

A Tight Bound for Conflict-free Coloring in terms of Distance to Cluster

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Abstract

Given an undirected graph $G = (V, E)$, a conflict-free coloring with respect to open neighborhoods (CFON coloring) is a vertex coloring such that every vertex has a uniquely colored vertex in its open neighborhood. The minimum number of colors required for such a coloring is the CFON chromatic number of G , denoted by $\chi_{ON}(G)$.

In previous work [WG 2020], we showed the upper bound $\chi_{ON}(G) \leq \text{dc}(G) + 3$, where $\text{dc}(G)$ denotes the distance to cluster parameter of G . In this paper, we obtain the improved upper bound of $\chi_{ON}(G) \leq \text{dc}(G) + 1$. We also exhibit a family of graphs for which $\chi_{ON}(G) > \text{dc}(G)$, thereby demonstrating that our upper bound is tight.

1 Introduction

Given a graph $G = (V, E)$, a *conflict-free coloring* is an assignment of colors to every vertex of G such that there exists a uniquely colored vertex in the open neighborhood of each vertex. This problem was motivated by the frequency assignment problem in cellular networks [6], where base stations and clients communicate with each other. It has also found applications in coding theory [9] and sensor networks [7, 10]. It is required that there exists a base station with a unique frequency in the neighborhood of each client. We formally define the problem as follows.

Definition 1 (Conflict-Free Coloring). A *CFON coloring* of a graph $G = (V, E)$ using k colors is an assignment $C : V(G) \rightarrow \{1, 2, \dots, k\}$ such that for every $v \in V(G)$, there exists an $i \in \{1, 2, \dots, k\}$ such that $|N(v) \cap C^{-1}(i)| = 1$. The smallest number of colors required for a CFON coloring of G is called the *CFON chromatic number of G* , denoted by $\chi_{ON}(G)$.

This problem has been studied from both algorithmic and structural perspectives [1, 2, 3, 4, 8, 11, 13]. For a detailed account, see the survey by Smorodinsky [13]. Combinatorial bounds on this problem have been studied with respect to vertex cover, treewidth, path-width, feedback vertex set and neighborhood diversity [3, 4, 8]. In this paper, we study the relation between CFON chromatic number and the distance to cluster parameter, which is formally defined as follows.

Definition 2 (Distance to Cluster). Let $G = (V, E)$ be a graph. The distance to cluster of G , denoted $\text{dc}(G)$, is the size of a smallest set $X \subseteq V$ such that $G[V \setminus X]$ is a disjoint union of cliques.

Reddy [12] showed that $2\text{dc}(G) + 1$ colors are sufficient to CFON color a graph G . This bound has been improved to $\text{dc}(G) + 3$ in [3]. In this paper, we further improve the bound to $\text{dc}(G) + 1$.

Theorem 3. For any graph G , we have $\chi_{ON}(G) \leq \max\{3, \text{dc}(G) + 1\}$.

Further, we show graphs for which $\text{dc}(G)$ colors are not sufficient, thereby demonstrating that the above bound is tight.

Theorem 4. For each value $d \geq 1$, there exist graphs G such that $\text{dc}(G) = d$ and $\chi_{ON}(G) > d$.

Theorem 4 is first proved in Section 2. The rest of the paper is devoted to the proof of Theorem 3.

1.1 Preliminaries

In this paper, we consider only simple, finite, undirected and connected graphs that have at least two vertices. If the graph has more than one connected component, Theorem 3 follows by its application to each component independently. Moreover, we assume that G does not have any isolated vertices as there is no CFON coloring for such graphs. We denote the set $\{1, 2, \dots, d\}$ by $[d]$. We use the function $C : V \rightarrow [d + 1]$ to denote the color assigned to a vertex. The open neighborhood of a vertex v , denoted by $N(v)$, is the set of vertices adjacent to v . The degree of a vertex v , denoted $\text{deg}(v)$ is defined as $|N(v)|$. Sometimes, we use the notation $\text{deg}_A(v) = |N(v) \cap A|$, where $A \subseteq V$. We use the notation $G[A]$ to denote the induced graph on the vertex set A . We use standard graph theoretic terminology from the textbook by Diestel [5].

During the coloring process, for each vertex v , we will designate a vertex as the *uniquely colored neighbor* of v , denoted by $U(v)$. The vertex $U(v)$ is a vertex $w \in N(v)$ such that $C(w) \neq C(x), \forall x \in N(v) \setminus \{w\}$. Sometimes, we will use the phrase “vertex v sees the vertex w as its uniquely colored neighbor” to refer to the same. We will also use the phrase “vertex v sees the color i ” instead of saying there exists a neighbor of v that is assigned the color i .

We will also frequently refer to a set X such that $G[V \setminus X]$ is a disjoint union of cliques. For the sake of brevity, instead of referring to a component or maximal clique K of $G[V \setminus X]$, we will say “ K is a clique in $G[V \setminus X]$ ”.

2 Lower Bound

In this section, we prove Theorem 4. We will see the existence of graphs G for which $\chi_{ON}(G) > \text{dc}(G)$.

Construction of graph G : Given a positive integer d , we construct the graph G such that $\text{dc}(G) = d$. It consists of three parts as described below.

- The set X is an independent set of d vertices v_1, v_2, \dots, v_d . Note that $|X| = d$, and $G[V \setminus X]$ will be a disjoint union of cliques.
- Singleton cliques $K_{(i,j)}, \forall 1 \leq i < j \leq d$. For each $K_{(i,j)}$, we have $N(K_{(i,j)}) = \{v_i, v_j\}$.

- A clique \widehat{K} that has $(d+1)2^d$ vertices. The vertices of the clique \widehat{K} consist of 2^d disjoint sets T^W , one corresponding to each subset $W \subseteq X$. For each $v \in T^W$, we have $N(v) \cap X = W$. Moreover, we have $|T^W| = d+1$, for each $W \subseteq X$.

Proof of Theorem 4. It can easily be noted that $\text{dc}(G) = d$. We will now see that $\chi_{\text{ON}}(G) > d$.

The singleton cliques $K_{(i,j)}$ force each vertex in X to be assigned a distinct color. WLOG, let $C(v_i) = i$ for each $v_i \in X$. The colors $1, 2, \dots, d$ are used exactly once in X . Now, we prove that d colors are not sufficient to color the clique \widehat{K} .

We first consider a vertex $u_1 \in T^\emptyset$. The vertex u_1 does not have any neighbors in X and hence has its uniquely colored neighbor from \widehat{K} . Let the vertex $w_1 \in \widehat{K}$ be the uniquely colored neighbor of u_1 . WLOG, let $C(w_1) = 1$. Now, consider the vertices in $T^{\{v_1\}}$. At least d vertices in $T^{\{v_1\}}$ see both v_1 and w_1 as their neighbors, and hence these vertices cannot have 1 as the unique color in their neighborhood. Let u_2 be such a vertex. WLOG, let $w_2 \in \widehat{K}$ be the vertex such that $C(w_2) = 2$ and w_2 acts as the uniquely colored neighbor for u_2 . Of the vertices in $T^{\{v_1, v_2\}}$, at least $d-1$ of them see w_1, w_2 in addition to v_1, v_2 as neighbors. Hence these vertices cannot have the colors 1 or 2 as the unique color in their neighborhood.

We continue this reasoning and show that there exists at least one vertex, say $u_{d+1} \in T^X$, that sees all the colors $\{1, 2, \dots, d\}$ at least twice in its neighborhood. Hence u_{d+1} cannot have any of the colors $1, 2, \dots, d$ as the unique color in its neighborhood. Hence we require a new color to CFON color G . \square

3 Upper Bound

In this section, we prove Theorem 3. Since it involves several cases and detailed analyses, we first present an overview of the proof, before getting into the details.

3.1 Overview of the Proof

Given a graph $G = (V, E)$, and a set of vertices $X \subseteq V$ such that $|X| = d$, we have that $G[V \setminus X]$ is a disjoint union of cliques. We require d colors, one for each vertex in X . This is because $G[V \setminus X]$ may contain $\binom{d}{2}$ singleton cliques, such that each of these cliques has degree 2, and adjacent to a pair of vertices in X . Since a clique can be CFON colored using at most 3 colors, it is easy to see that $d+3$ colors are sufficient to CFON color G when $G[X]$ is connected. Though it is less straightforward, the bound of $d+3$ can be extended to the case when $G[X]$ is not connected as well [3]. It is a challenge to further improve the bound to $d+1$. Our proof requires several cases and subcases since there does not seem to be a universal approach that leads to a desired coloring. The detailed case analysis is necessary because of the different forms the induced graph $G[X]$ can take.

Except for some special cases, we will color each vertex in X with a distinct color from $[d]$. Our coloring algorithm consists of two phases, an initial phase and a completion phase. In the initial phase, we color all the vertices of X , and identify uniquely colored neighbors for some vertices in X . The key requirement of this phase is to identify a *free color* f , which is a color in $[d]$ that will not serve as a unique color in the neighborhood of any vertex in X . This is straightforward in some cases, like when $G[X]$ has a component of size at least 3. The cases where all the vertices of $G[X]$ have degree 1 (Lemma 12), or all the vertices of $G[X]$ have degree 0 or 1 (Lemma 13) prove to be particularly challenging. In some of the

cases, this is accomplished by coloring one or two of the cliques in $G[V \setminus X]$. The full set of conditions that are to be satisfied by the initial phase is listed as the Rules of Lemma 9.

After the initial phase, we are ready to run the completion phase, which is executed in Lemma 9. The goal of the completion phase is to color the rest of the graph while retaining the uniquely colored neighbors of those vertices that had been identified in the initial phase. In the completion phase, we first identify uniquely colored neighbors for those vertices in X , for which it has not been identified. This involves coloring some of the vertices in $V \setminus X$, and hence may partially color some of the cliques in $G[V \setminus X]$. The cliques in $G[V \setminus X]$ are colored one by one. We have to use different approaches to color them, based on the number of vertices that are already colored in the clique. The general results are presented for the case when $d = |X| \geq 3$, and the cases $d = 1$ and $d = 2$ need to be treated differently.

The case $d = 1$ is straightforward, but the case $d = 2$ is somewhat involved in itself. These cases are presented first in Section 3.2 as they serve as “warm-ups” for the pattern and the type of arguments that will be used in the general $d \geq 3$ case. However, we reiterate that the general case involves a lot of cases and sub-cases that need to be treated separately and carefully.

3.2 The case when $|X| < 3$

We handle the cases $|X| = 1$ and $|X| = 2$ separately.

Lemma 5. Let $G = (V, E)$ be a graph and $X \subseteq V$ be a set of vertices such that $|X| = 1$ and $G[V \setminus X]$ is a disjoint union of cliques. Then $\chi_{ON}(G) \leq 3$.

Proof. We explain how to assign $C : V \rightarrow \{1, 2, 3\}$ such that C is a CFON coloring of G . Let $X = \{v_1\}$. We assign $C(v_1) = 1$.

Initial phase: We first choose a clique K in $G[V \setminus X]$. Since G is connected, there exists $v \in K$ such that $v_1 \in N(v)$. There are two cases.

- $|K| = 1$.
That is, $K = \{v\}$. We assign $C(v) = 2$. We have $U(v_1) = v$ and $U(v) = v_1$.
- $|K| \geq 2$.
We choose a vertex $v' \in K \setminus \{v\}$. We assign $C(v) = 2$, $C(v') = 3$, and the vertices (if any) in $K \setminus \{v, v'\}$ are assigned 1. We have $U(v_1) = U(v') = v$, and for all $y \in K \setminus \{v'\}$, we have $U(y) = v'$.

Completion phase: The lone vertex v_1 of X is already colored 1 and sees 2 as the unique color in its neighborhood. We have also colored one clique in $G[V \setminus X]$. For all the uncolored cliques K in $G[V \setminus X]$, we color K as per the applicable case.

- All the vertices in K see v_1 as their neighbor.
For all $y \in K$, assign $C(y) = 3$. The vertex v_1 acts as the uniquely colored neighbor for all $y \in K$.
- There exists $v \in K$ such that $v_1 \notin N(v)$.
Notice that if $|K| = 1$, then the lone vertex in K has to necessarily see v_1 as a neighbor. Hence in this case, we have $|K| \geq 2$. Choose a vertex $v' \in K \setminus \{v\}$. Assign $C(v) = 2$, $C(v') = 3$, and the vertices (if any) in $K \setminus \{v, v'\}$ the color 1.
We have $U(v') = v$ and for all $y \in K \setminus \{v'\}$, we have $U(y) = v'$.

□

Lemma 6. Let $G = (V, E)$ be a graph and $X \subseteq V$ be a set of vertices such that $|X| = 2$ and $G[V \setminus X]$ is a disjoint union of cliques. Then $\chi_{ON}(G) \leq 3$.

Proof. Let $X = \{v_1, v_2\}$. We explain how to assign $C : V \rightarrow \{1, 2, 3\}$ to get a CFON coloring of G . There are two cases depending on whether X is an independent set or not.

Case 1: X is an independent set. That is, $\{v_1, v_2\} \notin E(G)$. We have two subcases.

- Every vertex in $V \setminus X$ has at most one neighbor in X .

Initial phase: Since G is connected, there exists a clique K in $G[V \setminus X]$ such that $N(v_1) \cap K \neq \emptyset$ and $N(v_2) \cap K \neq \emptyset$. Let $v \in N(v_1) \cap K$ and $v' \in N(v_2) \cap K$. Notice that $v \neq v'$, otherwise we would be violating the subcase we are in.

We assign $C(v_1) = C(v_2) = 3$, $C(v) = 1$, $C(v') = 2$ and the remaining vertices (if any) in $K \setminus \{v, v'\}$ the color 3.

We get that $U(v_1) = v$ and $U(v_2) = v'$. Also $U(v) = v'$ and for each $y \in K \setminus \{v\}$, $U(y) = v$.

Completion phase: Now we color the remaining cliques in $G[V \setminus X]$. Let K be an uncolored clique. There are two possibilities.

- Each vertex $y \in K$ has $\deg_X(y) = 1$.

For each vertex $y \in K$, if $v_1 \in N(y)$, assign $C(y) = 2$. Else, assign $C(y) = 1$. The uniquely colored neighbor of each vertex in K is its lone neighbor in X .

- There exists $v \in K$ such that $\deg_X(v) = 0$.

Since G is connected, there exists $v' \in K$ such that $\deg_X(v') \neq 0$. WLOG let $v_1 \in N(v')$. We assign $C(v) = 1$, $C(v') = 2$ and the vertices (if any) in $K \setminus \{v, v'\}$ the color 3.

We get that $U(v) = v'$ and for each $y \in K \setminus \{v\}$, $U(y) = v$.

- There exists a vertex $v \in V \setminus X$ such that $\deg_X(v) = 2$.

Initial phase: Let $v \in K$, where K is a clique in $G[V \setminus X]$. We first assign $C(v_1) = 1$ and $C(v_2) = 2$.

If $|K| = 1$, we assign $C(v) = 1$ and we get that $U(v_1) = U(v_2) = v$ and $U(v) = v_1$.

Else $|K| \geq 2$ and we have the following cases.

- There exists a vertex $v' \in K \setminus \{v\}$ such that $v_2 \notin N(v')$.

We assign $C(v) = 2$, $C(v') = 3$ and the vertices (if any) in $K \setminus \{v, v'\}$ the color 1. We get that $U(v_1) = U(v_2) = v$, $U(v') = v$ and $U(y) = v'$ for all $y \in K \setminus \{v\}$.

- Else, for each $y \in K$, we have $v_2 \in N(y)$.

We assign $C(v) = 1$ and the remaining vertices in $K \setminus \{v\}$ with the color 3. We get that $U(v_1) = U(v_2) = v$ and $U(y) = v_2$ for all $y \in K$.

In each of the above case, both v_1 and v_2 have the same unique colors from $\{1, 2\}$. The other color in $\{1, 2\}$ does not serve as the unique color of v_1 and v_2 and we refer to it as the *free color*¹.

¹The notion of free color is used crucially in the proof of the general $|X| \geq 3$ case.

Completion phase: WLOG let v_1 and v_2 have the color 1 as the unique color, and 2 is the free color.

Now, we extend this coloring to the cliques $K \subseteq G[V \setminus X]$.

- There exists a vertex $v \in K$ such that $v_1 \in N(v)$.
Assign $C(v) = 3$ and the vertices (if any) in $K \setminus \{v\}$ the color 2. We have $U(v) = v_1$ and for all $y \in K \setminus \{v\}$, $U(y) = v$.
- None of the vertices in K see v_1 as a neighbor.
 - * All the vertices in K see v_2 as a neighbor.
Assign the color 3 to all the vertices in K . For each vertex $y \in K$, we have $U(y) = v_2$.
 - * There exists $v, v' \in K$ such that $v_2 \in N(v)$ and $v_2 \notin N(v')$.
Assign $C(v) = 3, C(v') = 1$ and the vertices (if any) in $K \setminus \{v, v'\}$ the color 2. We get that $U(v) = v'$ and $y \in K \setminus \{v\}$, $U(y) = v$.

Case 2: X is not an independent set. That is, $\{v_1, v_2\} \in E(G)$.

Before we explain how we CFON color the graph, we need to set up notation. A clique $K \subseteq G[V \setminus X]$ is called v_1 -seeing (v_2 -seeing) if for all vertices $y \in K$, we have $v_1 \in N(y)$ ($v_2 \in N(y)$).

- Each vertex $v \in V \setminus X$ that has $\deg_X(v) = 1$ appears in a clique K that is v_1 -seeing or v_2 -seeing.

In other words, each clique K in $G[V \setminus X]$ satisfies one of the two conditions: (i) K is v_1 -seeing or v_2 -seeing, or (ii) each vertex $v \in K$ has either $\deg_X(v) = 0$ or $\deg_X(v) = 2$.

Initial phase: There are two cases.

- There is a vertex $v \in V \setminus X$ such that $\deg_X(v) = 2$.
Let $v \in K$, where K is a clique in $G[V \setminus X]$. We assign $C(v_1) = 1, C(v_2) = 2, C(v) = 3$ and the vertices (if any) in $K \setminus \{v\}$ the color 2. We have $U(v_1) = U(v_2) = v, U(v) = v_1$ and for the vertices (if any) $y \in K \setminus \{v\}$, $U(y) = v$.
- For all $y \in V \setminus X$, we have $\deg_X(y) < 2$.

Since G is connected, there is a vertex v in each of the cliques in $G[V \setminus X]$ such that $\deg_X(v) = 1$. By the case definition, we have that each of the cliques must be v_1 -seeing or v_2 -seeing. If all the cliques were v_1 -seeing, it follows that $N(v_2) = \{v_1\}$. This means that $\text{dc}(G) = 1$ and this case has been addressed in Lemma 5. By an analogous argument, all the cliques cannot be v_2 -seeing as well. Hence there are cliques $K_1, K_2 \subseteq G[V \setminus X]$ such that K_1 is v_1 -seeing and K_2 is v_2 -seeing.

By the case definition, we have that for all vertices $y \in K_1$, $\deg_X(y) < 2$. Since K_1 is v_1 -seeing, it follows that for all $y \in K_1$, we have $N(y) \cap X = \{v_1\}$. We choose a vertex $v \in K_1$. We assign $C(v_1) = 1, C(v) = 3$, and the vertices (if any) in $K_1 \setminus \{v\}$, the color 2. We have $U(v_1) = v, U(v) = v_1$, and for the vertices (if any) $y \in K_1 \setminus \{v\}$, $U(y) = v$.

Similarly, for all $y \in K_2$, we have $N(y) \cap X = \{v_2\}$. We choose a vertex $v' \in K_2$ and assign $C(v_2) = 2, C(v') = 3$, and the vertices (if any) in $K_2 \setminus \{v'\}$, the color 1. We have $U(v_2) = v', U(v') = v_2$, and for the vertices (if any) $y \in K_2 \setminus \{v'\}$, $U(y) = v'$.

In each of the above cases, we have $C(v_1) = 1$, $C(v_2) = 2$, and the unique color seen by v_1 and v_2 is 3.

Completion phase: Now we color the remaining cliques in $G[V \setminus X]$. Let K be an uncolored clique. There are two possibilities.

- For each $y \in K$, we have $\deg_X(y) \geq 1$.
If K is v_1 -seeing, then for each vertex $y \in K$, assign $C(y) = 2$ and we get $U(y) = v_1$. If K is v_2 -seeing, then for each vertex $y \in K$, assign $C(y) = 1$ and we get $U(y) = v_2$. (If $\deg_X(y) = 2$ for each $y \in K$, then either of the above assignments work.)
- There exists a vertex $v \in K$ such that $\deg_X(v) = 0$.
Since G is connected, K has a vertex v' such that $\deg_X(v') > 0$. Since K is not v_1 -seeing or v_2 -seeing, it must be the case that $\deg_X(v') = 2$. We assign $C(v) = 3$, $C(v') = 1$ and the vertices (if any) in $K \setminus \{v, v'\}$ the color 2. We get that $U(v) = v'$ and for each $y \in K \setminus \{v\}$, $U(y) = v$.

- There exists a vertex $v \in V \setminus X$ that has $\deg_X(v) = 1$ that appears in a clique K that is neither v_1 -seeing nor v_2 -seeing.

Initial phase: WLOG, let $N(v) \cap X = \{v_1\}$. Since K is not v_1 -seeing, there exists $v' \in K$ such that $v_1 \notin N(v')$.

We assign $C(v_1) = 1$, $C(v_2) = 2$, $C(v) = 1$, $C(v') = 3$, and the vertices (if any) in $K \setminus \{v, v'\}$ the color 2. We have $U(v_1) = v$, $U(v_2) = v_1$, $U(v') = v$, and for each $y \in K \setminus \{v'\}$, $U(y) = v'$. Note that the unique color seen by both v_1 and v_2 is 1.

Completion phase: Now we color the remaining cliques in $G[V \setminus X]$. Let K be an uncolored clique. We have the following cases.

- There exists a vertex $v \in K$ such that $v_1 \in N(v)$.
Assign $C(v) = 3$, and the vertices (if any) in $K \setminus \{v\}$ the color 2. We have $U(v) = v_1$, and for any $y \in K \setminus \{v\}$, $U(y) = v$.
- None of the vertices in K see v_1 as a neighbor.
 - * For all $y \in K$, we have $N(y) \cap X = \{v_2\}$.
Assign the color 3 to all the vertices in K . All the vertices in K see v_2 as their uniquely colored neighbor.
 - * There is a vertex $v \in K$ such that $\deg_X(v) = 0$.
Since G is connected, $|K| \geq 2$. Choose a vertex $v' \in K \setminus \{v\}$. Assign $C(v) = 1$, $C(v') = 3$, and the vertices (if any) in $K \setminus \{v, v'\}$ the color 2. We have $U(v') = v$, and for $y \in K \setminus \{v'\}$, $U(y) = v'$.

□

3.3 The case when $|X| \geq 3$ and X is an independent set

We start handling the general case of $|X| \geq 3$. In this section, we prove the upper bound for the case when X is an independent set.

Theorem 7. Let $G(V, E)$ be a graph and $X \subseteq V$ be a set of vertices such that $|X| = d \geq 3$ and $G[V \setminus X]$ is a disjoint union of cliques. If X is an independent set, then $\chi_{ON}(G) \leq d + 1$.

In order to show the above theorem, we first prove Lemma 8, where we handle the case when every vertex in $V \setminus X$ has at most one neighbor in X . After this, we prove Lemma 10, where there is a vertex $v \in V \setminus X$ that has at least two neighbors in X . The proof of Lemma 10 uses Lemma 9, which also serves as the completion phase for all the remaining cases (including those where X is not an independent set).

Lemma 8. Let $G = (V, E)$ be a graph and $X \subseteq V$ be a set of vertices such that $|X| = d \geq 3$ and $G[V \setminus X]$ is a disjoint union of cliques. If X is an independent set and every vertex in $V \setminus X$ has at most one neighbor in X , then $\chi_{ON}(G) \leq d + 1$.

Proof. We explain how to assign $C : V \rightarrow [d + 1]$ such that C is a CFON coloring of G . Let $X = \{v_1, v_2, \dots, v_d\}$.

Initial phase: For each $v_i \in X$, assign $C(v_i) = d + 1$. For each $v_i \in X$, choose an arbitrary neighbor $w_i \in V \setminus X$ and assign $C(w_i) = i$. We get that $U(v_i) = w_i$. Now, each vertex in X is colored and has a uniquely colored neighbor.

Completion phase: Each uncolored singleton clique in $G[V \setminus X]$ is assigned the color $d + 1$. Note that all the singleton cliques have exactly one neighbor in X , and this neighbor is the uniquely colored neighbor. What remains to be addressed are cliques of size at least 2.

- **Clique $K \subseteq G[V \setminus X]$ with at least two colored vertices.** Color the uncolored vertices with the color $d + 1$. Let $v, v' \in K$ be two of the vertices that were colored prior to this step. Hence it follows that $C(v), C(v') \in [d]$ and $C(v) \neq C(v')$.

Since $\deg_X(y) \leq 1$ for all $y \in G[V \setminus X]$, one of v and v' will be the uniquely colored neighbor of all vertices in K .

- **Clique $K \subseteq G[V \setminus X]$ with exactly 1 colored vertex v .** Let $C(v) = j$ and hence $v_j \in N(v)$.

- If $|K| = 2$. Let $K = \{v, v'\}$.

- * If $v_j \notin N(v')$, we assign $C(v') = j$. We get that $U(v) = v'$ and $U(v') = v$.

- * Else, we have $N(v) \cap X = N(v') \cap X = \{v_j\}$. We assign $C(v')$ arbitrarily from $[d] \setminus \{j\}$. We get that $U(v) = v'$ and $U(v') = v$.

- Else if $|K| \geq 3$.

Let v' be arbitrarily chosen from $K \setminus \{v\}$. Since $|X| \geq 3$ and $\deg_X(v') \leq 1$, there exists a vertex $v_\ell \in X$, $v_\ell \neq v_j$ such that $v_\ell \notin N(v') \cap X$. Assign $C(v') = \ell$ and color the rest of the vertices in $K \setminus \{v, v'\}$ with the color $d + 1$. We get that for all vertices $w \in K$, either v or v' is a uniquely colored neighbor.

- **Clique $K \subseteq G[V \setminus X]$ with no colored vertices.** We first select two vertices $v, v' \in K$.

Since $\deg_X(v) \leq 1$, we can choose $j \in [d]$ such that $v_j \notin N(v) \cap X$. Since $\deg_X(v') \leq 1$ and $|X| \geq 3$, we can choose $\ell \in [d]$ such that $\ell \neq j$ and $v_\ell \notin N(v') \cap X$.

Assign $C(v) = j$, $C(v') = \ell$ and the rest of the vertices in $K \setminus \{v, v'\}$ the color $d + 1$. For all vertices $w \in K$, either v or v' is a uniquely colored neighbor.

□

We first state Lemma 9, which will serve as the completion phase for almost all the remaining cases (even for those where X is not an independent set). This lemma states that the graph can be CFON colored provided it has been partially colored satisfying certain rules.

We say that C is a *partial coloring* of V if C is an assignment of colors to a subset of vertices in V .

Lemma 9. Let $G = (V, E)$ be a graph with $X = \{v_1, v_2, \dots, v_d\} \subseteq V$ such that $d \geq 3$ and $G[V \setminus X]$ is a disjoint union of cliques. Further, $Y = \{v_i \in X : \deg_X(v_i) \geq 1\}$ and $C : V \rightarrow [d + 1]$ be a partial coloring that satisfies the below rules.

Then, C can be extended to a CFON coloring $\widehat{C} : V \rightarrow [d + 1]$ of all the vertices in V .

Rules

- (i) For all $v_i \in X$, $C(v_i) = i$.
- (ii) For some number of cliques K in $G[V \setminus X]$, all the vertices in K are colored using colors from $[d + 1]$. For the remaining cliques, none of the vertices are colored.
- (iii) All the vertices in Y and all the colored vertices in $V \setminus X$ have a uniquely colored neighbor. Moreover, some vertices in $X \setminus Y$ have a uniquely colored neighbor. Further, none of the vertices in X see $d + 1$ as the unique color in their neighborhood.
- (iv) The uniquely colored neighbor is identified for all the vertices in X whose entire neighborhood is colored.
- (v) There exists $1 \leq f \leq d$, such that (a) $v_f \in X$ has a uniquely colored neighbor and (b) for each vertex in X , the color f is not the unique color in its neighborhood. We refer to f as the *free color*.
- (vi) If a vertex $v_i \in X \setminus Y$ does not have a uniquely colored neighbor, then the color i is not assigned to any vertex in $V \setminus X$ yet.
- (vii) The colors in $[d] \setminus \{f\}$ are used at most at once in $V \setminus X$.

Before proving Lemma 9, we prove the upper bound when X is an independent set and there is a vertex $v \in V \setminus X$ that has at least two neighbors in X .

Lemma 10. Let $G(V, E)$ be a graph and $X \subseteq V$ be a set of vertices such that $|X| = d \geq 3$ and $G[V \setminus X]$ is a disjoint union of cliques. If X is an independent set and there exists a vertex $v \in V \setminus X$, such that $\deg_X(v) \geq 2$, then $\chi_{ON}(G) \leq d + 1$.

Proof. The goal here is to partially color some vertices of G so that the rules of Lemma 9 are satisfied. We then use Lemma 9 to extend the partial coloring and obtain a CFON coloring of G .

Let $X = \{v_1, v_2, \dots, v_d\}$. We explain how to assign $C : V \rightarrow [d + 1]$ such that C is a partial coloring that satisfies the rules of Lemma 9. For each vertex $v_i \in X$, we assign a distinct color $C(v_i) = i$. There are two cases depending on the neighborhood of vertices in $V \setminus X$.

- **There exists a singleton clique $K = \{v\}$ such that $\deg_X(v) \geq 2$.**

Let $N(v) \cap X = \{v_{i_1}, v_{i_2}, \dots, v_{i_m}\}$, with $m \geq 2$. We assign $C(v) = i_1$.

We get that $U(v) = v_{i_1}$ and for all $1 \leq j \leq m$, $U(v_{i_j}) = v$. The color i_2 will not be the unique color of any vertex in X and will be the free color.

- **All singleton cliques have degree equal to 1.** By assumption, there is a vertex $v \in V \setminus X$, such that $\deg_X(v) \geq 2$. It follows that $v \in K$ where K is a clique in $G[V \setminus X]$ and $|K| \geq 2$. Let $N(v) \cap X = \{v_{i_1}, v_{i_2}, \dots, v_{i_m}\}$, with $m \geq 2$.

Now we have two cases depending on whether there exist vertices in X whose neighbors belong to only K . We refer to these vertices as S_K . Formally, $S_K = \{v_i \in X : N(v_i) \subseteq K\} \setminus N(v)$. The vertices in $N(v) \cap X$ rely on v for their uniquely colored neighbor and hence does not require special attention.

- $S_K \neq \emptyset$.

First, we assign $C(v) = i_1$ and choose i_2 as the free color. For each vertex $v_j \in S_K$, we choose a vertex $w_j \in N(v_j)$. Note that by the definition of S_K , it follows that $w_j \in K$. We assign $C(w_j) = j$ if it is not already assigned (w_j could have been the chosen neighbor for some other vertex $v_{j'}$ as well). Now all the vertices in S_K have a uniquely colored neighbor.

- * If all vertices in K are colored because of the above coloring, every vertex in K has a uniquely colored neighbor. We get that $U(v) = v_{i_1}$. Each vertex $w \in K \setminus \{v\}$ is assigned a distinct color, say j , because it is adjacent to $v_j \in S_K$, which serves as its uniquely colored neighbor.

- * There exists at least one uncolored vertex in K .

- If there exists a uniquely colored neighbor for each uncolored vertex in K , assign the color $d + 1$ to all the uncolored vertices. We get that $U(v) = v_{i_1}$. The vertices $w_j \in K \setminus \{v\}$ rely on the corresponding v_j 's as mentioned above.
- Else, let $v' \in K$ be an uncolored vertex that does not see a uniquely colored neighbor. This implies that $N(v') \cap \{v_{i_1}, v_{i_2}, \dots, v_{i_m}\} = \{v_{i_1}\}$. We reassign $C(v) = i_2$, assign $C(v') = d + 1$, and designate i_1 as the free color instead of i_2 . We assign the color i_1 to the remaining uncolored vertices in K . We have $U(v') = v$ and $U(w) = v'$, for all $w \in K \setminus \{v'\}$.

- $S_K = \emptyset$.

In this case, there is no vertex in $X \setminus \{v_{i_1}, v_{i_2}, \dots, v_{i_m}\}$ that was relying on K for its uniquely colored neighbor. We first assign $C(v) = i_1$ and choose i_2 as the free color. We have the following cases.

- * There exists a vertex $v' \in K \setminus \{v\}$ such that $v_{i_1} \notin N(v')$. Assign $C(v') = d + 1$ and assign the remaining vertices of $K \setminus \{v, v'\}$ the color i_2 . For every vertex $w \in K \setminus \{v'\}$, $U(w) = v'$. Finally, we have $U(v') = v$.
- * Else, for every vertex $w \in K$, we have $v_{i_1} \in N(w) \cap X$. Reassign $C(v) = i_2$ and assign the color $d + 1$ to all the vertices in $K \setminus \{v\}$. The color i_1 is the redesignated free color. And for each $w \in K$, $U(w) = v_{i_1}$.

Now C is a partial color assignment satisfying all the conditions in the rules of Lemma 9. Hence by Lemma 9, we can extend C to a full CFON coloring of G that uses at most $d + 1$ colors. \square

We conclude this section with the proof of Lemma 9.

Proof of Lemma 9. For each colored vertex $w \in G$, $\widehat{C}(w) = C(w)$. We explain how to extend $\widehat{C} : V \rightarrow [d+1]$ to all vertices such that \widehat{C} is a CFON coloring of G . Let $X = \{v_1, v_2, \dots, v_d\}$ and $Y = \{v_i \in X : \deg_X(v_i) \geq 1\}$.

Process to identify uniquely colored neighbors for $X \setminus Y$: For every $v_j \in X \setminus Y$, that does not have a uniquely colored neighbor, choose an uncolored neighbor of v_j , say $w_j \in V \setminus X$ and assign $\widehat{C}(w_j) = j$. Rules (iv) and (vi) of Lemma 9 allow us to do this. Since v_j does not have a uniquely colored neighbor, rule (iv) implies that v_j has an uncolored neighbor, and as per rule (vi), no vertex in $V \setminus X$ is assigned the color j .

Observation: It is possible that all the neighbors of $v_j \in X \setminus Y$ may be colored by the above coloring process on other vertices $v_i \in X \setminus Y$ even before applying the process on v_j . In such a case, we choose an arbitrary neighbor of v_j that was already colored by this process and assign it as the uniquely colored neighbor for v_j . This neighbor acts as the uniquely colored neighbor for at least 2 vertices in $X \setminus Y$. This fact will be useful later.

Now, every vertex in $X \setminus Y$ has a uniquely colored neighbor. We now look at the previously uncolored cliques K in $G[V \setminus X]$. For each such clique K , we color K as per the applicable case below.

Case 1: K has no colored vertices

- $|K| = 1$. Let $K = \{w\}$.

We assign $\widehat{C}(w) = d + 1$. As all the neighbors of w are distinctly colored, we assign one of the neighbors as $U(w)$.

- $|K| \geq 2$. We have two subcases here.

- There exists a vertex $w \in K$, such that $N(w) \cap X = \emptyset$.

Choose another vertex $w' \in K \setminus \{w\}$ such that $N(w') \cap X \neq \emptyset$. We have two subcases.

- * $N(w') \cap X = \{v_f\}$, where $v_f \in X$ is the vertex that corresponds to the free color f .

Assign $\widehat{C}(w') = d + 1$ and $\widehat{C}(w) = c$, where $c \in [d] \setminus \{f\}$, chosen arbitrarily.

For all the vertices (if any) $x \in K \setminus \{w, w'\}$, assign $\widehat{C}(x) = f$.

We have $U(w') = w$ and for all vertices $x \in K \setminus \{w'\}$, we have $U(x) = w'$.

- * There exists a vertex $v_i \in N(w') \cap X$, where $v_i \neq v_f$.

Assign $\widehat{C}(w') = d + 1$. For all the vertices $x \in K \setminus \{w'\}$, assign $\widehat{C}(x) = f$.

We have $U(w') = v_i$ and for all vertices $x \in K \setminus \{w'\}$, we have $U(x) = w'$.

- For all $w \in K$, $N(w) \cap X \neq \emptyset$.

Assign all the vertices in K the color $d + 1$. For all the vertices in K , we assign one of the neighbors in X as the respective uniquely colored neighbor.

Case 2: K has exactly one colored vertex

WLOG, let $v \in K$ be such that $\widehat{C}(v) = j$. This implies that $v_j \in N(v) \cap X$.

- $|K| = 1$.

We have $U(v) = v_j$ and $U(v_j) = v$ (as was already assigned).

- $|K| = 2$. Let $K = \{v, v'\}$.

- $N(v') \cap X = \emptyset$.

We assign $\widehat{C}(v') = d + 1$. We get that $U(v) = v'$, $U(v') = v$ and $U(v_j) = v$.

- v' has a neighbor other than v_j in X . That is, $\exists v_k \in N(v') \cap X$, with $v_k \neq v_j$.
We assign $\widehat{C}(v') = d + 1$. We get that $U(v) = v'$, $U(v') = v_k$ and $U(v_j) = v$.
- $N(v') \cap X = \{v_j\}$.
 - * There exists a vertex $v_\ell \in X \setminus Y$, $v_\ell \neq v_j$ such that $U(v_\ell) = v$.
Note that in this case, no vertex in $V \setminus X$ is assigned the color ℓ . This is as noted in the observation at the beginning of the proof of this lemma. We reassign $\widehat{C}(v) = \ell$ and $\widehat{C}(v') = d + 1$. We have that $U(v) = v'$ and $U(v') = v$. Note that $U(v_j) = U(v_\ell) = v$, as before.
 - * No vertex in X other than v_j sees v as its uniquely colored neighbor and v has another neighbor in X besides v_j , say v_k .
We reassign $\widehat{C}(v) = d + 1$ and assign $\widehat{C}(v') = j$. We get that $U(v') = v$, $U(v) = v_k$ and we reassign $U(v_j) = v'$.
 - * $N(v) \cap X = N(v') \cap X = \{v_j\}$.
We argue this case based on whether v_j sees another uniquely colored vertex $w \in V \setminus X$, distinct from v , such that $\widehat{C}(w) \in [d] \setminus \{j, f\}$. Rule (vii) of Lemma 9 and the coloring processes followed in the rest of Lemma 9 ensure the following: If v_j sees $w \in V \setminus X$, such that $\widehat{C}(w) \in [d] \setminus \{j, f\}$, then it does not see any other $w' \in V \setminus X$ such that $\widehat{C}(w) = \widehat{C}(w')$.
If there exists such a vertex w , then reassign $\widehat{C}(v) = d + 1$ and assign $\widehat{C}(v') = d + 1$. We have $U(v_j) = w$, $U(v') = v$ and $U(v) = v'$.
Else, reassign $\widehat{C}(v)$ to an arbitrarily chosen value from $[d] \setminus \{j, f\}$. Note that such a value exists since $d \geq 3$. We assign $\widehat{C}(v') = d + 1$. We get that $U(v) = v'$, $U(v') = v$ and $U(v_j) = v$, as before.

- $|K| \geq 3$.

- There exists a vertex $v' \in K \setminus \{v\}$ such that $v_j \notin N(v')$.
We assign $\widehat{C}(v') = d + 1$ and the vertices in $K \setminus \{v, v'\}$ are colored with the free color f .
We get that $U(v') = v$ and for all $w \in K \setminus \{v'\}$, $U(w) = v'$. As before, $U(v_j) = v$.
- Every vertex in K is adjacent to v_j .
 - * There exists a vertex $v' \in K \setminus \{v\}$ such that $(N(v') \cap X) \setminus \{v_j, v_f\} \neq \emptyset$. Let $v_k \in N(v') \cap X$, where $v_k \neq v_j$ and $v_k \neq v_f$.
Assign $\widehat{C}(v') = d + 1$ and rest of the vertices in $K \setminus \{v, v'\}$ are colored with the free color f . We get that $U(v') = v_k$ and for all $w \in K \setminus \{v'\}$, $U(w) = v'$. As before, $U(v_j) = v$.
 - * For all $w \in K \setminus \{v\}$, $N(w) \cap X \subseteq \{v_j, v_f\}$.
We have the following cases.
 - There exists a vertex $v' \in K$ such that $v_f \notin N(v')$.
Choose $k \in [d] \setminus \{j, f\}$ arbitrarily. Note that such a value exists since $d \geq 3$. Choose two vertices v', v'' arbitrarily from $K \setminus \{v\}$ and assign $\widehat{C}(v') = k$, $\widehat{C}(v'') = d + 1$ and the remaining vertices in $K \setminus \{v, v', v''\}$ are colored with f .
We get that $U(v'') = v'$, and for all $w \in K \setminus \{v''\}$, $U(w) = v''$. As before $U(v_j) = v$.

- For all $w \in K \setminus \{v\}$, we have $N(w) \cap X = \{v_j, v_f\}$.
For all $w \in K \setminus \{v\}$, assign $\widehat{C}(w) = d + 1$. For all $w \in K \setminus \{v\}$, we have that $U(w) = v_f$. We have $U(v) = v_j$ and as before, $U(v_j) = v$.

Case 3: K has at least two colored vertices and there exists a vertex in K that is a uniquely colored neighbor for at least two vertices in $X \setminus Y$

Let $v' \in K$ be such that $U(v_j) = U(v_k) = v'$, where $v_j, v_k \in X \setminus Y$. The vertex v' was colored in the process to identify uniquely colored neighbors for vertices in $X \setminus Y$. WLOG, we may assume that v' was colored when assigning a unique colored neighbor for v_j . That is, $\widehat{C}(v') = j$. This also implies that no vertices in K are colored k . There are two cases here.

- All vertices in K are colored.
In this case, every vertex in K will have a uniquely colored neighbor. This is because every vertex in K would have been assigned a distinct color. If $w \in K$ is such that $\widehat{C}(w) = \ell$, then $U(w) = v_\ell \in X \setminus Y$.
- There exists an uncolored vertex $v \in K$. There are two subcases.
 - v is adjacent to v_k .
We assign $\widehat{C}(v) = d + 1$ and the remaining uncolored vertices in K (if any) are assigned the free color f .
We get that $U(v) = v_k$, and for all $w \in K \setminus \{v\}$, $U(w) = v$. Note that $U(v_j) = U(v_k) = v'$, as before.
 - v is not adjacent to v_k .
Reassign $\widehat{C}(v') = k$, assign $\widehat{C}(v) = d + 1$ and the remaining uncolored vertices in K (if any) are assigned the free color f .
We get that $U(v) = v'$, and for all $w \in K \setminus \{v\}$, $U(w) = v$. Note that $U(v_j) = U(v_k) = v'$, as before.

Case 4: K has at least two colored vertices and every colored vertex in K is the uniquely colored neighbor for exactly one vertex in $X \setminus Y$

Let $v, v' \in K$ be two colored vertices such that $\widehat{C}(v) = j$ and $\widehat{C}(v') = k$. The colors j and k are assigned because they are adjacent to v_j and v_k respectively, where $v_j, v_k \in X \setminus Y$. We have cases depending on the neighborhood of K .

- There exists a colored vertex in K that is adjacent to both v_j and v_k .
 - At least one of v or v' is adjacent to both v_j and v_k .
WLOG let that vertex be v . Reassign $\widehat{C}(v') = d + 1$ and assign the color f to the remaining uncolored vertices (if any).
We have that $U(v') = v_k$, and for all vertices $w \in K \setminus \{v'\}$, $U(w) = v'$. We reassign $U(v_k) = v$, while $U(v_j) = v$ as before.
 - There exists a colored vertex $v'' \in K \setminus \{v, v'\}$ such that $\{v_j, v_k\} \subseteq N(v'')$.
Let $\widehat{C}(v'') = \ell$ because it was the chosen neighbor for $v_\ell \in X \setminus Y$ in the coloring process stated in the beginning of this proof. We reassign $\widehat{C}(v) = d + 1$ and $\widehat{C}(v') = f$. It is important to note that because of the Case 4 definition, this reassignment does not affect the uniquely colored neighbors of vertices in $X \setminus \{v_j, v_k\}$.

The remaining uncolored vertices in K (if any) are assigned the color f .

We have $U(v) = v_j$, and for every vertex $w \in K \setminus \{v\}$, $U(w) = v$. We reassign $U(v_j) = U(v_k) = v''$, while $U(v_\ell) = v''$ as before.

- There exists an uncolored vertex $v'' \in K$ such that $\{v_j, v_k\} \subseteq N(v'')$.

We reassign $\widehat{C}(v) = d + 1$, $\widehat{C}(v') = f$ and assign $\widehat{C}(v'') = k$. The remaining uncolored vertices in K (if any) are assigned the color f .

We have $U(v) = v_j$, and for every vertex $w \in K \setminus \{v\}$, $U(w) = v$. We reassign $U(v_j) = U(v_k) = v''$.

- No vertex in K is adjacent to both v_j and v_k .

Assign the color f to the remaining uncolored vertices (if any). Every vertex in $K \setminus \{v, v'\}$ will see either v or v' as its uniquely colored neighbor. Also $U(v) = v_j$ and $U(v') = v_k$, while $U(v_j) = v$ and $U(v_k) = v'$, as before.

□

3.4 The case when $|X| \geq 3$ and X is not an independent set

In this section, we prove the upper bound when $|X| \geq 3$ and X is not an independent set.

Theorem 11. Let $G = (V, E)$ be a graph and $X \subseteq V$ be a set of vertices such that $|X| = d \geq 3$ and $G[V \setminus X]$ is a disjoint union of cliques. If X is not an independent set, then $\chi_{ON}(G) \leq d + 1$.

The proof of Theorem 11 involves a lot of cases. The cases when $G[X]$ is 1-regular and all the vertices in $G[X]$ have degree 0 or 1 needs particular care. We state these two cases below. The proofs of Lemmas 12 and 13, are proved in Sections 3.5 and 3.6 respectively. All the proofs in this section and subsequent sections will only deal with the initial phase, i.e., to achieve a partial coloring of the graph that satisfies the Rules of Lemma 9. The completion phase follows by Lemma 9.

Lemma 12. Let $G = (V, E)$ be a graph and $X \subseteq V$ be a set of vertices such that $|X| = d \geq 3$ and $G[V \setminus X]$ is a disjoint union of cliques. If $G[X]$ is 1-regular (perfect matching), then $\chi_{ON}(G) \leq d + 1$.

Lemma 13. Let $G = (V, E)$ be a graph and $X \subseteq V$ be a set of vertices, such that $|X| = d \geq 3$ and $G[V \setminus X]$ is a disjoint union of cliques. Moreover all the vertices in $G[X]$ have degree at most 1 and at least one vertex has degree 0. Then $\chi_{ON}(G) \leq d + 1$.

We first prove Theorem 11 assuming the above lemmas.

Proof of Theorem 11. Let $X = \{v_1, v_2, \dots, v_d\}$ and $Y = \{v_i \in X : \deg_X(v_i) \geq 1\}$. Also, let \mathcal{A} be the set of connected components of $G[X]$.

For each vertex $v_i \in X$, we assign a distinct color $C(v_i) = i$. We have the following cases depending on the components in $G[X]$.

- There exists a component $A \in \mathcal{A}$ such that $|A| \geq 3$, and $\exists v_j \in A$ with $\deg_X(v_j) = 1$.

Let $N(v_j) \cap X = \{v_k\}$. Since $|A| \geq 3$, there exists $v_\ell \in N(v_k) \cap X$ such that $v_\ell \neq v_j$. Every vertex in $A \setminus \{v_k\}$ chooses an arbitrary neighbor in A as its uniquely colored neighbor while we assign $U(v_k) = v_\ell$. The color j is not the unique color for any vertex in X . Hence we use j as the free color for the rest of the coloring.

For all $A' \in \mathcal{A} \setminus A$, where $|A'| \geq 2$, a vertex chooses one of its neighbors in A' as its uniquely colored neighbor. Thus all the vertices in Y have a uniquely colored neighbor.

At this point, we have partially colored G satisfying the rules of Lemma 9. Using Lemma 9, we can extend this to a full CFON coloring of G that uses $d + 1$ colors.

- There exists a $A \in \mathcal{A}$ such that $|A| \geq 3$, and all the vertices in A have degree at least 2 in $G[X]$.

Choose a vertex $v_j \in A$. Since every vertex in A has degree at least 2, it can be ensured that every vertex in A is assigned a uniquely colored neighbor other than v_j . We use the color j as the free color.

For all $A' \in \mathcal{A} \setminus A$, where $|A'| \geq 2$, a vertex chooses one of its neighbors in A' as its uniquely colored neighbor. Thus all the vertices in Y have a uniquely colored neighbor.

Since the rules of Lemma 9 are satisfied, we can extend this coloring to a CFON coloring of G that uses $d + 1$ colors.

- For all the components $A \in \mathcal{A}$, we have $|A| = 2$.

In this case, we have $X \setminus Y = \emptyset$. All the vertices $v_i \in X$ have $\deg_X(v_i) = 1$. We apply Lemma 12 to CFON color G with $d + 1$ colors.

- For all the components $A \in \mathcal{A}$, we have $|A| \leq 2$. Moreover there exists $A' \in \mathcal{A}$ such that $|A'| = 1$.

That is $X \setminus Y \neq \emptyset$. By assumption, X is not an independent set. Hence $Y \neq \emptyset$ as well. We apply Lemma 13 to CFON color G with $d + 1$ colors.

□

3.5 Proof of Lemma 12

Let $X = \{v_1, v_2, \dots, v_d\}$. Since each vertex $v_i \in X$ has $\deg_X(v_i) = 1$, we have that $d = |X|$ is an even number. This implies that $d \geq 4$. WLOG, we may assume that the edges in $G[X]$ are $\{v_1, v_2\}, \{v_3, v_4\}, \dots, \{v_{d-1}, v_d\}$. We explain how to assign $C : V \rightarrow [d + 1]$ such that C is a partial coloring that satisfies the rules of Lemma 9.

For each vertex $v_i \in X$, we assign the color $C(v_i) = i$. We have the following cases.

Case 1: There exists a vertex $v \in V \setminus X$ such that $\deg_X(v) = |X|$.

Let $v \in K_1$, where K_1 is a clique in $G[V \setminus X]$.

Subcase 1.1: K_1 is the only clique in $G[V \setminus X]$.

For all $v_i \in X \setminus \{v_1\}$, we reassign $C(v_i) = d + 1$. Assign $C(v) = 2$ and the remaining vertices (if any) in $K_1 \setminus \{v\}$ are assigned the color $d + 1$.

We get that $U(v) = v_1$ and for all $x \in V \setminus \{v\}$, $U(x) = v$. Thus the entire graph is CFON colored.

Subcase 1.2: There exists a clique $K_2 \neq K_1$, such that $K_2 = \{w\}$ and $\deg_X(w) \geq 2$.

- $N(w) \cap X$ contains a pair of adjacent vertices.

WLOG, let $v_1, v_2 \in N(w)$. We have cases based on the size of the clique K_1 .

– $|K_1| = 1$.

We assign $C(w) = 3$ and $C(v) = 1$. We have that $U(v) = U(w) = v_1$. Also, $U(v_2) = w$ and for each $v_i \in X \setminus \{v_2\}$, $U(v_i) = v$. We have color 4 as the free color.

– $|K_1| \geq 2$.

We consider an arbitrary vertex $v' \in K_1 \setminus \{v\}$. We have three cases based on the neighborhood of v' .

* $v_1 \notin N(v')$.

We assign $C(w) = 3$, $C(v) = 1$, $C(v') = d + 1$ and the vertices (if any) in $K_1 \setminus \{v, v'\}$ the color 4.

We have that $U(w) = v_2$, $U(v_2) = w$, and for each $v_i \in X \setminus \{v_2\}$, $U(v_i) = v$. Also $U(v') = v$ and for each $y \in K_1 \setminus \{v'\}$, $U(y) = v'$. We have color 4 as the free color.

* $v_1 \in N(v')$.

We assign $C(w) = 3$, $C(v) = 2$, $C(v') = d + 1$ and the vertices (if any) in $K_1 \setminus \{v, v'\}$ the color 4.

We have that $U(w) = v_2$, $U(v_1) = w$ and for each $v_i \in X \setminus \{v_1\}$, $U(v_i) = v$. Also $U(v') = v_1$ and for each $y \in K_1 \setminus \{v'\}$, $U(y) = v'$. We have color 4 as the free color.

• **None of the vertices in $N(w) \cap X$ are adjacent to each other.**

WLOG, let $v_1, v_3 \in N(w)$. We have cases based on the size of the clique K_1 .

– $|K_1| = 1$.

We assign $C(v) = 4$ and $C(w) = 1$. We have $U(w) = U(v) = v_1$. Also, $U(v_3) = w$, and for each $v_i \in X \setminus \{v_3\}$, $U(v_i) = v$. We have color 3 as the free color.

– $|K_1| \geq 2$.

We consider an arbitrary vertex $v' \in K_1 \setminus \{v\}$. We have two cases based on the neighborhood of v' .

* $v_2 \notin N(v')$.

We assign $C(w) = 3$, $C(v) = 2$, $C(v') = d + 1$ and the vertices (if any) in $K_1 \setminus \{v, v'\}$ the color 4.

We have that $U(w) = v_1$, $U(v_1) = w$, and for each $v_i \in X \setminus \{v_1\}$, $U(v_i) = v$. Also $U(v') = v$ and for each $y \in K_1 \setminus \{v'\}$, $U(y) = v'$. Color 4 is the free color.

* $v_2 \in N(v')$.

We assign $C(w) = 1$, $C(v) = 4$, $C(v') = d + 1$ and the vertices (if any) in $K_1 \setminus \{v, v'\}$ the color 3.

We have that $U(w) = v_1$, $U(v_3) = w$, and for each $v_i \in X \setminus \{v_3\}$, $U(v_i) = v$. Also $U(v') = v_2$ and for each $y \in K_1 \setminus \{v'\}$, $U(y) = v'$. Color 3 is the free color.

Subcase 1.3: There exists a clique $K_2 \neq K_1$, such that $K_2 = \{w\}$ and $\deg_X(w) = 1$.

WLOG let $N(w) = \{v_1\}$. We have cases based on the size of the clique K_1 .

• $|K_1| = 1$.

We assign $C(v) = 2$ and $C(w) = 3$. We have $U(v) = U(w) = v_1$. Also, $U(v_1) = w$, and for each $v_i \in X \setminus \{v_1\}$, $U(v_i) = v$. Color 4 is the free color.

• $|K_1| \geq 2$.

- There exists a vertex $v' \in K_1 \setminus \{v\}$, such that $v_2 \notin N(v')$.
We assign $C(v) = 2$, $C(w) = 3$, $C(v') = d + 1$ and the vertices (if any) in $K_1 \setminus \{v, v'\}$ the color 4.
We have that $U(w) = v_1$, $U(v_1) = w$, and for each $v_i \in X \setminus \{v_1\}$, $U(v_i) = v$. Also $U(v') = v$ and for each $y \in K_1 \setminus \{v'\}$, $U(y) = v'$. Color 4 is the free color.
- All the vertices in K_1 are adjacent to v_2 .
Choose a vertex $v' \in K_1 \setminus \{v\}$. Assign $C(w) = 4$, $C(v) = 1$, $C(v') = 3$ and the vertices (if any) in $K_1 \setminus \{v, v'\}$ the color $d + 1$.
We have that $U(w) = v_1$, $U(v_2) = v'$, and for each $v_i \in X \setminus \{v_2\}$, $U(v_i) = v$. Also for each $y \in K_1$, $U(y) = v_2$. Color 4 is the free color.

Subcase 1.4: There exists a clique $K_2 \neq K_1$, such that $|K_2| \geq 2$.

Since G is connected, there is an edge between X and K_2 . WLOG, we assume that v_1 is adjacent to $w \in K_2$. We now divide the cases based on the size of the clique K_1 .

- $|K_1| = 1$.
 - There exists a vertex $w' \in K_2 \setminus \{w\}$ such that $v_3 \notin N(w')$.
We assign $C(v) = 2$, $C(w) = 3$, $C(w') = d + 1$, and the vertices (if any) in $K_2 \setminus \{w, w'\}$ the color 4.
We have that $U(w') = w$, and for each $y \in K_2 \setminus \{w'\}$, $U(y) = w'$. Also $U(v) = v_1$, $U(v_1) = w$, and for each $v_i \in X \setminus \{v_1\}$, $U(v_i) = v$. Color 4 is the free color.
 - All the vertices in $K_2 \setminus \{w\}$ are adjacent to v_3 .
We assign $C(v) = 2$, $C(w) = 4$, and the vertices in $K_2 \setminus \{w\}$ the color $d + 1$.
We have that $U(v) = U(w) = v_1$ and for each $y \in K_2 \setminus \{w\}$, $U(y) = v_3$. Also $U(v_1) = w$, and for each $v_i \in X \setminus \{v_1\}$, $U(v_i) = v$. Color 1 is the free color.
- $|K_1| \geq 2$.
 - There exists a vertex $v' \in K_1 \setminus \{v\}$ such that $v_2 \notin N(v')$.
There are two subcases based on the neighborhood of the vertices in K_2 .
 - * There exists a vertex $w' \in K_2 \setminus \{w\}$ such that $v_3 \notin N(w')$.
We assign $C(v) = 2$, $C(v') = d + 1$, $C(w) = 3$, $C(w') = d + 1$ and the vertices (if any) in $K_1 \setminus \{v, v'\}$ and $K_2 \setminus \{w, w'\}$ the color 4.
We have that $U(v') = v$, and for each $y \in K_1 \setminus \{v'\}$, $U(y) = v'$. Also $U(w') = w$, and for each $y \in K_2 \setminus \{w'\}$, $U(y) = w'$. Moreover, $U(v_1) = w$, and for each $v_i \in X \setminus \{v_1\}$, $U(v_i) = v$. Color 4 is the free color.
 - * All the vertices in $K_2 \setminus \{w\}$ are adjacent to v_3 .
We assign $C(v) = 2$, $C(v') = d + 1$, $C(w) = 4$, the vertices (if any) in $K_1 \setminus \{v, v'\}$ the color 1 and the vertices in $K_2 \setminus \{w\}$ the color $d + 1$.
We have that $U(v') = v$, and for each $y \in K_1 \setminus \{v'\}$, $U(y) = v'$. Also $U(w) = v_1$, and for each $y \in K_2 \setminus \{w\}$, $U(y) = v_3$. Moreover, $U(v_1) = w$, and for each $v_i \in X \setminus \{v_1\}$, $U(v_i) = v$. Color 1 is the free color.
 - All the vertices in K_1 are adjacent to v_2 .
We choose $v' \in K_1 \setminus \{v\}$ arbitrarily. We assign $C(v) = 1$, $C(v') = 3$ and the vertices (if any) in $K_1 \setminus \{v, v'\}$ the color 4. We leave the clique K_2 uncolored for now.

We have that for each $y \in K_1$, $U(y) = v_2$. Moreover, $U(v_2) = v'$, and for each $v_i \in X \setminus \{v_2\}$, $U(v_i) = v$. Color 4 is the free color.

Case 2: For all $y \in V \setminus X$, we have $\deg_X(y) < |X|$. And there is a vertex $v \in V \setminus X$ such that $\deg_X(v) \geq 2$.

Let $v \in K$ where K is a clique in $G[V \setminus X]$. Recall that since $G[X]$ is 1-regular, it follows that $|X| \geq 4$. We have the following cases depending on the neighborhood of v .

- **$N(v) \cap X$ contains a pair of adjacent vertices.**

WLOG, let $v_3, v_4 \in N(v)$. Since $\deg_X(v) < |X|$, WLOG we assume $v_1 \notin N(v)$.

- $|K| = 1$.

We assign $C(v) = 2$. We get that $U(v) = v_3$ and $U(v_3) = U(v_4) = v$. For each $v_i \in X \setminus \{v_3, v_4\}$, the uniquely colored neighbor is the lone neighbor of v_i in X . The color 3 is the free color.

- $|K| \geq 2$.

- * There exists a vertex $v' \in K \setminus \{v\}$ such that $v_2 \notin N(v')$.

We assign $C(v) = 2$, $C(v') = d + 1$ and the vertices (if any) in $K \setminus \{v, v'\}$ the color 4.

We have that $U(v') = v$, and for all $y \in K \setminus \{v'\}$, $U(y) = v'$. Also $U(v_3) = U(v_4) = v$, and for each $v_i \in X \setminus \{v_3, v_4\}$, the uniquely colored neighbor is the lone neighbor of v_i in X . We have color 4 as the free color.

- * All vertices in $K \setminus \{v\}$ are adjacent to v_2 .

Let v' be arbitrarily chosen from $K \setminus \{v\}$. We assign $C(v) = 1$, $C(v') = 3$ and the vertices (if any) in $K \setminus \{v, v'\}$ the color $d + 1$.

We have that $U(v) = v_4$, and for all $y \in K \setminus \{v\}$, $U(y) = v_2$. Also $U(v_2) = v'$, $U(v_3) = U(v_4) = v$, and for all $v_i \in X \setminus \{v_2, v_3, v_4\}$, the uniquely colored neighbor is the lone neighbor of v_i in X . We have color 4 as the free color.

- **None of the vertices in $N(v) \cap X$ are adjacent to each other.**

WLOG $v_2, v_4 \in N(v)$ which implies $v_1, v_3 \notin N(v)$.

- $|K| = 1$.

We assign $C(v) = 2$. We get that $U(v) = v_2$ and $U(v_2) = U(v_4) = v$. For each $v_i \in X \setminus \{v_2, v_4\}$, the uniquely colored neighbor is the lone neighbor of v_i in X . The color 3 is the free color.

- $|K| \geq 2$.

- * There exists a $v' \in K \setminus \{v\}$ such that $v_2 \notin N(v')$.

We assign $C(v) = 2$, $C(v') = d + 1$ and the vertices (if any) in $K \setminus \{v, v'\}$ the color 3.

We get that $U(v') = v$ and for all $y \in K \setminus \{v'\}$, $U(y) = v'$. Moreover, $U(v_2) = U(v_4) = v$, and for each $v_i \in X \setminus \{v_2, v_4\}$, the uniquely colored neighbor is the lone neighbor of v_i in X . The color 3 is the free color.

- * Every vertex in K is adjacent to v_2 .

We assign $C(v) = 4$ and the vertices in $K \setminus \{v\}$ the color $d + 1$.

We get that for all $y \in K$, $U(y) = v_2$. Moreover, $U(v_2) = U(v_4) = v$, and for each $v_i \in X \setminus \{v_2, v_4\}$, the uniquely colored neighbor is the lone neighbor of v_i in X . The color 3 is the free color.

Case 3: Every vertex $v \in V \setminus X$ has $\deg_X(v) \leq 1$.

Since G is connected, there is a $v \in V \setminus X$ such that $\deg_X(v) = 1$. Let $v \in K$ where K is a clique in $G[V \setminus X]$. WLOG let $N(v) \cap X = \{v_1\}$.

- $|K| = 1$.

We assign $C(v) = 1$. We get that $U(v) = v_1$. Also, $U(v_1) = v$ and for each $v_i \in X \setminus \{v_1\}$, the uniquely colored neighbor is the lone neighbor of v_i in X . The color 2 is the free color.

- $|K| \geq 2$.

- There exists a vertex $v' \in K \setminus \{v\}$ such that $v_1 \notin N(v')$.

We assign $C(v) = 1$, $C(v') = d + 1$ and the vertices (if any) in $K \setminus \{v, v'\}$ the color 2.

We get that $U(v') = v$ and for all $y \in K \setminus \{v'\}$, $U(y) = v'$. Moreover, $U(v_1) = v$, and for each $v_i \in X \setminus \{v_1\}$, the uniquely colored neighbor is the lone neighbor of v_i in X . The color 2 acts as the free color.

- All vertices in K are adjacent to v_1 .

We assign $C(v) = 3$ and the vertices in $K \setminus \{v\}$ the color $d + 1$.

We get that for all $y \in K$, $U(y) = v_1$. Moreover, $U(v_1) = v$, and for each $v_i \in X \setminus \{v_1\}$, the uniquely colored neighbor is the lone neighbor of v_i in X . The color 2 acts as the free color.

In Subcase 1.1, G has been assigned a full CFON coloring using $d + 1$ colors. In all the other cases (and subcases therein), G has been partially colored satisfying the rules of Lemma 9. The uncolored cliques $K \in G[V \setminus X]$ can be colored with the application of Lemma 9, yielding a full CFON coloring of G that uses $d + 1$ colors.

3.6 Proof of Lemma 13

Let $X = \{v_1, v_2, \dots, v_d\}$ and $Y = \{v_i \in X : \deg_X(v_i) \geq 1\}$. By the conditions in the statement of the lemma, we have $X \setminus Y \neq \emptyset$ and for each vertex $v \in Y$, $\deg_X(v) = \deg_Y(v) = 1$. For each vertex $v_i \in X$, we assign the color $C(v_i) = i$.

High Level Idea: We have four cases depending on how the vertices in $V \setminus X$ interact with $X \setminus Y$ and Y . In each case, we choose a vertex $v \in K$ for some clique $K \subseteq G[V \setminus X]$. We assign colors to the vertices in K such that all the vertices in K and $N(v) \cap X$ have a uniquely colored neighbor, while satisfying the rules of Lemma 9. In particular, we identify a free color from the above partial coloring. We use Lemma 9 to color the remaining vertices and obtain a CFON coloring of G .

The key obstacle here is that while coloring the clique K , we could end up assigning the free color or the color $d + 1$ to multiple vertices of K . There could exist vertices $v_i \in X \setminus Y$, such that $N(v_i) \subseteq K$ and all the vertices in $N(v_i)$ are assigned the free color and the color $d + 1$. This may leave the vertex v_i without a uniquely colored neighbor. Hence, while coloring K , we need to handle these vertices separately. Let S_K be the set of such vertices.

Formally, $S_K = \{v_i \in X \setminus Y : N(v_i) \subseteq K\} \setminus N(v)$. The vertices in $N(v) \cap (X \setminus Y)$ rely on v for their uniquely colored neighbor and hence does not require special attention.

Lemma 14 shows that we can color K in such a way that all the vertices in S_K have a uniquely colored neighbor, and satisfying all the rules of Lemma 9. Lemma 14 will be

proved after completing the proof of Lemma 13. For now, we shall assume Lemma 14 and proceed.

Lemma 14. Let $G = (V, E)$, X, Y be as above. Let $v \in K$ where K is a clique in $G[V \setminus X]$ such that $|K| \geq 2$. Let $C(v_i) = i$ for all $v_i \in X$ and all the vertices in $K \setminus \{v\}$ are uncolored. Suppose $C(v)$ is assigned and the free color f is identified, in such a way that v relies on a color other than f as the unique color in its neighborhood. Then K can be colored in such a way that all the vertices in S_K have a uniquely colored neighbor, and satisfying all the rules of Lemma 9.

It will be convenient to denote an application of Lemma 14 by the 4-tuple, $(v, C(v), f, K)$. For example, we will say “applying Lemma 14 to $(v, 1, 3, K)$ ” to denote an application of Lemma 14 where $v \in K$, $C(v) = 1$ and 3 is the free color.

We have four cases based on the neighborhoods of the vertices in $V \setminus X$.

Case 1: There exists a vertex $v \in V \setminus X$ such that $|N(v) \cap (X \setminus Y)| \geq 2$.

Let $v \in K$, where K is a clique in $G[V \setminus X]$. WLOG let $v_1, v_2 \in X \setminus Y$ such that $v_1, v_2 \in N(v)$.

- $|K| = 1$.

We assign $C(v) = 1$ and we get that $U(v) = v_1$, $U(v_1) = U(v_2) = v$. We get the color 2 as the free color.

- $|K| \geq 2$.

- $S_K \neq \emptyset$.

We assign $C(v) = 1$ and we get that $U(v) = v_1$ and for all $v_i \in N(v) \cap (X \setminus Y)$, we have $U(v_i) = v$. We now apply Lemma 14 to $(v, 1, 2, K)$ ensuring that K is colored, while taking the remaining vertices of S_K into account. Color 2 is the free color.

- $S_K = \emptyset$.

- * There exists a vertex $v' \in K \setminus \{v\}$ such that $v_1 \notin N(v')$.

Assign $C(v) = 1$, $C(v') = d + 1$ and the vertices (if any) in $K \setminus \{v, v'\}$ the color 2.

We get that $U(v') = v$ and for all $y \in K \setminus \{v'\}$, $U(y) = v'$. Also $U(v_1) = U(v_2) = v$. Color 2 is the free color.

- * Every vertex in K is adjacent to v_1 .

Assign $C(v) = 2$ and the vertices in $K \setminus \{v\}$ the color $d + 1$.

We get that $U(y) = v_1$ for all $y \in K$. Also $U(v_1) = U(v_2) = v$. Color 1 is the free color.

In all the above cases, for each $v_i \in Y$, the uniquely colored neighbor is the lone neighbor of v_i in X .

Case 2: There exists a vertex $v \in V \setminus X$ such that $|N(v) \cap (X \setminus Y)| = 1$ and $N(v) \cap Y$ contains a pair of vertices that are adjacent to each other.

Since Case 1 is already addressed, we assume for all $y \in V \setminus X$, we have $|N(y) \cap (X \setminus Y)| \leq 1$.

Let $v \in K$, where K is a clique in $G[V \setminus X]$. WLOG let $N(v) \cap (X \setminus Y) = \{v_1\}$, and let $v_2, v_3 \in N(v) \cap Y$ such that $\{v_2, v_3\} \in E(G)$.

- $|K| = 1$.

We assign $C(v) = 1$. We get that $U(v) = v_1$ and $U(v_1) = U(v_2) = U(v_3) = v$. We have the color 2 as the free color.

- $|K| = 2$.

Let $K = \{v, v'\}$. We have the following cases.

- $S_K \neq \emptyset$.

We assign $C(v) = 1$ and we get that $U(v) = v_1, U(v_1) = U(v_2) = U(v_3) = v$. We apply Lemma 14 to $(v, 1, 2, K)$. We have the color 2 as the free color.

- $S_K = \emptyset$.

Subcase 1: $v_1 \notin N(v')$.

We assign $C(v) = 1$ and $C(v') = d + 1$. We get that $U(v) = v_1, U(v') = v$ and $U(v_1) = U(v_2) = U(v_3) = v$. We have color 2 as the free color.

Subcase 2: $v_1 \in N(v')$. That is, $N(v') \cap (X \setminus Y) = \{v_1\}$.

We first check if there exists a clique $\widehat{K} \subseteq G[V \setminus X]$ such that $N(v_2) \cap \widehat{K} \neq \emptyset$ or $N(v_3) \cap \widehat{K} \neq \emptyset$.

If there is no such clique \widehat{K} , we reassign $C(v_3) = 2$, assign $C(v) = 3$ and $C(v') = d + 1$. We get that $U(v) = v_1$ and $U(v_1) = U(v_2) = U(v_3) = U(v') = v$. Color 2 is the free color.

Else, there exists a clique \widehat{K} such that $N(v_2) \cap \widehat{K} \neq \emptyset$ or $N(v_3) \cap \widehat{K} \neq \emptyset$. WLOG let $N(v_3) \cap \widehat{K} \neq \emptyset$. We assign $C(v) = 2$ and $C(v') = d + 1$. Now the vertex v_3 does not have a uniquely colored neighbor. Let $w \in N(v_3) \cap \widehat{K}$.

If $S_{\widehat{K}} \neq \emptyset$, we assign $C(w) = 3$. We have $U(v) = U(v') = v_1, U(v_1) = U(v_2) = v, U(v_3) = w$ and $U(w) = v_3$. Due to the case definition, $|N(w) \cap (X \setminus Y)| \leq 1$. For the lone vertex $v_i \in N(w) \cap (X \setminus Y)$ (if it exists), we have $U(v_i) = w$. We now apply Lemma 14 to $(w, 3, 1, \widehat{K})$ to color the remaining vertices \widehat{K} taking care of the vertices in $S_{\widehat{K}}$. We have color 1 as the free color.

Else if $S_{\widehat{K}} = \emptyset$, we do the following².

- * There exists a vertex $w' \in \widehat{K} \setminus \{w\}$, such that $v_3 \notin N(w')$.

We assign $C(w) = 3, C(w') = d + 1$ and the vertices (if any) in $\widehat{K} \setminus \{w, w'\}$ the color 1.

We get that $U(v) = U(v') = v_1, U(v_1) = U(v_2) = v, U(v_3) = w, U(w') = w$ and for all vertices $x \in \widehat{K} \setminus \{w'\}, U(x) = w'$. We have the color 1 as the free color.

- * For each $x \in \widehat{K}$, we have $v_3 \in N(x)$.

We assign $C(w) = 1$ and the rest of the vertices (if any) in $\widehat{K} \setminus \{w\}$ the color $d + 1$.

We get that $U(v) = U(v') = v_1, U(v_1) = U(v_2) = v, U(v_3) = w$, and for all vertices $x \in \widehat{K}, U(x) = v_3$. Color 3 is the free color.

- $|K| \geq 3$.

²One may wonder about the possibility of vertices $v_i \in X \setminus Y$ such that $N(v_i) \subseteq K \cup \widehat{K}$, and be concerned that these vertices v_i do not feature in S_K or $S_{\widehat{K}}$. We note that there are no such vertices v_i . This is because, we have $N(v) \cap (X \setminus Y) = N(v') \cap (X \setminus Y) = \{v_1\}$ in order to be in Subcase 2.

- $S_K \neq \emptyset$.
We assign $C(v) = 1$ and we get that $U(v) = v_1, U(v_1) = U(v_2) = U(v_3) = v$. We apply Lemma 14 to $(v, 1, 2, K)$. We have the color 2 as the free color.
- $S_K = \emptyset$.
 - * There exists a vertex $v' \in K \setminus \{v\}$ such that $v_1 \notin N(v')$.
We assign $C(v) = 1, C(v') = d + 1$ and the vertices in $K \setminus \{v, v'\}$ the color 3.
We get that $U(v') = v$ and for all $y \in K \setminus \{v'\}, U(y) = v'$. Also $U(v_1) = U(v_2) = U(v_3) = v$. We have the color 3 as the free color.
 - * Every vertex in K is adjacent to v_1 .
Choose two vertices $v', v'' \in K \setminus \{v\}$ and assign $C(v) = 1, C(v') = 2, C(v'') = d + 1$ and the vertices (if any) in $K \setminus \{v, v', v''\}$ the color 3.
We get that $U(v'') = v'$ and for all $y \in K \setminus \{v''\}, U(y) = v''$. Also $U(v_1) = U(v_2) = U(v_3) = v$. We have the color 3 as the free color.

In each of the above cases, for each $v_i \in Y \setminus \{v_2, v_3\}$, the uniquely colored neighbor is the lone neighbor of v_i in X .

Case 3: There exists a vertex $v \in V \setminus X$ such that $|N(v) \cap (X \setminus Y)| = 1$ and $|N(v) \cap Y| \geq 1$. Moreover, none of the vertices in $N(v) \cap Y$ are adjacent to each other.

Let $v \in K$ for a clique $K \subseteq G[V \setminus X]$. WLOG let $v_1 \in N(v) \cap (X \setminus Y)$ and $v_2 \in N(v) \cap Y$. Let v_3 be the lone neighbor of v_2 in Y . It follows that $v_3 \notin N(v)$.

- $|K| = 1$.
We assign $C(v) = 1$ and we get that $U(v) = v_1, U(v_1) = U(v_2) = v$ and $U(v_3) = v_2$. Color 3 is the free color.
- $|K| \geq 2$.
 - $S_K \neq \emptyset$.
We assign $C(v) = 1$ and we get that $U(v) = v_1, U(v_1) = U(v_2) = v$ and $U(v_3) = v_2$. We apply Lemma 14 to $(v, 1, 3, K)$. Color 3 is the free color.
 - $S_K = \emptyset$.
 - * There exists a vertex $v' \in K \setminus \{v\}$ such that $v_2 \in N(v')$.
We assign $C(v) = 1, C(v') = d + 1$ and the vertices (if any) in $K \setminus \{v, v'\}$ the color 3.
We get that $U(v') = v_2$ and for all $y \in K \setminus \{v'\}, U(y) = v'$. Also $U(v_1) = U(v_2) = v$ and $U(v_3) = v_2$. We have color 3 as the free color.
 - * None of the vertices in $K \setminus \{v\}$ are adjacent to v_2 .
We assign $C(v) = 2$ and the vertices in $K \setminus \{v\}$ the color $d + 1$. We get that $U(v) = v_1$ and for all $y \in K \setminus \{v\}, U(y) = v$. Also $U(v_1) = U(v_2) = v$ and $U(v_3) = v_2$. We have color 3 as the free color.

In each of the above cases, for each $v_i \in Y \setminus \{v_2, v_3\}$, the uniquely colored neighbor is the lone neighbor of v_i in X .

Case 4: For each $y \in V \setminus X$ such that $|N(y) \cap (X \setminus Y)| = 1$, we have $|N(y) \cap Y| = 0$.

Since Case 1 is addressed, we assume that each vertex in $V \setminus X$ has at most 1 neighbor in $X \setminus Y$.

Since G is connected and since³ $|X \setminus Y| \geq 1$, we can choose a clique $K \subseteq G[V \setminus X]$ with

³This is where we make use of the assumption that $X \setminus Y$ is nonempty.

distinct vertices⁴ $v, v' \in K$ such that $N(v) \cap (X \setminus Y) \neq \emptyset$ and $N(v') \cap Y \neq \emptyset$. WLOG let $v_1 \in N(v) \cap (X \setminus Y)$ and $v_2 \in N(v') \cap Y$. Let v_3 be the lone neighbor of v_2 in Y . It follows that $v_2, v_3 \notin N(v)$.

- $S_K = \emptyset$.

We assign $C(v) = 3$, $C(v') = d + 1$ and the vertices (if any) in $K \setminus \{v, v'\}$ the color 1. We get that $U(v') = v_2$ and for all $y \in K \setminus \{v'\}$, $U(y) = v'$. Also $U(v_1) = v$, $U(v_2) = v_3$ and $U(v_3) = v_2$. Color 1 is the free color.

- $S_K \neq \emptyset$.

We assign $C(v) = 3$ and we have $U(v_1) = v$, $U(v_2) = v_3$ and $U(v_3) = v_2$.

Recall that $S_K = \{v_i \in X \setminus Y : N(v_i) \subseteq K\} \setminus N(v)$. For each $v_i \in S_K$, choose a vertex $w_i \in N(v_i)$, assign $C(w_i) = i$ and let $U(v_i) = w_i$. Since each vertex in $V \setminus X$ has at most 1 neighbor in $X \setminus Y$, it also follows that $N(v_i) \cap N(v_{i'}) = \emptyset$ for any two vertices $v_i, v_{i'} \in S_K$.

Because of the condition of Case 4, $N(v') \cap (X \setminus Y) = \emptyset$. Assign $C(v') = d + 1$, and assign the color 1 to all the remaining uncolored vertices (if any) in K . We have $U(v') = v_2$ and for all $y \in K \setminus \{v'\}$, $U(y) = v'$. Color 1 is the free color⁵.

In each of the above cases, for each $v_i \in Y \setminus \{v_2, v_3\}$, the uniquely colored neighbor is the lone neighbor of v_i in X .

We have concluded the four cases. In each of the cases we have a free color f . We use Lemma 9 to get a uniquely colored neighbor for remaining vertices in $X \setminus Y$, the cliques in $G[V \setminus X]$ and thereby obtain a CFON coloring.

Lemma 14 (Restated). Let $G = (V, E)$ be a graph and $X = \{v_1, v_2, \dots, v_d\} \subseteq V$ be a set of vertices such that $G[V \setminus X]$ is a disjoint union of cliques. Let $Y = \{v_i \in X : \deg_X(v_i) \geq 1\}$. Let $v \in K$ where K is a clique in $G[V \setminus X]$ such that $|K| \geq 2$. Let $C(v_i) = i$ for all $v_i \in X$ and all the vertices in $K \setminus \{v\}$ are uncolored. Suppose $C(v)$ is assigned and the free color f is identified, in such a way that v relies on a color other than f as the unique color in its neighborhood. Then K can be colored in such a way that all the vertices in S_K have a uniquely colored neighbor, and satisfying all the rules of Lemma 9.

Proof. Recall that $S_K = \{v_i \in X \setminus Y : N(v_i) \subseteq K\} \setminus N(v)$. Let $S_K = \{v_{j_1}, v_{j_2}, \dots, v_{j_m}\}$, for some $m \geq 1$. For each $v_i \in S_K$, choose an uncolored vertex $w_i \in N(v_i)$, assign $C(w_i) = i$ and let $U(v_i) = w_i$. If all the vertices in $N(v_i)$ are colored, we arbitrarily choose a vertex in $N(v_i)$ as $U(v_i)$. WLOG let the colors used in K because of the above process be $\{j_1, j_2, \dots, j_{m'}\}$ where $m' \leq m$. Note that the vertex v is colored prior to the application of this lemma, and has a uniquely colored neighbor as well. Hence we do not talk about $C(v)$ and $U(v)$ in this proof.

We have the following cases based on the number of uncolored vertices in K .

- All the vertices in K are colored.

Each vertex in $K \setminus \{v\}$ was colored because it was chosen as w_i by some $v_i \in S_K$. Hence $U(w_i) = v_i$.

⁴Because of the definition of Case 4, it follows that $v \neq v'$.

⁵One may wonder why we did not apply Lemma 14 to $(v, 3, 1, K)$ in this situation. This is because Lemma 14 requires v to rely on a color other than the free color as the unique color in its neighborhood. There is no assignment that meets this requirement.

- K contains exactly one uncolored vertex.

Let the uncolored vertex in K be v' . If v' has a uniquely colored neighbor, we assign $C(v') = d + 1$. Now, every vertex in K has a uniquely colored neighbor.

If v' does not have a uniquely colored neighbor, we have two cases depending on the number of colors used in K .

- $m = m'$.

This means that each vertex $v_i \in S_K$ chose a neighbor $w_i \in K$ and assigned the color i to it. So v' sees each of the colors j_1, j_2, \dots, j_m twice in its neighborhood. This means that $v_{j_1}, v_{j_2}, \dots, v_{j_m} \in N(v')$. Recall that f is the free color, where $1 \leq f \leq d$, and hence $v_f \in X$ sees a color other than f as the unique color in its neighborhood. We do the following to obtain a uniquely colored neighbor for v' :

- * There exists a vertex $v'' \in K \setminus \{v, v'\}$ such that $v_f \in N(v'') \cap X$.
Let $C(v'') = k$ due to a vertex $v_k \in N(v'') \cap S_K$. Assign $C(v') = k$ and reassign $C(v'') = d + 1$. We have that $U(v') = v_k$, $U(v'') = v_f$ and we reassign $U(v_k) = v'$.

- * None of the vertices in $K \setminus \{v, v'\}$ is adjacent to v_f .
Note that $v_f \notin N(v')$, else v_f would have served as a uniquely colored neighbor for v' .

Choose a vertex $v'' \in K \setminus \{v, v'\}$. Suppose $C(v'') = k$ and this implies that $v_k \in N(v'') \cap S_K$. We reassign $C(v'') = f$ and assign $C(v') = d + 1$. We have $U(v') = v_k$ and we reassign $U(v_k) = v'$. The assignment of $d + 1$ as the unique color in the neighborhood of v_k is an exception. However, this is fine as $N(v_k)$ is contained in K , and does not interact with any other cliques in $G[V \setminus X]$.

- $m > m'$.

This implies that there exists a vertex $v_j \in S_K$ such that the color j is not given to any vertex in K . So v_j must be seeing a vertex $v'' \in K \setminus \{v'\}$ as its uniquely colored neighbor.

We claim that $v_j \notin N(v')$. If $v_j \in N(v')$, then v_j is the lone vertex in $N(v')$ that is colored j , and hence is a uniquely colored neighbor for v' . As per the scope of this case, v' does not have a uniquely colored neighbor. This is a contradiction.

We reassign $C(v'') = j$ and assign $C(v') = d + 1$. We get that $U(v') = v''$.

- K contains at least two uncolored vertices.

We first check if there exists an uncolored vertex v' in K such that v' has a uniquely colored neighbor other than v_f . If such a v' exists, then we assign $C(v') = d + 1$ and the remaining uncolored vertices in K the free color f . For all $w \in K$ such that $C(w) = f$, we have $U(w) = v'$.

If such a vertex v' does not exist, we have the following cases based on the relation between m and m' .

- $m = m'$.

Choose a colored vertex $w \in K \setminus \{v\}$ and an uncolored vertex $v' \in K$. Suppose $C(w) = j$, which means that $v_j \in N(w)$. Since v' does not see a uniquely colored neighbor other than v_f , it is the case that $v_j \in N(v')$.

We reassign $C(w) = f$, assign $C(v') = d+1$ and the remaining uncolored vertices (if any) in K the free color f . We get that $U(v') = v_j$ and $U(v_j) = v'$. All the vertices in $K \setminus \{v'\}$ will have v' as their uniquely colored neighbor.

The assignment of $d+1$ as the unique color in the neighborhood of v_j is an exception. However, this is fine as $N(v_j)$ is contained in K , and does not interact with any other cliques in $G[V \setminus X]$.

– $m > m'$.

This implies that there exists a vertex $v_j \in S_K$ such that the color j was not used in K . This also implies that none of the uncolored vertices in K have v_j in their neighborhood. This is because if v_j had an uncolored neighbor in K , then that neighbor would have been colored j in the coloring process performed at the beginning of this proof.

We choose two uncolored vertices $v', v'' \in K$ and assign $C(v') = d+1$, $C(v'') = j$ and the remaining uncolored vertices (if any) the color f .

We get that $U(v') = v''$ and for all other vertices $w \in K \setminus \{v'\}$ will have $U(w) = v'$.

□

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