex in the set $V(H) = \cup_{i=0}^{k} B_i$ is uncolored so far. Consider a vertex v in C_k. From Observation 6, v has at least one neighbor in each of the sets $B_0, B_1, \ldots, B_k$, and hence $|e_v| \geq 4 \log \Delta + 1$. Further, each hyperedge in this hypergraph intersects at most Δ^2 other hyperedges. Substituting $t = 2 \log \Delta$ and $\Gamma = \Delta^2$ in Theorem 7, we can see that the conflict-free chromatic number of the hypergraph H is $O(\log^2 \Delta)$. We ensure that the set of colors we use to color the points in the hypergraph H is disjoint from the set $\{c_0, c_1, \ldots, c_k\}$. Now, consider the graph G. For each vertex $v \in \cup_{i=0}^{k} B_i = V(H)$, we assign the color it obtained while coloring H. This means that every vertex in C_k now has some color appearing exactly once in its closed neighborhood in G and thereby satisfying the 'nice' property mentioned above. Finally, use a new color (that has not been used so far) to color all the uncolored vertices in G. $\square$

Algorithmic note: The construction of maximal distance-3^+ sets is performed by repeatedly selecting vertices that are at distance 3 or more. Thus the sets A_i, B_i and C_i can be computed in deterministic polynomial time. The randomized algorithm for finding the desired coloring for the hypergraph, in the proof of Theorem 7 in [PT09], is obtained using an algorithmic version of the Local Lemma. Deterministic algorithms have been devised for the Local Lemma [CGH13, Har19], and we could use these deterministic algorithms in place of the randomized algorithm used in [PT09]. By applying Theorem 1.1 (1) from [Har19], we get a deterministic polynomial time algorithm to find a conflict-free coloring of a hypergraph that uses $O(t\Gamma^{(1+\delta)/t} \log \Gamma)$ colors, where t and Γ are as defined in the statement of Theorem 7 and $\delta > 0$ is a constant. This weaker bound suffices to get a conflict-free coloring of the hypergraph H (defined inside the proof of Theorem 3) that uses $O(\log^2 \Delta)$ colors. We thus have a deterministic polynomial time algorithm that yields a CFCN-coloring for G using $O(\log^2 \Delta)$ colors.

References

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