Extremal Results on Conflict-free Coloring*

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Abstract

A conflict-free open neighborhood coloring of a graph is an assignment of colors to the vertices such that for every vertex there is a color that appears exactly once in its open neighborhood. For a graph G, the smallest number of colors required for such a coloring is called the conflict-free open neighborhood (CFON) chromatic number and is denoted by $\chi_{ON}(G)$. By considering closed neighborhood instead of open neighborhood, we obtain the analogous notions of conflict-free closed neighborhood (CFCN) coloring, and CFCN chromatic number (denoted by $\chi_{CN}(G)$). The notion of conflict-free coloring was introduced in 2002, and has since received considerable attention.

We study CFON and CFCN colorings and show the following results. In what follows, Δ denotes the maximum degree of the graph.

- We show that if G is a $K_{1,k}$-free graph then $\chi_{ON}(G) = O(k \ln \Delta)$. Dębski and Przybyło in [JGT 2021] had shown that if G is a line graph, then $\chi_{CN}(G) = O(\ln \Delta)$. As an open question, they had asked if their result could be extended to claw-free ($K_{1,3}$-free) graphs, which is a superclass of line graphs. Since $\chi_{CN}(G) \leq 2\chi_{ON}(G)$, our result answers their open question. It is known that there exist separate families of $K_{1,k}$-free graphs with $\chi_{ON}(G) = \Omega(\ln \Delta)$ and $\chi_{ON}(G) = \Omega(k)$.

*We note that one of the results in this submission, Theorem 23, had already appeared as part of [4] in the conference 47th International Symposium on Mathematical Foundations of Computer Science (MFCS 2022). The article [4] was authored by a subset of authors of this submission.

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- For a $K_{1,k}$-free graph G on n vertices, we show that $\chi_{CN}(G) = O(\ln k \ln n)$. This bound is asymptotically tight for some values of k since there are graphs G with $\chi_{CN}(G) = \Omega(\ln^2 n)$ [Glebov, Szabó, Tardos, CPC 2014].
- Let $\delta \geq 0$ be an integer. We define $f_{CN}(\delta)$ as follows:

  $$f_{CN}(\delta) = \max\{\chi_{CN}(G) : G \text{ is a graph with minimum degree at least } \delta\}.$$

  It is easy to see that $f_{CN}(\delta') \geq f_{CN}(\delta)$ when $\delta' < \delta$. Let c be a positive constant. It was shown [Dębski and Przybyło, JGT 2021] that $f_{CN}(c\Delta) = \Theta(\ln \Delta)$. In this paper, we show (i) $f_{CN}(\frac{c\Delta}{\ln^\epsilon \Delta}) = O(\ln^{1+\epsilon} \Delta)$, where ϵ is a constant such that $0 \leq \epsilon \leq 1$ and (ii) $f_{CN}(c\Delta^{1-\epsilon}) = \Omega(\ln^2 \Delta)$, where ϵ is a constant such that $0 < \epsilon < 0.003$. Together with the known [Bhyravarapu, Kalyanasundaram and Mathew, JGT 2021] upper bound $\chi_{CN}(G) = O(\ln^2 \Delta)$, this implies that $f_{CN}(c\Delta^{1-\epsilon}) = \Theta(\ln^2 \Delta)$.

1 Introduction

For a hypergraph $\mathcal{H} = (V, \mathcal{E})$ and a positive integer k, a coloring $f : V \to [k]$ is a *conflict-free coloring* (or *CF coloring*) of $\mathcal{H}$ if for every $E \in \mathcal{E}$, some vertex in E gets a color that is different from the color received by every other vertex in E. The minimum k such that $f : V \to [k]$ is a CF coloring of $\mathcal{H}$ is called the *Conflict-Free chromatic number* (or *CF chromatic number*) of $\mathcal{H}$. We shall use $\chi_{CF}(\mathcal{H})$ to denote the CF chromatic number of $\mathcal{H}$. The notion of CF coloring has been extensively studied in the context of 'neighborhood hypergraphs' of graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the set of neighbors of v in G is called the *open neighborhood* of v. We use $N_G(v)$ to denote this. The *closed neighborhood* of v, denoted by $N_G[v]$, is $\{v\} \cup N_G(v)$.

Definition 1 (Conflict-free open neighborhood chromatic number). *A conflict-free coloring concerning the open neighborhoods of G is an assignment of colors to $V(G)$ such that every vertex has a uniquely colored vertex in its open neighborhood. We call such a coloring a* Conflict-Free Open Neighborhood coloring *(or* CFON coloring*). The minimum number of colors required for a CFON coloring of G is called the* Conflict-Free Open Neighborhood chromatic number *(or* CFON chromatic number*), denoted by $\chi_{ON}(G)$.*

Definition 2 (Conflict-free closed neighborhood chromatic number). *A conflict-free coloring concerning the closed neighborhoods of G is an assignment of colors to $V(G)$ such that every vertex has a uniquely colored vertex in its closed neighborhood. We call such a coloring a* Conflict-Free Closed Neighborhood coloring *(or* CFCN coloring*). The minimum number of colors required for a CFCN coloring of G is called the* Conflict-Free Closed Neighborhood chromatic number *(or* CFCN chromatic number*), denoted by $\chi_{CN}(G)$.*

It is easy to see that every proper coloring is a CFCN coloring since each vertex serves as its own uniquely colored neighbor. Hence we have $\chi_{CN}(G) \leq \chi(G)$. The following result connects CFON and CFCN chromatic numbers of a graph G.

Proposition 3 (Inequality 1.3 in [18]). $\chi_{CN}(G) \leq 2\chi_{ON}(G)$.

The above follows by modifying a CFON coloring to obtain a CFCN coloring. We may have to duplicate every color in the worst case, as the unique color seen by a vertex in its open neighborhood could be its own color.

Conflict-free coloring was introduced by Even et al. [10] in the year 2002. Since its introduction, CF coloring of hypergraphs, CFON and CFCN coloring of graphs have been extensively studied [6, 11, 18, 2, 7, 3, 16, 8, 13, 12, 19]. The interested reader may refer to the survey by Smorodisnky [20] on conflict-free coloring. Abel et al. [1] showed that it is NP-complete to determine if a planar graph has a 'partial' CFCN coloring with one color (in a partial CFCN coloring, we color only a subset of the vertices such that every vertex sees a unique color in its closed neighborhood). Conflict-free colorings have been studied on various geometric intersection graphs such as interval graphs, unit disk graphs, unit square graphs etc. [4, 16, 20]. The problem has also been studied on geometric graphs such as intersection graphs of pseudo disks [13] and string graphs [12].

Conflict-free coloring and its variants have found applications in frequency assignment problem in cellular networks, battery consumption aspects of sensor networks, RFID protocols, and the vertex ranking (or, ordered coloring) problem which finds applications in VLSI design, operations research, etc. [20]. Recently, an application was discovered in the PICOD problem which is a problem from coding theory [14, 15].

2 Definitions and notations

For a positive integer k, we use $[k]$ to denote the set $\{1, 2, \ldots, k\}$. Throughout this paper, we consider only graphs that are simple, finite, and undirected. For a graph G, we use $V(G)$ to denote its vertex set and $E(G)$ to denote its edge set. In the introduction section, we had defined the open and closed neighborhoods, denoted respectively $N_G(v)$ and $N_G[v]$, for a vertex v in $V(G)$. We shall use $d_G(v)$ to denote the degree of v in G. That is, $d_G(v) = |N_G(v)|$. For a positive integer k, we shall use $K_{1,k}$ to denote the complete bipartite graph with 1 vertex in one part and k vertices in the other part. A graph is *$K_{1,k}$-free* if it does not contain $K_{1,k}$ as an induced subgraph. Graphs that are $K_{1,3}$-free are also known by the name *claw-free* graphs. The *claw number* of a graph G is defined to be the largest k for which G contains $K_{1,k}$ as an induced subgraph.

Given a graph G, the *line graph* of G, denoted by $L(G)$, is the graph defined as follows: The vertex set of $L(G)$ is $V(L(G)) = E(G)$ and two vertices e_1, e_2 of $L(G)$ are adjacent to each other if and only if in the original graph G, the edges e_1 and e_2 share an end point.

Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$, the *degree* of an element $v \in V$, denoted by $d_{\mathcal{H}}(v)$, is the number of hyperedges that v is present in. The *maximum degree of the hypergraph* $\mathcal{H}$ is $\max\{d_{\mathcal{H}}(v) \ : \ v \in V\}$.

3 Our contributions and open questions

Dębski and Przybyło in [7] showed that for a graph G with maximum degree Δ, the CFCN chromatic number of its line graph is $\chi_{CN}(L(G)) = O(\ln \Delta)$. Note that line graphs are a subclass of claw-free graphs (or $K_{1,3}$-free graphs). The following example implies that the upper bound of $O(\ln \Delta)$ from [7] is asymptotically tight.

Example 4. *Consider $L(K_n)$, the line graph of the complete graph on n vertices. In [7], it was shown that $\chi_{CN}(L(K_n)) = \Omega(\ln n)$. Since $\chi_{CN}(G) \leq 2\chi_{ON}(G)$ (Proposition 3), this implies $\chi_{ON}(L(K_n)) = \Omega(\ln n)$.*

Let us first discuss the dependence of the CFON chromatic number of a graph on its claw number k and maximum degree Δ. Example 4 is a family of graphs whose maximum degree is $2n - 4$ and claw number is 2. This means that $\chi_{ON}(G)$ cannot be a function of the form $k \cdot h(\Delta)$ where $h(\Delta) = o(\ln \Delta)$. On the other hand, Example 5, given below, is a family of graphs where $\Delta = n - 1$ and $k = n - 1$. This means that $\chi_{ON}(G)$ cannot be a function of the form $g(k) \cdot \ln \Delta$, where $g(k) = O(k^{1-\epsilon})$, for an $\epsilon > 0$.

Example 5. *Let K_n^* be the $K_{1,n}$-free graph with maximum degree $n - 1$ obtained by subdividing every edge of K_n exactly once. It is known (see [18]) that $\chi_{ON}(K_n^*) = n$.*

We complement the above observations with an upper bound of $\chi_{ON}(G) = O(k \ln \Delta)$. This implies an upper bound $\chi_{CN}(G) = O(k \ln \Delta)$ by Proposition 3. Our result, proved in Section 5, generalizes the upper bound of $\chi_{CN}(G) = O(\ln \Delta)$ [7] for line graphs. As mentioned before, line graphs are a subclass of claw-free graphs. In many of the practical applications that motivate conflict-free coloring, the underlying graphs happen to be geometric intersection graphs such as unit disk graphs, unit square graphs, etc. [16, 20]. These graph classes are usually $K_{1,k}$-free for some constant k. For instance, unit disk graphs are $K_{1,6}$-free.

It was posed as an open question in [7] if the $O(\ln \Delta)$ upper bound could be generalized to claw-free graphs. Our $O(k \ln \Delta)$ upper bound answers this question in the affirmative. Though Examples 4 and 5 imply the existence of graphs G for which $\chi_{ON}(G) = \Omega(k)$ and $\chi_{ON}(G) = \Omega(\ln \Delta)$, it is of interest to know whether the upper bound of $O(k \ln \Delta)$ is tight.

Open Question 6. *Are there $K_{1,k}$-free graphs G with maximum degree Δ for which $\chi_{ON}(G) = \Omega(k \ln \Delta)$?*

In Section 5, we show that if G is a $K_{1,k}$-free graph on n vertices, then $\chi_{CN}(G) = O(\ln k \ln n)$. This bound is asymptotically tight for some values of k as it was shown in [11] that there exist graphs G on n vertices with $\chi_{CN}(G) = \Omega(\ln^2 n)$. This still leaves the possibility of the following improvement:

Open Question 7. *Can a bound of $O(\ln k \ln \Delta)$ be obtained for $\chi_{CN}(G)$, for $K_{1,k}$-free graphs G with maximum degree Δ?*

Now we turn our attention to CFCN chromatic number for graphs of a specified minimum degree. Let Δ denote the maximum degree of the graph under consideration and let c be any positive constant. Let $\delta \geq 0$ be an integer. We define

$$f_{CN}(\delta) := \max\{\chi_{CN}(G) \ : \ G \text{ is a graph with minimum degree at least } \delta\}.$$

It is easy to see that $f_{CN}(\delta') \geq f_{CN}(\delta)$ when $\delta' < \delta$. Dębski and Przybyło in [7] showed that $f_{CN}(c\Delta) = \Theta(\ln \Delta)$. In Section 6, we show that $f_{CN}(c\frac{\Delta}{\ln^{\epsilon}\Delta}) = O(\ln^{1+\epsilon}\Delta)$, where $0 \leq \epsilon \leq 1$. A natural open question is if this bound is tight.

Open Question 8. *Is* $f_{CN}(c\frac{\Delta}{\ln^{\epsilon}\Delta}) = \Omega(\ln^{1+\epsilon}\Delta)$*?*

Further, in Section 7, we show that $f_{CN}(c\Delta^{1-\epsilon}) = \Omega(\ln^2\Delta)$, for $0 < \epsilon < 0.003$. It was shown by Bhyravarapu, Kalyanasundaram, and Mathew [2] that for any graph G, $\chi_{CN}(G) = O(\ln^2\Delta)$. Combining both, we get $f_{CN}(c\Delta^{1-\epsilon}) = \Theta(\ln^2\Delta)$. An affirmative answer to Open Question 8 will help us understand the function f_{CN} in its full range.

Analogous to the function f_{CN}, we can define a function f_{ON} as

$$f_{ON}(\delta) := \max\{\chi_{ON}(G) \ : \ G \text{ is a graph with minimum degree at least } \delta\}.$$

Like in the case of CFCN coloring, we have that $f_{ON}(\delta') \geq f_{ON}(\delta)$ when $\delta' < \delta$. The results in [7] imply[1] that $f_{ON}(c\Delta) = \Theta(\ln \Delta)$. It was shown by Pach and Tardos [18] that for any graph G with minimum degree $c \log \Delta$, we have $\chi_{ON}(G) = O(\ln^2\Delta)$. Combining this with our result in Section 7, we have $f_{ON}(c\Delta^{1-\epsilon}) = \Theta(\ln^2\Delta)$, where $0 < \epsilon < 0.003$. What is the value of $f_{ON}(\delta)$, when $\delta = o(\ln \Delta)$? It is known that $\chi_{ON}(G) \leq \Delta + 1$, for any graph G. This bound is tight as $\chi_{ON}(K_n^*) = n$. Thus, $f_{ON}(c) = \Theta(\Delta)$. This leaves us with the following open question.

Open Question 9. *What is the value of* $f_{ON}(\delta)$ *when* $\delta = o(\ln \Delta)$ *and* δ *is not any absolute constant?*

Theorem 1.2 in [18] implies that $f_{ON}(\delta) = O(\delta \cdot \Delta^{\frac{2}{\delta}} \cdot \ln \Delta)$. However, it is not clear whether this bound is tight in the range of values of δ that we are interested in.

4 Auxiliary results

In this section, we state a few known auxiliary lemmas and theorems that will be used later. We state the local lemma that will be used in the proof of Lemma 13 and Chernoff bound that will be used in the proof of Theorem 24.

[1]The article [7] explicitly discusses only CFCN chromatic number. However, the proof techniques of the upper bound extend to yield an identical upper bound for CFON chromatic number. The lower bound in [7] implies a similar lower bound for CFON chromatic number by an application of Proposition 3.

Lemma 10 (*The Local Lemma,* [9]). *Let $A_1, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most d, and that $Pr[A_i] \leq p$ for all $i \in [n]$. If $4pd \leq 1$, then $Pr[\cap_{i=1}^{n} \overline{A_i}] > 0$.*

Theorem 11 (Chernoff Bound, Corollary 4.6 in [17]). *Let $X_1, \ldots, X_n$ be independent Poisson trials such that $Pr[X_i] = p_i$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$. For $0 < \delta < 1$, $Pr[|X - \mu| \geq \delta\mu] \leq 2e^{-\mu\delta^2/3}$.*

The theorem below gives an upper bound to the CF chromatic number of a hypergraph in terms of its maximum degree.

Theorem 12 (Theorem 1.1(b) in [18]). *Let $\mathcal{H}$ be a hypergraph and let Δ be the maximum degree of any vertex in $\mathcal{H}$. Then, $\chi_{CF}(\mathcal{H}) \leq \Delta + 1$.*

Finally, we prove the following lemma for CF chromatic number of near uniform hypergraphs. We will use this lemma in the proofs of Theorems 14 and 23.

Lemma 13. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph where (i) every hyperedge intersects with at most Γ other hyperedges, and (ii) for every hyperedge $E \in \mathcal{E}$, $r \leq |E| \leq \ell r$, where $\ell \geq 1$ is some integer and $r \geq 2\log_2(4\Gamma)$. Then, $\chi_{CF}(\mathcal{H}) \leq e\ell r$, where e is the base of natural logarithm.*

Proof. For each vertex in V, assign a color that is chosen independently and uniformly at random from a set of $e\ell r$ colors. We will first show that the probability of this coloring being bad for an edge is small, and then use Local Lemma to show the existence of conflict-free coloring for $\mathcal{H}$ using at most $e\ell r$ colors.

Consider a hyperedge $E \in \mathcal{E}$ with $m := |E|$. By assumption, we have $r \leq m \leq \ell r$. Let A_E denote the bad event that E is colored with $\leq |E|/2$ colors. Note that if A_E does not occur, then E is colored with $> |E|/2$ colors, hence there is at least one color that appears exactly once in E.

$$\begin{aligned}
Pr[A_E] &\leq \binom{e\ell r}{m/2}\left(\frac{m/2}{e\ell r}\right)^m \\
&\leq \left(\frac{e^2\ell r}{m/2}\right)^{m/2}\left(\frac{m/2}{e\ell r}\right)^m \qquad \left(\text{since } \binom{n}{k} \leq \left(\frac{en}{k}\right)^k\right) \\
&= \frac{(m/2)^{m/2}}{(\ell r)^{m/2}} = \left(\frac{m}{2\ell r}\right)^{m/2} \\
&\leq (1/2)^{m/2} \leq \frac{1}{4\Gamma}.
\end{aligned}$$

Here the penultimate inequality follows since $m \leq \ell r$, and the last inequality follows since $m \geq 2\log_2(4\Gamma)$.

We apply the Local Lemma (Lemma 10) on the events A_E for all hyperedges $E \in \mathcal{E}$. Since each hyperedge intersects with at most Γ other hyperedges, and $4 \cdot \frac{1}{4\Gamma} \cdot \Gamma \leq 1$, we get $Pr[\cap_{E\in\mathcal{E}}(\overline{A}_E)] > 0$. That is, there is a conflict free coloring of $\mathcal{H}$ that uses at most $e\ell r$ colors. This completes the proof of the lemma. □

5 $K_{1,k}$-free graphs

In this section, we show improved upper bounds for CFON and CFCN chromatic numbers on $K_{1,k}$-free graphs.

Theorem 14. *Let G be a $K_{1,k}$-free graph with maximum degree $\Delta \geq 2$ having no isolated vertices. Then, $\chi_{ON}(G) = O(k \ln \Delta)$.*

Proof. Let A be a maximal independent set of G. Let $A_1 := \{v \in A : d_G(v) \leq 12k \ln \Delta\}$ and $A_2 := A \setminus A_1$. Let $X := \bigcup_{v \in A_1} N_G(v)$.

Next, we obtain G' by removing all the vertices from G that belong to $A \cup X$. In other words, $G' = G[V \setminus (A \cup X)]$. Since A is a maximal independent set in G, every $v \in V \setminus (A \cup X)$ has a neighbor, say w, in A. Since no vertex in A_1 has a neighbor in $V(G')$, $w \in A_2$. Thus we have the following:

Observation 15. *Every vertex in $V(G')$ has a neighbor in A_2.*

We start with a proper coloring of G', say $h : V(G') \longrightarrow [s] = \{1, 2, \ldots, s\}$ that uses at most $\Delta + 1$ colors. Let $L_1, L_2, \ldots, L_s$ be the color classes with respect to the coloring h, where $s \leq \Delta + 1$.

Observation 16. *For every $2 \leq i \leq s$, we may assume that every vertex $v \in L_i$ has a neighbor in each L_j, $1 \leq j < i$. If v has no neighbor in L_j, $j < i$, we can move v to L_j.*

Observation 17. *Since G is $K_{1,k}$-free, any vertex in G has at most $k - 1$ neighbors in L_i, for every $i \in [s]$.*

Observation 18. *Consider a subset $\widehat{A} \subseteq A$, and let $\widehat{\mathcal{H}} = (\widehat{V}, \widehat{\mathcal{E}})$ be defined as follows: $\widehat{V} = \bigcup_{v \in \widehat{A}} N_G(v)$ and $\widehat{\mathcal{E}} = \{N_G(v) : v \in \widehat{A}\}$. Since G is $K_{1,k}$-free and A is an independent set, the maximum degree of $\widehat{\mathcal{H}}$ is at most $k - 1$.*

If $s > 12 \ln \Delta$, then we define $B := L_1 \cup L_2 \cup \cdots \cup L_{12 \ln \Delta}$ and $C = V(G') \setminus B$. Otherwise, we define $B := L_1 \cup L_2 \cup \cdots \cup L_s$ and $C = \emptyset$.

We obtain the desired CFON coloring of G by conflict-free coloring five hypergraphs, $\mathcal{H}_1, \ldots, \mathcal{H}_5$, which are defined below. Note that the set of colors we use to color each hypergraph $\mathcal{H}_i$ is disjoint from the set of colors we use to color any other hypergraph $\mathcal{H}_j$, $1 \leq i < j \leq 5$.

- Suppose $C \neq \emptyset$. We define a hypergraph $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$, where $V_1 = B$ and $\mathcal{E}_1 = \{N_{G'}(v) \cap B : v \in C\}$. The following observation follows from Observations 16 and 17.

  Observation 19. *Every vertex in C has at least $12 \ln \Delta$ neighbors in B. Further, for every $v \in V(G)$, $|N_G(v) \cap B| \leq 12(k - 1) \ln \Delta$.*

  So for each $E \in \mathcal{E}_1$, we have $12 \ln \Delta \leq |E| \leq 12(k-1) \ln \Delta$. By applying Lemma 13 to $\mathcal{H}_1$, with $\ell = k-1, r = 12 \ln \Delta$, and $\Gamma \leq \Delta^2$, we get $\chi_{CF}(\mathcal{H}_1) \leq e \cdot (k-1) \cdot 12 \ln \Delta$.

Here, Γ denotes the number of other hyperedges a given hyperedge $E := N_{G'}(v) \cap B$, for some $v \in C$, is overlapping with. Since the maximum degree is Δ, and since $E \subseteq N_G(v)$, it follows that $\Gamma \leq \Delta^2$. The conflict-free coloring of hypergraph $\mathcal{H}_1$ ensures that all the vertices in C, see a unique color in their open neighborhood.

- Similarly, we define a hypergraph $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$, where $V_2 = A_2$ and $\mathcal{E}_2 = \{N_G(v) \cap A_2 : v \in B\}$. By Observation 15, every vertex in B has a neighbor in A_2. This ensures that the hypergraph $\mathcal{H}_2$ does not have empty hyperedges. By Observation 17, the maximum degree of $\mathcal{H}_2$ is at most $(k-1) \cdot 12 \ln \Delta$. Hence by Theorem 12, we have $\chi_{CF}(\mathcal{H}_2) \leq (k-1) \cdot 12 \ln \Delta + 1$. This conflict-free coloring of $\mathcal{H}_2$ ensures that every $v \in B$ sees a unique color in its open neighborhood.

- Let $\mathcal{H}_3 = (V_3, \mathcal{E}_3)$ be a hypergraph, where $V_3 = A_1$ and $\mathcal{E}_3 = \{N_G(v) \cap A_1 : v \in X\}$. By choice of vertices in A_1, the maximum degree of $\mathcal{H}_3$ is at most $12k \ln \Delta$. Hence by Theorem 12, $\chi_{CF}(\mathcal{H}_3) \leq 12k \ln \Delta + 1$. The conflict-free coloring of $\mathcal{H}_3$ ensures that every $v \in X$ sees a unique color in its open neighborhood.

Now we need to handle the needs of the vertices in A. We first partition A as follows: let $A_X = \{v \in A : N_G(v) \cap X \neq \emptyset\}$ and $A_{\overline{X}} = A \setminus A_X$. From their definitions, (A_1, A_2) and $(A_X, A_{\overline{X}})$ are two partitions of A that satisfy $A_1 \subseteq A_X$, and thereby $A_{\overline{X}} \subseteq A_2$. Also, no vertex in $A_{\overline{X}}$ has a neighbor in X. Since $A_{\overline{X}} \subseteq A_2$ and every vertex in A_2 has degree greater than $12k \ln \Delta$, we have the following observation.

Observation 20. *For every $v \in A_{\overline{X}}$, (i) $N_G(v) \subseteq B \cup C$, and (ii) $|N_G(v)| > 12k \ln \Delta$.*

- We define a hypergraph $\mathcal{H}_4 = (V_4, \mathcal{E}_4)$, where $V_4 = X$ and $\mathcal{E}_4 = \{N_G(v) \cap X : v \in A_X\}$. By the definition of A_X, the hypergraph $\mathcal{H}_4$ does not have empty hyperedges. By Observation 18, the maximum degree of $\mathcal{H}_4$ is at most $k-1$. Hence by Theorem 12, $\chi_{CF}(\mathcal{H}_4) \leq k$. This coloring addresses the requirements of the vertices in A_X.

- What is left to be addressed are the requirements of the vertices in $A_{\overline{X}}$. We first claim that, if $C = \emptyset$, then $A_{\overline{X}} = \emptyset$. Assume, for the sake of contradiction, that $v \in A_{\overline{X}}$. Then, by Observation 20, $|N_G(v)| > 12k \ln \Delta$ and $N_G(v) \subseteq B$. By Observation 19, $|N_G(v)| \leq 12(k-1) \ln \Delta$ which is a contradiction.

  Now we may assume that $C \neq \emptyset$. Then, we construct a hypergraph $\mathcal{H}_5 = (V_5, \mathcal{E}_5)$, where $V_5 = C$ and $\mathcal{E}_5 = \{N_G(v) \cap C : v \in A_{\overline{X}}\}$. From Observations 19 and 20, it follows that $N_G(v) \cap C \neq \emptyset$, for every $v \in A_{\overline{X}}$. By Observation 18, we know that the maximum degree of $\mathcal{H}_5$ is at most $k-1$. Therefore, by Theorem 12, $\chi_{CF}(\mathcal{H}_5) \leq k$.

Note that we have addressed the needs of all the vertices in G. Also, each vertex is colored at most once in the above. There may be vertices that are left uncolored because they did not feature in any of the hypergraphs. We can assign all these vertices a new color, obtaining a conflict-free coloring of G that uses at most $57k \ln \Delta + 2k + 3$ colors. $\square$

The following result gives an upper bound for CFCN chromatic number of $K_{1,k}$-free graphs.

Theorem 21. *Let G be a $K_{1,k}$-free graph with n vertices. Then, $\chi_{CN}(G) = O(\ln k \ln n)$.*

Proof. We first give a brief overview of the proof. We use an approach similar to the one used by Pach and Tardos for the proof of Theorem 1.6 in [18]. Using a probabilistic approach, we show the existence of a subset I_1^* of a maximal independent set. By coloring all the vertices of I_1^* with the color 1, we can ensure that a $c/\log_2 k$ fraction of the vertices of the graph sees a uniquely colored neighbor. Since the need of the vertices in I_1^* is satisfied by themselves, we will not recolor them in the later coloring process. We can repeat this process $O(\ln k \ln n)$ times to ensure that all the vertices of G see a uniquely colored neighbor.

We now describe in detail how we pick the random independent sets. Let $G_1 = G$. Let S_1 be a maximal independent set in G_1. Pick an integer i uniformly at random from the set $\{0, 1, \ldots, \lfloor \log_2 k \rfloor\}$. Select $I_1 \subseteq S_1$ by picking v into I_1 with probability 2^{-i} independently, for each vertex $v \in S_1$. Note that the integer i can be equal to 0 with probability $\frac{1}{\lfloor \log_2 k \rfloor + 1}$. In that case, every $v \in S_1$ is chosen into I_1 with probability 1.

$$\forall v \in S_1, Pr[v \text{ is chosen into } I_1] \geq \frac{1}{\lfloor \log_2 k \rfloor + 1} . \tag{1}$$

Color every vertex in I_1 with color 1. Let $A_1 = \{v \in V(G_1) \setminus S_1 : |N_{G_1}(v) \cap I_1| = 1\}$. For a vertex $w \in V(G_1) \setminus S_1$, we define $d_w := |N_{G_1}(w) \cap S_1|$. Note that for any $w \in V(G_1) \setminus S_1$, we have $k - 1 \geq d_w \geq 1$ as (1) G is $K_{1,k}$-free, and (2) S_1 is a maximal independent set in G_1. For a vertex $w \in V(G_1) \setminus S_1$, what is the probability that $w \in A_1$ (or $|N_{G_1}(w) \cap I_1| = 1$)? Let A_1^w denote the event that $w \in A_1$. Let $p = \frac{1}{2^{\lfloor \log_2 d_w \rfloor}}$. Below, we estimate the probability of the event A_1^w.

$$\begin{aligned} Pr[A_1^w] &= \sum_{x=0}^{\lfloor \log_2 k \rfloor} Pr[i = x] \cdot Pr[A_1^w | i = x] \\ &\geq Pr[i = \lfloor \log_2 d_w \rfloor] \cdot Pr[A_1^w | i = \lfloor \log_2 d_w \rfloor] \\ &= \frac{1}{\lfloor \log_2 k \rfloor + 1} \left(d_w \cdot p(1-p)^{d_w - 1} \right) . \end{aligned}$$

We analyze the above expression for different values of d_w. When $d_w = 1$, we can check that $Pr[A_1^w | i = \lfloor \log_2 d_w \rfloor] = 1$. When $2 \leq d_w \leq 4$, we can verify using direct calculations that $d_w \cdot p(1-p)^{d_w - 1} > 0.02$. For the remaining values of d_w, we have that

$$d_w \cdot p(1-p)^{d_w - 1} \geq \left(1 - \frac{2}{d_w}\right)^{d_w - 1} \geq \left(1 - \frac{2}{d_w}\right)^{d_w} \geq e^{-2}\left(1 - \frac{4}{d_w}\right) \geq c,$$

where $c = 0.02$. The first inequality holds since $\frac{1}{d_w} \leq p \leq \frac{2}{d_w}$, and the third inequality holds since $(1 + x/n)^n \geq e^x(1 - x^2/n)$ for $n \geq 1$, $|x| \leq n$. The last inequality holds when

$d_w \geq 5$. Thus we get the following:

$$\forall w \in V(G_1)\backslash S_1, Pr[w \text{ is chosen into } A_1] \geq \frac{c}{\lfloor \log_2 k \rfloor + 1}\,. \tag{2}$$

From equations (1) and (2), we have the expected cardinality of $I_1 \cup A_1$ is at least $\frac{c \cdot |V(G_1)|}{\lfloor \log_2 k \rfloor + 1}$. By the probabilistic method, this implies the existence of I_1^* such that at least $\frac{c \cdot |V(G_1)|}{\lfloor \log_2 k \rfloor + 1}$ vertices of G_1 see a uniquely colored neighbor. Color all the vertices of I_1^* with color 1.

Let G_2 be the subgraph of G_1 induced on the vertices that do not have a uniquely colored neighbor. We can repeat the same construction and argument for G_2, ensuring that $\frac{c \cdot |V(G_2)|}{\lfloor \log_2 k \rfloor + 1}$ vertices of G_2 see a uniquely colored neighbor.

After r such rounds, there will be at most $n\big(1 - \frac{c}{\log_2 k}\big)^r$ many vertices of the graph that do not have a uniquely colored neighbor. By setting $r > (\ln n \log_2 k)/c$, we get that there are < 1 vertices that do not have a uniquely colored neighbor. Since we use one new color per round, we need $(\ln n \log_2 k)/c = O(\ln k \ln n)$ colors. □

6 Graphs with high minimum degree

We first prove the following lemma, which will be used in the proof of Theorem 23.

Lemma 22. *Let Δ denote the maximum degree of a graph G. It is given that every vertex in G has degree at least $\frac{c\Delta}{\ln^{\epsilon} \Delta}$ for some $\epsilon \geq 0$ and c is a constant. Then, there exists $A \subseteq V(G)$ such that for every vertex $v \in V(G)$,*

$$108 \ln(2\Delta) < |N_G(v) \cap A| < \frac{180}{c} \ln^{1+\epsilon}(2\Delta).$$

Proof. We construct a random subset A of $V(G)$ as described below. Each $v \in V(G)$ is independently chosen into A with probability $\frac{144 \ln^{1+\epsilon}(2\Delta)}{c\Delta}$. For a vertex $v \in V(G)$, let X_v be a random variable that denotes $|N_G(v) \cap A|$. Then, $\mu_v := E[X_v] = \frac{144 \ln^{1+\epsilon}(2\Delta)}{c\Delta} d_G(v) \geq 144 \ln(2\Delta)$. Since $d_G(v) \leq \Delta$, we also have $\mu_v \leq \frac{144 \ln^{1+\epsilon}(2\Delta)}{c}$. Let B_v denote the event that $|X_v - \mu_v| \geq \frac{\mu_v}{4}$. Applying Theorem 11 with $\delta = 1/4$, we get $Pr[B_v] = Pr[|X_v - \mu_v| \geq \frac{\mu_v}{4}] \leq 2e^{-\frac{\mu_v}{48}} \leq 2e^{-\frac{144 \ln(2\Delta)}{48}} = \frac{2}{(2\Delta)^3}$. The event B_v is mutually independent of all but those events B_u where $N_G(u) \cap N_G(v) \neq \emptyset$. Hence, every event B_v is mutually independent of all but at most Δ^2 other events. Applying Lemma 10 with $p = Pr[B_v] \leq \frac{2}{(2\Delta)^3}$ and $d = \Delta^2$, we have $4 \cdot \frac{2}{(2\Delta)^3} \cdot \Delta^2 \leq 1$. Thus, there is a non-zero probability that none of the events B_v occur. In other words, for every v, it is possible to have $\frac{3}{4}\mu_v < X_v < \frac{5}{4}\mu_v$. Using the upper and lower bounds of μ_v we computed above, we can say that there exists an A such that, for every v, $108 \ln(2\Delta) < |N_G(v) \cap A| < \frac{180}{c} \ln^{1+\epsilon}(2\Delta)$. □

The following theorem provides improved upper bounds for χ_{ON} in terms of its maximum degree for graphs G that have high minimum degrees.

Theorem 23. *Let G be a graph with maximum degree Δ. It is given that every vertex in G has a degree at least $\frac{c\Delta}{\ln^{\epsilon} \Delta}$ for some $\epsilon \geq 0$ and c is a constant. Then, $\chi_{ON} = O(\ln^{1+\epsilon} \Delta)$.*

Proof. Apply Lemma 22 to find an $A \subseteq V(G)$ such that for every $v \in V(G)$, $108\ln(2\Delta) < |N_G(v) \cap A| < \frac{180}{c}\ln^{1+\epsilon}(2\Delta)$. Construct a hypergraph $\mathcal{H} = (A, \mathcal{E})$ where $\mathcal{E} = \{N_G(v) \cap A : v \in V(G)\}$. Every $E \in \mathcal{E}$ satisfies $2\log_2(4\Delta^2) < 108\ln(2\Delta) < |E| < \frac{180}{c}\ln^{1+\epsilon}(2\Delta)$. Applying Lemma 13 with $r = 108\ln(2\Delta)$ and $\ell = \frac{5}{3c}\log^{\epsilon}(2\Delta)$, we get $\chi_{CF}(\mathcal{H}) \leq \frac{490}{c}\ln^{1+\epsilon}(2\Delta)$. By assigning an unused color to the vertices in $V(G) \setminus A$, we can extend a conflict-free coloring of $\mathcal{H}$ to a CFON coloring for G. □

7 A lower bound

Let $\delta \geq 0$ be an integer. Recall that, $f_{CN}(\delta) = \max\{\chi_{CN}(G) : G \text{ has minimum degree at least } \delta\}$. From the definition, it follows that $f_{CN}(\delta') \geq f_{CN}(\delta)$, when $\delta' < \delta$. As discussed earlier, with Δ denoting the maximum degree of the graph under consideration and with c denoting any positive constant, we know that $f_{CN}(c\Delta) = \Theta(\ln\Delta)$. In this section, in Theorem 24, we show that $f_{CN}(c\Delta^{1-\epsilon}) = \Omega(\ln^2\Delta)$, where $0 < \epsilon < 0.003$ is a constant. Combined with the known upper bound $\chi_{CN}(G) = O(\ln^2\Delta)$ for any graph G, due to [2], we have $f_{CN}(c\Delta^{1-\epsilon}) = \Theta(\ln^2\Delta)$. In order to show that $f_{CN}(c\Delta^{1-\epsilon}) = \Omega(\ln^2\Delta)$, we need to show the existence of a graph with minimum degree $\Omega(\Delta^{1-\epsilon})$ having CFCN chromatic number $\Omega(\ln^2\Delta)$. We use the same random graph model used by Glebov, Szabó, and Tardos in [11] and show that such a graph exists with positive probability. Our proof is an extension of the proof of Theorem 4 in [11] as it builds on the ideas presented there.

Let $A \subseteq V(G)$, for a graph G. We define $N_G^{(1)}(A) := \{v \in V(G)\backslash A : |N_G(v) \cap A| = 1\}$ to be the set of vertices outside A that have exactly one neighbor in A.

Theorem 24. *There exists a graph G with maximum degree Δ and minimum degree $\Delta^{1-\epsilon}$, where $0 < \epsilon < 0.003$ is a constant, such that $\chi_{CN}(G) = \Omega(\ln^2\Delta)$.*

Proof Sketch. *We start our proof by constructing a random graph G on n vertices. Over all the choices of colorings, we bound the number of vertices that gets taken care of, i.e., see a uniquely colored vertex in its closed neighborhood. Our proof is divided into three lemmas. Lemma 25 provides a bound asymptotically almost surely (a.a.s.) on the number of vertices that serve as their own uniquely colored neighbor. Lemmas 26 and 27 together a.a.s. bound the number of vertices that are taken care of by their neighboring vertices. Finally, it follows that independent of the choice of coloring used, a.a.s., the total number of vertices that are taken care of either by themselves or by their neighboring vertices is less than the total number of vertices n.*

Let $\epsilon_0 = \frac{\epsilon}{3}$. Below, we describe the construction of a random graph G on n vertices. We use V to denote $V(G)$. For the sake of simplicity, we assume that $\lfloor \ln n \rfloor$ divides n. We partition the vertex set V into parts $L_1, \ldots, L_{\lfloor \ln n \rfloor}$ of size $\frac{n}{\lfloor \ln n \rfloor}$ each. We define the weight of a vertex $x \in L_i$ to be $w_x = (1-\epsilon_0)^i$. For any $x \in L_i$, $y \in L_j$, we put an edge between x and y with probability $w_x w_y = (1-\epsilon_0)^{i+j}$. We define the weight of a set $S \subseteq V$ as,

$$w(S) = \sum_{v \in S} w_v.$$

Let $f : V(G) \longrightarrow [\epsilon_0^3 \ln^2 \Delta]$ be a coloring (not necessarily proper) of the vertices of G. We say that a vertex x is *taken care of* by a vertex w under the coloring f if

1. $w \in N_G[x]$, and

2. $f(w)$ is distinct from $f(y)$, for every $y \in N_G[x] \setminus \{w\}$. When $w = x$, we say that a vertex x is *taken care of by itself* under f.

For each color class of the above coloring, the vertices that are taken care of by themselves form an independent set. The below lemma provides a bound on such vertices.

Lemma 25. *For the graph G constructed, the independence number $\alpha(G) \leq n^{0.003}$ asymptotically almost surely.*

Proof. We know that every edge in G is present with probability at least $(1-\epsilon_0)^{2\ln n}$. Suppose $Pr[\alpha(G) > n^{0.003}]$ does not tend to 0 as n tends to infinity. Then, we can say that for a graph $H \in \mathcal{G}(n, (1-\epsilon_0)^{2\ln n})$, $Pr[\alpha(H) > n^{0.003}]$ too does not tend to 0 as $n \longrightarrow \infty$. Here $\mathcal{G}(n, (1-\epsilon_0)^{2\ln n})$ denotes the Erdős-Rényi graph on n vertices where each edge is chosen with probability $(1-\epsilon_0)^{2\ln n}$. But this contradicts a known result (see Theorem 11.25 (ii) in Bollobás' book [5]) that the largest independent set in $\mathcal{G}(n,p)$ has size at most $2\frac{\ln(np)}{p}$ a.a.s. when $2.27/n \leq p \leq 1/2$. This would imply that the following holds a.a.s.:

$$\begin{aligned}
\alpha(H) &\leq 2\frac{\ln n}{(1-\epsilon_0)^{2\ln n}} = \frac{2\ln n}{n^{2\ln(1-\epsilon_0)}} = n^{\frac{\ln(2\ln n)}{\ln n} - 2\ln(1-\epsilon_0)} = n^{-2\ln(1-\epsilon_0)\left[-\frac{\ln(2\ln n)}{2(\ln n)\cdot(\ln(1-\epsilon_0))}+1\right]} \\
&= n^{-2\ln(1-\epsilon_0)\left[1-o(1)\right]} = n^{2\ln\left(\frac{1}{(1-\epsilon_0)}\right)\left[1-o(1)\right]} \leq n^{\ln\left(\frac{1}{(1-\epsilon_0)^2}\right)} < n^{0.003}.
\end{aligned}$$

In the above, the last inequality follows since $\ln(1/(1-\epsilon_0)^2)$ increases when ϵ_0 increases. By assumption, we have $\epsilon_0 < 0.001$, and hence $\ln(1/(1-\epsilon_0)^2) < \ln(1/0.999^2) < 0.003$. $\square$

Let S be a color class of the coloring f. Let $x \in L_i$ and suppose $x \notin S$. We shall use $p(x,S)$ to denote the probability that x is taken care of by some vertex in the color class S. We have,

$$\begin{aligned} p(x,S) = Pr[|N_G(x)\cap S| = 1] &= \sum_{s\in S} Pr[N_G(x)\cap S = \{s\}] \\ &= \sum_{s\in S} w_s w_x \prod_{y\in S\setminus\{s\}} (1 - w_y w_x) \\ &< w_x \sum_{s\in S} w_s \exp\left(- \sum_{y\in S\setminus\{s\}} w_y w_x \right) \\ &= w_x \sum_{s\in S} w_s \exp(-w(S)w_x + w_s w_x) \\ &\le w_x w(S) e^{-w_x w(S) + (1-\epsilon_0)}, \end{aligned}$$

where the third line follows since $1 - t \le e^{-t}$. It can be verified that the function ze^{-z} has a unique maximum at $z = 1$. Thus we get the following:

$$p(x,S) < e^{-\epsilon_0} . \tag{3}$$

We say that a set S is *heavy* if $w(S) > \sqrt{n}$; otherwise we call S a *light* set. Note that any vertex has weight at least $(1-\epsilon_0)^{\ln n} = n^{\ln(1-\epsilon_0)} > n^{-2\epsilon_0}$ (when $0 < \epsilon_0 < 0.5$, we have that $\ln(1-\epsilon_0) \ge \frac{-\epsilon_0}{1-\epsilon_0} > -2\epsilon_0$). For any set S, we have $w(S) > |S| \cdot n^{-2\epsilon_0}$. Thus we have

$$|S| < n^{0.5+2\epsilon_0}, \text{ when } S \text{ is a light set } (w(S) \le \sqrt{n}). \tag{4}$$

The below lemma provides a bound on the number of vertices that are taken care of by a heavy set.

Lemma 26. *For each heavy set $S \subseteq V$ of the graph G, asymptotically almost surely $|N^{(1)}(S)| < n^{0.6}$. That is, a.a.s., at most $n^{0.6}$ vertices that are not in S are taken care of by S.*

Proof. Consider a heavy subset $S \subseteq V$. Since S is a heavy set, $w(S) > n^{0.5}$. Now fix a set $A \subseteq V\setminus S$ with $|A| \ge n^{0.6}$. The probability that all elements $x \in A$ have exactly one neighbor in S is estimated below.

$$\begin{aligned} Pr[N^{(1)}(S) \supseteq A] &= \prod_{x\in A} p(x,S) \\ &\le \prod_{x\in A} w_x w(S) e^{-w_x w(S)+1} \\ &< \left(n^{(0.5-2\epsilon_0)} e^{(-n^{0.5-2\epsilon_0}+1)} \right)^{n^{0.6}} \\ &= \exp\left((0.5-2\epsilon_0)\ln n - n^{0.5-2\epsilon_0} + 1 \right)^{n^{0.6}} \\ &= \exp\left(- n^{(1.1-2\epsilon_0)} \left(- \frac{(0.5-2\epsilon_0)n^{0.6}\ln n}{n^{(1.1-2\epsilon_0)}} + 1 - \frac{n^{0.6}}{n^{(1.1-2\epsilon_0)}} \right) \right) \\ &\le \exp\left(- n^{(1.1-2\epsilon_0)} \left(1 - o(1) \right) \right). \end{aligned}$$

The inequality in the third line follows from the observation that $w_x w(S) > (1-\epsilon_0)^{\ln n} \cdot n^{0.5} \geq n^{0.5-2\epsilon_0}$ and that ze^{-z} is decreasing in the interval $[1,\infty)$. Taking union over the possible $2^n \cdot 2^n$ choices for S and A, we see that the probability of S taking care of A tends to 0. $\square$

The next lemma bounds the number of vertices that can be taken care of by light sets.

Lemma 27. *Let $r = \lfloor \epsilon_0^3 \ln^2 n \rfloor$. For all pairwise disjoint light sets $S_1, \ldots, S_r \subseteq V$, we have asymptotically almost surely $|\bigcup_{i=1}^r N^{(1)}(S_i)| < n - n^{0.7}$.*

Proof. We first fix light subsets $S_1, \ldots, S_r$ of V. Since each S_i is a light set, $w(S_i) \leq \sqrt{n}$. We first need the following claim.

Claim 28. *We have $\sum_{i=1}^r p(x, S_i) > \epsilon_0 \ln n$ for at most half of the vertices $x \in V$.*

Proof. Assume the contrary. We then have

$$\frac{n}{2} \cdot \epsilon_0 \ln n \leq \sum_{x \in V} \sum_{i=1}^{r} p(x, S_i) = \sum_{i=1}^{r} \sum_{x \in V} p(x, S_i) \; . \tag{5}$$

When S_i is fixed, we have seen that $p(x, S_i) \leq w_x w(S_i) e^{-w_x w(S_i) + (1-\epsilon_0)}$. When $x \in L_j$, we get

$$p(x, S_i) \leq (1-\epsilon_0)^j w(S_i) e^{-(1-\epsilon_0)^j w(S_i) + 1} = z_j e^{-z_j + 1},$$

where the first inequality follows by dropping $-\epsilon_0$ from the exponent. We set $z_j = (1-\epsilon_0)^j w(S_i)$ to get the second equality. Observe that

$$\sum_{x \in V} p(x, S_i) = \sum_{j=1}^{\ln n} \sum_{x \in L_j} p(x, S_i) \leq \frac{n}{\ln n} \sum_{j=1}^{\ln n} z_j e^{-z_j+1} \leq \frac{n}{\ln n} \sum_{j=1}^{\infty} z_j e^{-z_j+1}. \tag{6}$$

We will now upper bound $\sum_{j=1}^{\infty} z_j e^{-z_j+1}$ by considering three ranges for z_j. Notice that $z_j > 0$ when S_i is nonempty.

- When $0 < z_j \leq 1$, we have $z_j e^{-z_j+1} \leq e z_j$. Therefore, $\sum_{j: z_j \leq 1} \left(z_j e^{-z_j+1}\right) \leq \sum_{j: z_j \leq 1} \left(e z_j\right) \leq e\big(1 + (1-\epsilon_0) + (1-\epsilon_0)^2 + \cdots\big) = \frac{e}{1-(1-\epsilon_0)} = \frac{e}{\epsilon_0}$.
- When $1 < z_j < 2$, we have the following summation: $\sum_{j: 1 < z_j < 2} \left(z_j e^{-z_j+1}\right)$. It can be verified that $z_j e^{-z_j+1} < 1$ in the range $1 < z_j < 2$. It can also be noted that the number of terms in this summation is at most $\frac{1}{\epsilon_0}$. Thus we get that the terms sum to at most $\frac{1}{\epsilon_0}$.
- When $z_j \geq 2$, we have $\sum_{j: z_j \geq 2} \left(z_j e^{-z_j+1}\right) \leq \frac{1}{\epsilon_0} \int_1^{\infty} z e^{-z+1} \, dz = \frac{2}{\epsilon_0}$. The factor of $\frac{1}{\epsilon_0}$ is due to the fact that $z_j - z_{j+1} = z_j - z_j(1-\epsilon_0) = z_j \epsilon_0 > \epsilon_0$. Thus, the number of $z_j's$ that lie between a and $a+1$, for any integer $a \geq 2$, is at most $\frac{1}{\epsilon_0}$. A straightforward integration by parts gives us that $\int_1^{\infty} z e^{-z+1} \, dz = 2$.

Combining equations (5), (6), and the above bound, gives us the following.

$$\frac{n}{2}\cdot\epsilon_0\ln n \leq \sum_{i=1}^{r}\frac{n}{\ln n}\sum_{j=1}^{\infty} z_j e^{-z_j+1} \leq r\frac{n}{\ln n}\left(\frac{e}{\epsilon_0}+\frac{1}{\epsilon_0}+\frac{2}{\epsilon_0}\right) \leq \frac{6rn}{\epsilon_0\ln n}. \tag{7}$$

Rearranging terms in inequality (7), we get $r \geq \frac{\epsilon_0^2\ln^2 n}{12}$. This contradicts the fact that $r = \epsilon_0^3\ln^2 n \leq 0.001\cdot\epsilon_0^2\ln^2 n = \frac{\epsilon_0^2\ln^2 n}{1000}$.

This completes the proof of the claim that $\sum_{i=1}^{r} p(x,S_i) > \epsilon_0\ln n$ for at most half the vertices $x\in V$. $\square$

Let $V'\subseteq V$ be the set of those vertices $x\in V$ for which $\sum_{i=1}^{r} p(x,S_i)\leq\epsilon_0\ln n$. Then, $|V'|\geq n/2$. Now fix a set $B\subseteq V$ with $|B| = n^{0.7}$. In the below calculation, we bound the probability that all the vertices $x\in V\setminus B$ have exactly one neighbor in at least one of the S_i's. This probability is given by

$$\begin{aligned}\prod_{x\in V\setminus B}\left(1-\prod_{i=1}^{r}\Big(1-p(x,S_i)\Big)\right) &\leq \prod_{x\in V'\setminus B}\left(1-\prod_{i=1}^{r}\Big(1-p(x,S_i)\Big)\right)\\ &\leq \exp\left(-\sum_{x\in V'\setminus B}\prod_{i=1}^{r}\Big(1-p(x,S_i)\Big)\right)\\ &\leq \exp\left(-\sum_{x\in V'\setminus B} e^{-f(\epsilon_0)\sum_{1=1}^{r}p(x,S_i)}\right),\ f(\epsilon_0)\text{ is defined below.}\end{aligned}$$

The first inequality follows by restricting the scope of vertices, and the second inequality follows by using the fact that $1-x\leq e^{-x}$. For the third inequality, let us set $f(\epsilon_0) := e^{\epsilon_0}\ln\big(\frac{1}{1-e^{-\epsilon_0}}\big)$. From equation (3), we get that $p(x,S_i)\in[0,e^{-\epsilon_0}]$. When a number z is chosen from the range $[0,1)$, we can verify that $\big(\ln\big(\frac{1}{1-z}\big)\big)/z$ is an increasing function on z. So $\big(\ln\big(\frac{1}{1-e^{-\epsilon_0}}\big)\big)/e^{-\epsilon_0}\geq\big(\ln\big(\frac{1}{1-p(x,S_i)}\big)\big)/p(x,S_i)$. Rearranging this, it follows that $e^{-f(\epsilon_0)p(x,S_i)}\leq 1-p(x,S_i)$. We continue our computation below.

$$\begin{aligned}\exp\left(-\sum_{x\in V'\setminus B} e^{-f(\epsilon_0)\sum_{1=1}^{r}p(x,S_i)}\right) &\leq \exp\left(-\Big(\frac{n}{2}-n^{0.7}\Big)e^{-f(\epsilon_0)\epsilon_0\ln n}\right)\\ &\leq \exp\left(-\Big(\frac{n}{2}-n^{0.7}\Big)e^{-0.01\ln n}\right)\\ &= \exp\Big(-0.5n^{0.99}+n^{0.69}\Big)\\ &\leq \exp\left(-n^{0.99}\Big(\frac{1}{2}-o(1)\Big)\right).\end{aligned} \tag{8}$$

The first inequality follows since for vertices x of V', we have $\sum_{i=1}^{r} p(x,S_i)\leq\epsilon_0\ln n$. Below we explain the second inequality. We have $f(\epsilon_0)\cdot\epsilon_0 = \epsilon_0\cdot e^{\epsilon_0}\ln\big(\frac{1}{1-e^{-\epsilon_0}}\big)$, where $0<\epsilon_0<0.001$. Using $e^{-x}\leq 1-\frac{x}{2}$, for $x\in[0,1]$, we get $f(\epsilon_0)\cdot\epsilon_0\leq\epsilon_0\cdot e^{\epsilon_0}\ln\big(\frac{2}{\epsilon_0}\big)$. Since

$x \cdot e^x \ln\left(\frac{2}{x}\right)$ is an increasing function when $x \in (0,1]$ and considering that $\epsilon_0 \in (0, 0.001)$, we have, $f(\epsilon_0) \cdot \epsilon_0 \leq 0.001 \cdot e^{0.001} \ln\left(\frac{2}{0.001}\right) < 0.01$.

Finally, since $|B| = n^{0.7}$ and using the size bound on light sets given in equation (4), we note that the number of choices for sets $S_1, \ldots, S_r$, and B is at most

$$\begin{aligned}\binom{n}{n^{0.7}}\binom{n}{n^{0.5+2\epsilon_0}}^r &< n^{n^{0.7}}\left(n^{n^{0.5+2\epsilon_0}}\right)^r \\ &= e^{n^{0.7}\ln n}\left(e^{(n^{0.5+2\epsilon_0})r\ln n}\right) \\ &= e^{n^{0.7}\ln n + n^{(0.5+2\epsilon_0)}r\ln n} \\ &= e^{O(n^{0.7}\ln n)}.\end{aligned}$$

From equation (8) and the above calculations, we can see that any r light sets $S_1, S_2, \ldots, S_r$ can take care of at most $n - n^{0.7}$ vertices a.a.s. $\square$

The below lemma shows that the graph G constructed has the desired minimum degree with high probability.

Lemma 29. *For the graph G constructed, the minimum degree of G is $\Omega(\Delta^{1-\epsilon})$ asymptotically almost surely.*

Proof. We first calculate the expected degree of a vertex $x \in L_j$. Let $d_G(x)$ denote the degree of x.

$$\begin{aligned}\mu(x) := \mathbb{E}[d_G(x)] &= \frac{n}{\ln n}\left[(1-\epsilon_0)^{j+1} + (1-\epsilon_0)^{j+2} + \cdots + (1-\epsilon_0)^{j+\ln n}\right] \\ &\geq \frac{n}{\ln n}(1-\epsilon_0)^{j+1} \qquad (9) \\ &= (1-\epsilon_0)\cdot\frac{n}{\ln n}\cdot(1-\epsilon_0)^{\ln(e^j)} \\ &= (1-\epsilon_0)\cdot\frac{n}{\ln n}\cdot(e^j)^{\ln(1-\epsilon_0)} \\ &\geq (1-\epsilon_0)\cdot\frac{n}{\ln n}\cdot(e^j)^{-2\epsilon_0}, \qquad (10)\end{aligned}$$

where we use the fact that $\ln(1-\epsilon_0) \geq -2\epsilon_0$, as noted in the discussion preceding the statement of Lemma 26. Using the Chernoff bound given in Theorem 11 for any $0 < \alpha < 1$, we get

$$\begin{aligned}Pr[|d_G(x) - \mu(x)| \geq \alpha\mu(x)] &\leq 2e^{\frac{-\alpha^2\mu(x)}{3}} \\ &\leq 2e^{\frac{-\alpha^2 n}{\ln n}\cdot\frac{(1-\epsilon_0)^{j+1}}{3}},\end{aligned}$$

where the last inequality follows by equation (9). For an $x \in L_j$, let A_x^j denote the event that $|d_G(x) - \mu(x)| \geq \alpha\mu(x)$. We have shown that $Pr[A_x^j] \leq 2e^{\frac{-\alpha^2 n}{\ln n}\cdot\frac{(1-\epsilon_0)^{j+1}}{3}}$. Let A^j denote the event $\bigcup_{x\in L_j} A_x^j$. Below we calculate the probability of A^1, using the union bound.

$$\begin{aligned}
Pr[A^1] &\le \sum_{x\in L_1} Pr[A^1_x] \\
&\le \frac{2n}{\ln n}\cdot e^{\frac{-\alpha^2 n}{\ln n}\cdot\frac{(1-\epsilon_0)^2}{3}} \\
&\le 2n\cdot\exp\left(-\frac{\alpha^2 n}{\ln n}\frac{0.999^2}{3}\right) && \text{(since } \epsilon_0 < 0.001\text{)} \\
&\le \exp\left(\ln(2n) - \frac{\alpha^2 n}{4\ln n}\right). && \text{(since } 0.999^2/3 > 1/4\text{)}
\end{aligned}$$

Observe that, $Pr[A^{j+1}] \le e^{1-\epsilon_0} Pr[A^j]$, for $\forall 1 \le j < \ln n$. So we have the following for $1 \le j \le \ln n$.

$$Pr[A^j] \le e^{j-1}\exp\left(\ln(2n) - \frac{\alpha^2 n}{4\ln n}\right).$$

Let $\overline{A^j}$ denote the complement of the event A^j. We have

$$\begin{aligned}
Pr[\overline{A^1}\cap\cdots\cap\overline{A^{\ln n}}] &= 1 - Pr[A^1\cup\cdots\cup A^{\ln n}] \\
&\ge 1 - \left[\frac{e^{\ln(2n)}}{e^{\frac{\alpha^2 n}{4\ln n}}} + e\cdot\frac{e^{\ln(2n)}}{e^{\frac{\alpha^2 n}{4\ln n}}} + \cdots + e^{\ln n - 1}\cdot\frac{e^{\ln(2n)}}{e^{\frac{\alpha^2 n}{4\ln n}}}\right] \\
&\ge 1 - n\cdot\left[\frac{e^{\ln(2n)}}{e^{\frac{\alpha^2 n}{4\ln n}}}\right] \\
&= 1 - o(1).
\end{aligned}$$

We have thus shown that, for every vertex x in G, $|d_G(x) - \mu(x)| < \alpha\mu(x)$ a.a.s. The vertex with the maximum degree is from L_1 a.a.s. and similarly, the vertex with the minimum degree is from $L_{\ln n}$ a.a.s. Now we calculate an upper bound for $\mu(x)$, for $x \in L_1$.

$$\begin{aligned}
\mu(x) &= \frac{n}{\ln n}\left[(1-\epsilon_0)^{1+1} + (1-\epsilon_0)^{1+2} + \cdots + (1-\epsilon_0)^{1+\ln n}\right] \\
&\le \frac{n}{\ln n}\frac{(1-\epsilon_0)^2}{\epsilon_0}.
\end{aligned}$$

From the above upper bound and the lower bound (computed in equation (9)) on $\mu(x)$ when $x \in L_1$, we have $\Delta = \Theta\left(\frac{n}{\ln n}\right)$ a.a.s. Now, consider an $x \in L_{\ln n}$. From the lower bound for $\mu(x)$ that we computed at the beginning of this proof in equation (10), we have, $\mu(x) \ge (1-\epsilon_0)\cdot\frac{n}{\ln n}\cdot n^{-2\epsilon_0}$. Thus, the minimum degree of G is $\Omega\left(\frac{n^{1-2\epsilon_0}}{\ln n}\right)$ a.a.s. That is, the minimum degree of G is $\Omega(\Delta^{1-\epsilon})$ a.a.s. (as $\epsilon = 3\epsilon_0$). □

To summarize the proof of Theorem 24, we note that any coloring $f : V(G) \longrightarrow [r]$ that uses $r = \epsilon_0^3 \ln^2\Delta$ many colors cannot take care of all the vertices of G. For a fixed color class, Lemma 25 bounds the number of vertices of that color class that can take care of themselves. Lemma 25 provides a bound of $n^{0.003}$. Thus across the r color classes, the number of vertices that take care of themselves is at most $rn^{0.003}$. The number of vertices that are taken care of by a fixed heavy set is bounded by Lemma 26 to $n^{0.6}$. Thus the total number of vertices that are taken care of by heavy color classes is at most $rn^{0.6}$. Lemma 27 bounds the number of vertices taken care of by all the light color classes to $n - n^{0.7}$. Summing up, we note that all the n vertices cannot be taken care of.

Lemma 29 shows that the minimum degree of G is $\Omega(\Delta^{1-\epsilon})$, completing the proof of Theorem 24.

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