Conflict-free Coloring on Subclasses of Perfect graphs and Bipartite graphs*

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Abstract

A *Conflict-Free Open Neighborhood coloring*, abbreviated CFON* coloring, of a graph $G = (V, E)$ using k colors is an assignment of colors from a set of k colors to a subset of vertices of V such that every vertex sees some color exactly once in its open neighborhood. The minimum k for which G has a CFON* coloring using k colors is called the *CFON* chromatic number* of G, denoted by $\chi^*_{ON}(G)$. The analogous notion for closed neighborhood is called CFCN* coloring and the analogous parameter is denoted by $\chi^*_{CN}(G)$. The problem of deciding whether a given graph admits a CFON* (or CFCN*) coloring that uses k colors is NP-complete. Below, we describe briefly the main results of this paper.

- We show that it is NP-hard to determine the CFCN* chromatic number of chordal graphs. We also show the existence of a family of chordal graphs G that requires $\Omega(\sqrt{\omega(G)})$ colors to CFCN* color G, where $\omega(G)$ represents the size of a maximum clique in G.
- We give a polynomial time algorithm to compute $\chi^*_{ON}(G)$ for interval graphs G. This answers in positive the open question posed by Reddy [Theoretical Comp. Science, 2018] to determine whether CFON* chromatic number can be computed in polynomial time for interval graphs.
- We explore biconvex graphs, a subclass of bipartite graphs, and give a polynomial time algorithm to compute their CFON* chromatic number.

Keywords: Conflict-free Coloring, Graph Coloring, Interval Graphs, Chordal Graphs, Bipartite Graphs.

1 Introduction

Given a coloring of a graph $G = (V, E)$, we say a vertex $v \in V(G)$ *sees* a color c if there exists a neighbor of v that is assigned the color c. A *Conflict-Free Open Neighborhood coloring*,

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abbreviated CFON* coloring, of a graph $G = (V, E)$ using k colors is an assignment of colors to a subset of vertices such that every vertex sees some color exactly once in its open neighborhood. The minimum k for which G has a CFON* coloring using k colors is called the *CFON* chromatic number* of G, denoted by $\chi^*_{ON}(G)$. [1] The analogous notion for closed neighborhood is called CFCN* coloring and the analogous parameter is denoted by $\chi^*_{CN}(G)$. It is known (see for instance, Equation 1.3 from [27]) that if G has no isolated vertices, then $\chi^*_{CN}(G)$ is at most twice $\chi^*_{ON}(G)$. Given a graph G and an integer $k \in \mathbb{N}$, the *CFON* coloring problem* is the problem of determining if $\chi^*_{ON}(G) \leq k$. The CFON* variant is considered to be harder than the CFCN* variant, see for instance the remarks in [24, 27]. As an example, consider the complete graph K_n on n vertices. The chromatic number of K_n is $\chi(K_n) = n$ while the conflict-free chromatic numbers are $\chi^*_{CN}(K_n) = 1$ and $\chi^*_{ON}(K_n) = 2$. It is sufficient to color one vertex (resp. two vertices) of K_n to obtain a CFCN* (resp. CFON*) coloring.

The notion of conflict-free coloring was introduced by Even, Lotker, Ron and Smorodinsky in 2004, motivated by the frequency assignment problem in wireless communication [16]. The conflict-free coloring problem on graphs was introduced and first studied by Cheilaris [11] and Pach and Tardos [27]. Conflict-free coloring has found applications in the area of sensor networks [19, 26] and coding theory [25]. Since its introduction, the problem has been extensively studied, see for instance [1, 3, 6, 8, 11, 20, 21, 27, 29]. The decision version of the CFON* coloring problem and many of its variants are known to be NP-complete [1, 20]. In [20], Gargano and Rescigno showed that the optimization version of the CFON* coloring problem is hard to approximate within a factor of $n^{1/2-\epsilon}$, unless $\mathsf{P} = \mathsf{NP}$. Fekete and Keldenich [17] and Hoffmann et al. [23] studied a conflict-free variant of the chromatic Art Gallery Problem, which is about guarding a simple polygon P using a finite set of colored point guards such that each point $p \in P$ sees at least one guard whose color is distinct from all the other guards visible from p.

The conflict-free coloring problem has been studied on several graph classes like planar graphs [1], graphs of bounded degree [27], geometric intersection graphs like interval graphs [5, 12, 28], unit disk intersection graphs and unit square intersection graphs [4, 18], split graphs [4, 28], distance hereditary graphs [4], etc. The problem has been studied from a parameterized complexity perspective and is fixed-parameter tractable when parameterized by tree-width [2, 8], neighborhood diversity, distance to cluster [28], or the combined parameters clique-width and the number of colors [4, 5].

1.1 Our Contribution and Discussion

In this paper, we consider the problems of determining χ^*_{ON} and χ^*_{CN} on some subclasses of perfect graphs and bipartite graphs. Some of the popular subclasses of perfect graphs include chordal graphs, split graphs, interval graphs and cographs. Given a cograph G, we can determine $\chi^*_{ON}(G)$ and $\chi^*_{CN}(G)$ in polynomial time [5]. Moreover, it is known that $\chi^*_{ON}(G), \chi^*_{CN}(G) \leq 2$. For a split graph G, $\chi^*_{CN}(G)$ can be computed in polynomial time while determining $\chi^*_{ON}(G)$ is NP-hard [5]. Further, it is known that $\chi^*_{CN}(G) \leq 2$ whereas there exists a family of split graphs G' such that $\chi^*_{ON}(G') = \Theta(\sqrt{n})$.

In general, we use n to denote the number of vertices of the input graph. We denote by $\omega(G)$, the size of a largest clique in the graph G. Since split graphs are a subclass of chordal graphs, determining χ^*_{ON} on chordal graphs is NP-hard. It is known that $\omega(G)$ colors are sufficient and necessary to properly color (any pair of adjacent vertices are assigned distinct colors) a chordal graph G. Since a proper coloring is also a CFCN* coloring, we have that $\chi^*_{CN}(G) \leq \omega(G)$.

[1] It is also known by the name 'partial conflict-free chromatic number' as only a subset of vertices are assigned colors. The '(full) conflict-free chromatic number' of a graph, which requires assigning colors to all the vertices, is at most one more than its partial conflict-free chromatic number.

Therefore we ask the following questions on chordal graphs: (i) whether $\chi^*_{CN}(G) \in O(1)$, similar to the case when the graph is a split graph or an interval graph, and (ii) whether $\chi^*_{CN}(G)$ can be computed in polynomial time. We answer both the questions in the negative by exhibiting a family of chordal graphs that require $\Omega(\sqrt{\omega(G)})$ colors in any CFCN* coloring of G. Then we show that it is NP-hard to determine if $\chi^*_{CN}(G) = 1$. We state the results formally below and the proofs are presented in Section 3. Chordal graphs are formally defined at the beginning of the section.

Theorem 1. *Given a chordal graph G, it is* NP*-hard to determine if $\chi^*_{CN}(G) = 1$.*

Theorem 2. *There exists an infinite sequence of chordal graphs G_k, on an increasing number of vertices, such that $\chi^*_{CN}(G_k) = \Omega(\sqrt{\omega(G_k)})$.*

Next, we consider interval graphs. For an interval graph G, it is known that $\chi^*_{CN}(G) \leq 2$ and the problem of determining $\chi^*_{CN}(G)$ is polynomial time solvable [18]. It was shown that $\chi^*_{ON}(G) \leq 3$ for interval graphs and that the bound is tight [5]. It was asked in [28] if there is a polynomial time algorithm that, given an interval graph G, computes $\chi^*_{ON}(G)$. We answer this in the affirmative and give polynomial time characterization algorithms for interval graphs G that decide if $\chi^*_{ON}(G) \in \{1, 2, 3\}$. Formally, we have the following theorem, the proof for which is presented in Section 4. Interval graphs are formally defined at the beginning of this section.

Theorem 3. *Given an interval graph G, we can determine $\chi^*_{ON}(G)$ in time $O(n^{20})$.*

Towards the end, we consider a subclass of bipartite graphs called biconvex graphs. It is easy to see that $\chi^*_{CN}(G) \leq 2$ for a bipartite graph G. On the contrary, there exist bipartite graphs (for instance, subdivision of a clique), for which $\chi^*_{ON}(G) = \Theta(\sqrt{n})$. It is NP-complete to decide if a planar bipartite graph G has $\chi^*_{ON}(G) \in \{1, 2, 3\}$ [1]. We show that $\chi^*_{ON}(G) \in \{1, 2\}$ for a biconvex graph G. To decide whether $\chi^*_{ON}(G) = 1$ or $\chi^*_{ON}(G) = 2$ for a biconvex graph, we use characterization algorithms, similar to those in interval graphs. The results and the formal definition of biconvex graphs are presented in Section 5.

Theorem 4. *Given a biconvex graph G, we can determine $\chi^*_{ON}(G)$ in time $O(n^5)$.*

Note: Theorems 1 and 2 are new results whereas Theorems 3 and 4 appeared in [7].

2 Preliminaries

Throughout the paper, we consider simple undirected graphs without any isolated vertices (for graphs with isolated vertices there is no CFON* coloring). For standard terminology related to graph theory, we refer to the textbook by Diestel [14]. For a vertex $v \in V(G)$, its *open neighborhood*, denoted by $N_G(v)$, is the set of neighbors of v in G. The *closed neighborhood* of v, denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. For a set of vertices S, we denote $N_G(S) = \cup_{v \in S} N_G(v)$. We drop the subscript when it is clear from the context.

In the introduction, we defined conflict-free colorings to be an assignment of colors to a *subset* of the vertices of the graph. For the sake of convenience, we will use the color 0 to denote uncolored vertices. The "color" 0 cannot serve as a unique color in the neighborhood of any vertex.

3 Chordal graphs

In this section, we consider conflict-free colorings on chordal graphs.

Definition 5. *A* chord *of a cycle is an edge whose endpoints are vertices of the cycle but is not part of the cycle. A* chordal graph *is a graph in which every cycle of length at least 4 has a chord.*

It is known that determining the CFON* chromatic number on chordal graphs is NP-hard, which follows from the NP-hardness result on split graphs [5]. In this section, we explore bounds on $\chi^*_{CN}(G)$ for chordal graphs G. Since a proper coloring is also a CFCN* coloring and a chordal graph G can be properly colored using $\omega(G)$ (the size of a largest clique) colors, it follows that $\chi^*_{CN}(G) \leq \omega(G)$. It is natural to ask if this upper bound can be improved to a constant. We answer this in the negative. We also study the complexity of determining the CFCN* chromatic number of chordal graphs.

Definition 6 (Perfect Independent Dominating Set [10])**.** *Given a graph G, a* perfect independent dominating set *is a set of vertices $S \subseteq V(G)$ such that S is an independent set and every vertex outside S has exactly one neighbor in S. That is, for each $v \in V(G)$, $|N[v] \cap S| = 1$.*

Given a graph G, the Perfect Independent Dominating Set (PIDS, in short) problem asks if G has a perfect independent dominating set. It is known that PIDS is NP-hard on chordal graphs [10].

Proof of Theorem 1. We give a reduction from the Perfect Independent Dominating Set problem on chordal graphs. Consider an instance G of PIDS where G is chordal. We show that there exists a perfect independent dominating set of G if and only if $\chi^*_{CN}(G) = 1$.

Let $S \subseteq V(G)$ be a perfect independent dominating set of G. We now give an assignment $f : V(G) \rightarrow \{0, 1\}$. We assign $f(v) = 1$ for each $v \in S$ and assign the color 0 to all vertices in $V(G) \setminus S$. Since every vertex $v \in V(G)$ has exactly one neighbor in $N[v] \cap S$, the coloring f is a CFCN* coloring.

Let $f : V(G) \rightarrow \{0, 1\}$ be a CFCN* coloring of G. We obtain a perfect independent dominating set S of G by picking vertices that are assigned the color 1 in G.

Since PIDS is NP-hard on chordal graphs [10], it is NP-hard to determine if the CFCN* chromatic number of a chordal graph is 1. □

We now show the existence of chordal graphs with large CFCN* chromatic number. We first look at the following lemma.

Lemma 7. *Let H be a graph such that $\chi^*_{CN}(H) \geq k$. Consider a graph G, which contains two disjoint copies of H, say H_1 and H_2. Let $X \subseteq V(G)$ such that X is disjoint from $V(H_1) \cup V(H_2)$. Further let vertices in X be adjacent to each vertex of H_1 and H_2, and $N_G(V(H_1)) \setminus V(H_1) = N_G(V(H_2)) \setminus V(H_2) = X$. Then in any CFCN* coloring of G using k colors, there exists a vertex $w \in V(H_1) \cup V(H_2)$ such that each uniquely colored neighbor of w belongs to X.*

Proof. Suppose for the sake of contradiction that $c : V(G) \rightarrow \{1, \ldots, k\}$ is a CFCN* coloring of G where each vertex of H_1 (resp. H_2) has a uniquely colored neighbor from H_1 (resp. H_2). This means that c restricted to H_1 (resp. H_2) is a CFCN* coloring of H_1 (resp. H_2). Since $\chi^*_{CN}(H_1) = \chi^*_{CN}(H_2) \geq k$, each of the colors in $\{1, 2, \ldots, k\}$ appear at least twice in the neighborhood of each vertex in X. This contradicts the assumption that c is a CFCN* coloring of G. □

Theorem 8. *There exists a family of chordal graphs G_k such that $\chi^*_{CN}(G_k) \geq k$.*

Proof. We construct graphs G_k, where $k \geq 1$, in an inductive fashion satisfying the property that G_k cannot be CFCN* colored using $k - 1$ colors. Let the graph G_1 be isomorphic to K_2. For each $k \geq 1$, the graph G_{k+1} is constructed as follows:

- Add a set $B = \{v_1, v_2, \ldots, v_{k+1}\}$ of *bottom vertices* with the vertices in B being pairwise adjacent (thereby forming a clique),
- For each nonempty $X \subseteq B$, add two disjoint copies of G_k, say G_1^X and G_2^X, and make each vertex of G_1^X and G_2^X adjacent to every vertex in X.

An illustration of the graph G_3 is given in Figure 1. We use induction on k to show that G_k does not have a CFCN* coloring using $k-1$ colors. The hypothesis is clearly true for the base case where the graph is G_1. We assume that the hypothesis is true for G_k. Suppose for the sake of contradiction that G_{k+1} is CFCN* colorable using k colors. From Lemma 7, in any CFCN* coloring of G_{k+1} using k colors, we have that for each nonempty $X \subseteq B$ there is a vertex in $G_1^X \cup G_2^X$ whose each uniquely colored neighbor belongs to X. Consider the set $X_1 = B$. Because of Lemma 7, there is a vertex from X_1 that is a uniquely colored neighbor of some vertex from $G_1^{X_1} \cup G_2^{X_1}$. Without loss of generality, let the vertex be v_1. Now consider the set $X_2 = X_1 \setminus \{v_1\}$. Again from Lemma 7, there is a vertex in X_2 that acts as a uniquely colored neighbor of some vertex from $G_1^{X_2} \cup G_2^{X_2}$. Without loss of generality let that vertex from X_2 be v_2. We repeat this process until we reach the set $X_{k+1} = X_k \setminus \{v_1, v_2, v_3, \ldots, v_k\} = \{v_{k+1}\}$. By the same argument, we infer that v_{k+1} is a uniquely colored neighbor of some vertex in $G_1^{X_{k+1}} \cup G_2^{X_{k+1}}$.

Now, we show that this leads to a contradiction, by showing that no two vertices in $\{v_1, v_2, \ldots, v_{k+1}\}$ can be assigned the same color. Suppose that there exist two vertices v_i and v_j of the same color, where $1 \leq i < j \leq k+1$. Recall that v_i was chosen as a uniquely colored neighbor of some vertex $w \in G_1^{X_i} \cup G_2^{X_i}$. Since $X_j \subseteq X_i$, w sees both v_i and v_j which are assigned the same color, contradicting our inference that v_i is a uniquely colored neighbor of w.

All that remains to show is that G_{k+1} is a chordal graph. We show this by induction. The base case $G_1 = K_2$ is a chordal graph. Suppose that G_{k+1}, for some $k \geq 1$, is not a chordal graph. Among the cycles of length at least 4, let C be a cycle of shortest length that does not have a chord. It is easy to see that C does not contain more than two vertices from B. We have the following cases depending on the size of $C \cap B$.

- C contains exactly two vertices from B.

  Let the two vertices be v_i and v_j. It must be the case that v_i and v_j are consecutive in C, otherwise C contains a chord. Then the vertices of $C - \{v_i, v_j\}$ should come from a copy of G_k, say H, that is adjacent to both v_i and v_j. According to the construction, v_i and v_j are adjacent to every vertex in H and thus C cannot be a shortest cycle without a chord.

- C contains exactly one vertex from B.

  Then the other vertices of C belong to one copy of G_k. The arguments are similar to those in the above case.

- C contains no vertex from B.

  Then C contains vertices from a copy of G_k which by induction is chordal, and thus a contradiction.

□

Claim 9. *The clique number of G_k is $\omega(G_k) = \frac{k(k+1)}{2} + 1$.*

Proof. We use induction to prove the bound. Let ω_k denote the size of the maximum sized clique in G_k. The graph G_1 is isomorphic to K_2 and thus $\omega_1 = 2$, satisfying the base case. Any maximum clique in G_k contains the set of k bottom vertices, say B_k, and the vertices of a maximum clique in a copy of G_{k-1} that is adjacent to each of the vertices in B_k. We cannot

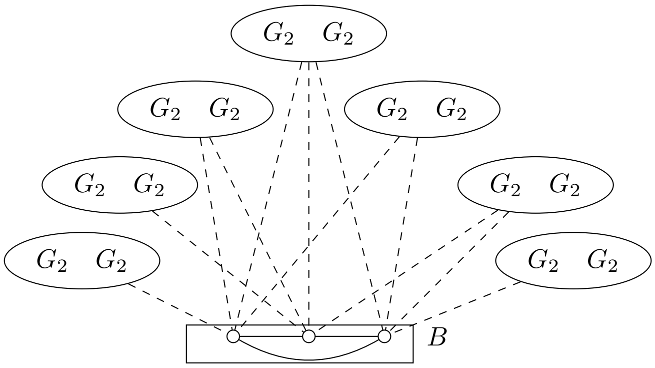

Figure 1: Illustration of the graph G_3. The dashed line between a vertex, say v, from B and an ellipse containing two copies of G_2 indicate that v is adjacent to every vertex inside the ellipse.

have vertices from two different copies of G_{k-1} as they are not adjacent to each other. Thus $\omega_k = |B_k| + \omega_{k-1} = k + (k-1)k/2 + 1 = k(k+1)/2 + 1$. □

The proof of Theorem 2 follows from Theorem 8 and Claim 9.

4 Interval graphs

In this section, we show that the problem of determining the CFON* chromatic number of a given interval graph is polynomial time solvable. It was shown in [5, 28] that, for an interval graph G, $\chi^*_{ON}(G) \leq 3$ and that there exists an interval graph that requires three colors. The complexity of the problem on interval graphs was posed as an open question in the above papers. We show that, given an interval graph G, it is possible to decide in polynomial time whether $\chi^*_{ON}(G)$ is 1, 2 or 3.

Definition 10 (Interval Graphs)**.** *A graph $G = (V, E)$ is called an* interval graph *if there exists a set of intervals on the real line such that the following holds: (i) there is a bijection between the intervals and the vertices and (ii) there exists an edge between two vertices if and only if the corresponding intervals intersect.*

The main ingredient of the algorithm is the use of *multi-chain ordering* property on interval graphs. Before defining the multi-chain ordering property, we look at some prerequisites.

Definition 11 (Chain Graph [15])**.** *A bipartite graph $G = (A, B)$ is a* chain graph *if and only if for any two vertices $u, v \in A$, either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$.*

Proposition 12. *If $G = (A, B)$ is a chain graph as defined above, it follows that for any two vertices $u, v \in B$, either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$.*

As a consequence, we can order the vertices in B in the decreasing order of their degrees. If there are multiple vertices of the same degree, we arbitrarily order these vertices. If $b_1 \in B$ appears before $b_2 \in B$ in the ordering, then it follows that $N(b_2) \subseteq N(b_1)$.

Definition 13 (Multi-chain Ordering [9, 15])**.** *Let $L_0, L_1, \ldots, L_p$ be a partition of the vertices of the graph G. We say these layers form a* multi-chain ordering *of G if*

- $|L_0| = 1$,

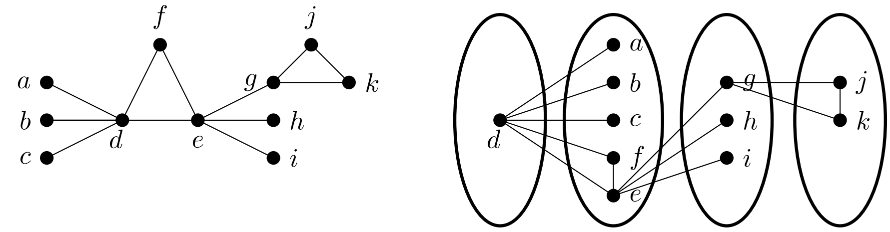

Figure 2: A graph G (on the left) and a multi-chain ordering of G (on the right).

- *the layer L_i, where $0 \leq i \leq p$, represents the set of vertices that are at a distance i from the vertex in L_0, and*
- *for every two consecutive layers L_i and L_{i+1}, where $i \in \{0, 1, \ldots, p-1\}$, we have that the vertices in L_i and L_{i+1}, and the edges connecting these layers form a chain graph.*

Note that p here denotes the largest integer such that L_p is non-empty.

For a given graph G, it is possible to check for the existence of a multi-chain ordering in polynomial time by trying out all the possibilities of the starting vertex in L_0. A illustration of a multi-chain ordering is given in Figure 2. Notice that G is an interval graph.

Theorem 14 (Theorem 2.5 of [15])**.** *All connected interval graphs admit multi-chain orderings.*

We give a characterization of interval graphs that require one color and two colors in polynomial time in Theorem 17 and Theorem 19 respectively. Given an interval graph G, the algorithms decide if G is CFON* colorable using one color or two colors. If G is not CFON* colorable using one color or two colors, we conclude that G is CFON* colorable using three colors (since it is known that for an interval graph G, $\chi^*_{ON}(G) \leq 3$). One of the key ideas used in Theorem 19 (to decide if G can be CFON* colored using two nonzero colors) is sort of a bootstrapping idea. After narrowing down the possibilities, we need to test if a given subgraph can be colored using the colors $\{0, 1\}$ so as to obtain a CFON* coloring. To solve this, we use Theorem 17.

Before we proceed to the main theorems of this section, we observe the following on a graph G that admits multi-chain ordering.

Observation 15. *If G admits a multi-chain ordering, then every distance layer L_i, for $0 \leq i < p$, contains a vertex v such that $N(v) \supseteq L_{i+1}$.*

Proof. Consider a multi-chain ordering of G. For any two consecutive distance layers L_i and L_{i+1}, it can be seen that each vertex in L_{i+1} has a neighbor in L_i. This, together with the fact that L_i and L_{i+1} form a chain graph, imply that there is a vertex $v \in L_i$ such that $N(v) \supseteq L_{i+1}$. □

Observation 16. *In any CFON* coloring of G that uses one color, at most one vertex in each L_i is assigned the color 1.*

Proof. Consider a layer L_i of the graph. As per Observation 15, there is a $v \in L_i$ such that $N(v) \supseteq L_{i+1}$. If two vertices in L_{i+1} are colored 1, then the vertex $v \in L_i$ does not have a uniquely colored neighbor. Hence in all the layers $L_1, L_2, \ldots$ up to the last layer L_p, we have that at most one vertex is assigned the color 1. Since L_0 has only one vertex, the statement is trivially true for L_0. □

Theorem 17. *Given an interval graph $G = (V, E)$, we can decide in time $O(n^5)$ if $\chi^*_{ON}(G) = 1$.*

Proof. Let $L_0, L_1, \ldots, L_p$ be the distance layers that form a multi-chain ordering of G. Let $L_0 = \{v_0\}$. If there is a CFON* coloring that uses 1 color, then from Observation 16, at most one vertex in each layer is assigned the color 1. There are two possibilities for a layer L_i: either it has no vertex colored 1, or it has exactly one vertex that is colored 1. In the former case, there is a unique coloring for L_i when none of the vertices in L_i are assigned the color 1. In the latter case, we have $|L_i|$ many colorings (of L_i) where each coloring has exactly one vertex with color 1 (and the rest are assigned 0). In total, we have at most $|L_i| + 1$ colorings for each L_i. We call all such colorings *valid*.

The task is to find if there is a sequence of colorings assigned to each layer of G such that we have a CFON* coloring. Notice that the vertices in L_i can possibly have neighbors only in the layers L_{i-1}, L_i, and L_{i+1}. The question of deciding whether the vertices in L_i have a uniquely colored neighbor entirely depends on the colorings assigned to these three layers. We use a dynamic programming based approach to verify the existence of a CFON* coloring for G.

We now construct a layered companion hypergraph $\mathcal{G} = (V', \mathcal{E})$ with vertices in $p + 1$ layers. Each layer T_i of $\mathcal{G}$ corresponds to the layer L_i of G where $i \in [p] \cup \{0\}$. Each vertex in layer T_i of $\mathcal{G}$ corresponds to a valid coloring of vertices in L_i of G. Hence the number of vertices in each layer T_i of $\mathcal{G}$ is equal to $|L_i| + 1$. We now explain how the hyperedges $\mathcal{E}$ of $\mathcal{G}$ are determined.

For $1 \leq i \leq p - 1$, the vertices $x \in T_{i-1}$, $y \in T_i$, $z \in T_{i+1}$ form a hyperedge $\{x, y, z\}$ if the corresponding colorings, when assigned to L_{i-1}, L_i and L_{i+1} respectively, ensures that every vertex in L_i has a uniquely colored neighbor. We also have hyperedges $\{y, z\}$, where $y \in T_0$ and $z \in T_1$ are colorings such that when y and z are assigned to L_0 and L_1 respectively, the vertex in L_0 sees a uniquely colored neighbor. Similarly, we have hyperedges $\{x, y\}$, where $x \in T_{p-1}$ and $y \in T_p$ are colorings such that when x and y are assigned to L_{p-1} and L_p respectively, all the vertices in L_p see a uniquely colored neighbor.

Since the number of valid colorings is $|L_i| + 1$ for the layer L_i, the total number of valid colorings across all layers is at most $2n$. The total number of potential hyperedges to check is at most $O(n^3)$. Once we fix valid colorings x_{i-1}, x_i, x_{i+1} for L_{i-1}, L_i, L_{i+1} respectively, we can check in $O(|L_i| \cdot n) \leq O(n^2)$ time if $\{x_{i-1}, x_i, x_{i+1}\} \in \mathcal{E}$. Hence we need $O(n^5)$ time to construct $\mathcal{G}$.

To obtain a CFON* coloring for G from the hypergraph $\mathcal{G}$, we need to construct a sequence of colorings $x_0 \in T_0$, $x_1 \in T_1, \ldots,$ $x_p \in T_p$ such that $\{x_0, x_1\} \in \mathcal{E}$, $\{x_{i-1}, x_i, x_{i+1}\} \in \mathcal{E}$ for all $1 \leq i \leq p - 1$, and finally $\{x_{p-1}, x_p\} \in \mathcal{E}$. For this, we use Lemma 18, stated and proved below. Note that each $|T_i| = |L_i| + 1 \leq n + 1$, and number of layers is at most n. This gives us that the parameters in Lemma 18, $\alpha \leq n + 1$ and $\beta \leq n$. Hence it takes at most $O(n^4)$ time to decide if G has a CFON* coloring that uses 1 color. The construction of $\mathcal{G}$ takes $O(n^5)$ time and dominates the running time. ☐

Lemma 18. *Suppose there is a layered hypergraph $\mathcal{G} = (V', \mathcal{E})$ with layers $T_0, T_1, T_2, \ldots, T_p$, where $|T_i| \leq \alpha$, for $0 \leq i \leq p$ and $p \leq \beta$. The layers partition the vertex set, i.e., $\cup_{i=0}^{p} T_i = V'$. Suppose further that all the hyperedges in $\mathcal{E}$ are of size 2 or 3 and are of the following form: the hyperedges contain one vertex each from three consecutive layers, or contain one vertex each from T_0 and T_1, or contain one vertex each from T_{p-1} and T_p. We can determine if there exists a sequence $x_0 \in T_0$, $x_1 \in T_1, \ldots,$ $x_p \in T_p$ such that $\{x_0, x_1\} \in \mathcal{E}$, $\{x_{i-1}, x_i, x_{i+1}\} \in \mathcal{E}$ for all $1 \leq i \leq p - 1$, and finally $\{x_{p-1}, x_p\} \in \mathcal{E}$ in $O(\alpha^3 \beta)$ time.*

Proof. For each vertex $x_1 \in T_1$, we store a list of predecessors $x_0 \in T_0$ such that $\{x_0, x_1\} \in \mathcal{E}$. For $1 \leq i \leq p - 1$, we do the following at each vertex $x_i \in T_i$. We look at the list of predecessors stored. If x_{i-1} is a listed predecessor of x_i, then we search for all the hyperedges $\{x_{i-1}, x_i, z\}$, where $z \in T_{i+1}$. If we find such a hyperedge $\{x_{i-1}, x_i, x_{i+1}\} \in \mathcal{E}$, then we store x_i as a predecessor in the list at x_{i+1}. Finally, for each $x_p \in T_p$, we check if there is a listed predecessor $z \in T_{p-1}$ of x_p such that $\{z, x_p\} \in \mathcal{E}$. If there is any such $x_p \in T_p$ for which this holds, then there exists a sequence as desired in the statement of the lemma.

Note that the general step involves going through a list of size at most α at each vertex x_i. For each listed predecessor x_{i-1}, there are potentially at most α hyperedges of the form $\{x_{i-1}, x_i, z\}$ to check, where $z \in T_{i+1}$. We need to do this for all the vertices (at most α of them) of T_i. This gives a time complexity of $O(\alpha^3)$ at the i-th layer. Since there are β layers, the total running time is $O(\alpha^3\beta)$. □

We now proceed to the next result that decides in polynomial time whether $\chi^*_{ON}(G) = 2$.

Theorem 19. *Given an interval graph G, we can decide in time $O(n^{20})$ if $\chi^*_{ON}(G) = 2$.*

Proof. The idea of this proof is similar to the proof of Theorem 17. Let $L_0, L_1, \ldots, L_p$ be the distance layers that form a multi-chain ordering of G. For a layer L_i, we had $|L_i| + 1$ colorings to consider in Theorem 17. Unlike in Theorem 17, we have more colorings to consider since the vertices can get the colors $\{0, 1, 2\}$. We have the following types of colorings in each layer L_i, for $i \geq 1$:

Type 1: All the vertices in L_i are assigned the color 0. There is only one coloring of L_i of this type.

Type 2: Exactly one vertex is assigned the color 1 or 2 while the rest are assigned the color 0. The number of colorings is $2|L_i|$.

Type 3: Both the colors 1 and 2 appear exactly once and the rest are assigned the color 0. The number of colorings is $|L_i|(|L_i| - 1) \leq |L_i|^2$.

Type 4: One of the colors 1 or 2 appears at least twice while the other color appears exactly once. The remaining vertices are assigned the color 0.

Type 5: Exactly one of the colors 1 or 2 appears at least twice and all the other vertices are assigned the color 0.

Type 6: Both the colors 1 or 2 appears at least twice and all the other vertices are assigned the color 0.

Notice that we cannot have a Type 6 coloring for any L_i. Consider layer L_i with $i \geq 1$. Note that by Observation 15, there is a vertex $v \in L_{i-1}$ such that $N(v) \supseteq L_i$. Hence we cannot have a Type 6 coloring in L_i where there are at least two vertices with color 1 and at least two vertices with color 2. This would imply that the $v \in L_{i-1}$ does not have a uniquely colored neighbor. Hence, the layers L_i, for $1 \leq i \leq p$, cannot have a Type 6 coloring. Since L_0 has only one vertex, this case does not arise for L_0 as well.

Notice that the number of colorings of Types 1, 2, 3 are polynomial in $|L_i|$ while the number of colorings of Types 4 and 5 are exponential in $|L_i|$. Hence we cannot consider all the possible colorings exhaustively. We instead consider a polynomial subset of Type 4 and Type 5 colorings which are representatives of all possible Type 4 and Type 5 colorings. We now explain how to obtain these representative colorings.

Let us first consider a Type 4 coloring f of L_i. WLOG, let the coloring have at least two vertices colored 1, and exactly one vertex colored 2. All the remaining vertices are colored 0. We call the lone vertex that is colored 2 as the *special vertex* of L_i with respect to f. Consider the vertices of L_i in a nonincreasing order of their degrees with respect to L_{i-1}. Let this ordering be called σ_i. For example, vertex v appears ahead of u in σ_i if $\deg_{L_{i-1}}(v) > \deg_{L_{i-1}}(u)$. If there are multiple vertices of the same degree, we arbitrarily order these vertices. The first two vertices that are colored 1 as per σ_i are called *left important vertices* of L_i with respect to the coloring f.

Similarly, we define the ordering of the vertices of L_i, in the nonincreasing order of their degrees with respect to L_{i+1}. If there are multiple vertices of the same degree, we arbitrarily order these vertices. Let this ordering be called τ_i. The first two vertices that are colored 1 as per τ_i are called *right important vertices* of L_i with respect to the coloring f.

For a Type 4 coloring with exactly one vertex colored 1, and at least two vertices colored 2, a similar argument to the above applies by swapping colors 1 and 2. That is, the left important and right important vertices will be those colored 2, and the special vertex will be the lone vertex colored 1. We can define left important and right important vertices with respect to Type 5 colorings as well.

Observation 20. *Let $f : V(G) \to \{0,1,2\}$ be a coloring of G which is a Type 4 coloring, when restricted to L_i. Let $x_1^i, x_2^i \in L_i$ be the left important vertices with respect to f such that $f(x_1^i) = f(x_2^i) = 1$.*

Consider the vertices $X = \{x \in L_i \ : \ x$ appears after x_2^i in $\sigma_i, f(x) \in \{0,1\}\}$. Suppose $u, u' \in L_{i-1}$ such that u has a uniquely colored neighbor and u' has no uniquely colored neighbor with respect to f. Let f' be a coloring of G such that $f(v) = f'(v)$ when $v \notin X$, and $f'(v) \in \{0,1\}$ when $v \in X$. Then u will have a uniquely colored neighbor and u' will not have a uniquely colored neighbor with respect to f'.

Proof. Let us consider a vertex $u \in L_{i-1}$ that had a uniquely colored neighbor with respect to f. Suppose the uniquely colored neighbor was w and $f(w) = 2$. Since the set of vertices colored 2 by f' is the same as the set of vertices colored 2 by f, w will continue to be the unique neighbor of u colored 2.

Now suppose $f(w) = 1$. If $w \notin L_i$, then u does not see any vertex in $f^{-1}(1) \cap L_i$. In particular, u is not adjacent to $x_1^i, x_2^i \in L_i$. Since the bipartite graph between L_{i-1} and L_i is a chain graph, and since all the vertices in X appear after x_2^i in σ_i, it follows that u is not adjacent to any vertex in X. Since the only vertices that are colored differently in f and f' are those in X, it follows that w continues to be the uniquely colored neighbor of u in f' as well. If $w \in L_i$, then it follows that $w = x_1^i$, $x_2^i \notin N(u)$ and $N(u) \cap X = \emptyset$. In this case as well, $w = x_1^i$ continues to be the uniquely colored neighbor of u with respect to f'.

Now consider a vertex $u' \in L_{i-1}$ that did not have a uniquely colored neighbor with respect to f. The only ways in which u' may obtain a uniquely colored neighbor in f' is due to the recoloring of a vertex $x \in X \cap N(u')$. However, since $x \in N(u')$, the multi-chain ordering implies that x_1^i, $x_2^i \in N(u')$. Since u' is adjacent to two vertices colored 1, the recoloring of vertices in X using the colors $\{0,1\}$ cannot introduce a uniquely colored neighbor for u' in f'. ☐

Similarly, we have the following observation.

Observation 21. *Let $f : V(G) \to \{0,1,2\}$ be a coloring of G which is a Type 4 coloring, when restricted to L_i. Let $y_1^i, y_2^i \in L_i$ be the right important vertices with respect to f such that $f(y_1^i) = f(y_2^i) = 1$.*

Consider the vertices $X = \{x \in L_i \ : \ x$ appears after y_2^i in $\tau_i, f(x) \in \{0,1\}\}$. Suppose $u, u' \in L_{i+1}$ such that u has a uniquely colored neighbor and u' has no uniquely colored neighbor with respect to f. Let f' be a coloring of G such that $f(v) = f'(v)$ when $v \notin X$, and $f'(v) \in \{0,1\}$ when $v \in X$. Then u will have a uniquely colored neighbor and u' will not have a uniquely colored neighbor with respect to f'.

Note that the Observations 20 and 21 continue to hold in the "color-inverted" setting: i.e., when we have a Type 4 coloring where at least two vertices are colored 2 and exactly one vertex that is colored 1. Analogous observations also hold when f is a Type 5 coloring.

Let f be a coloring of L_i which is of Type 4 or 5, with at least two vertices colored 1. Let x_1^i, x_2^i be the left important vertices and y_1^i, y_2^i be the right important vertices with respect to f. This implies that $x_1^i, x_2^i, y_1^i, y_2^i$ are assigned the color 1, and the vertices that precede x_2^i in σ_i are colored 0 (with the exception of x_1^i, and possibly the special vertex which is colored 2), and vertices that precede y_2^i in τ_i are colored 0 (again with the exception of y_1^i, and possibly the special vertex). It may be the case that $\{x_1^i, x_2^i\} \cap \{y_1^i, y_2^i\} \neq \emptyset$. The main consequence of the above observations is that the the colors of the remaining vertices have no impact on the vertices in L_{i-1} and L_{i+1} having a uniquely colored neighbor.

Given a Type 4 or Type 5 coloring f of L_i, we compute the set of "indifferent" vertices X_i as follows:

$$\begin{aligned} X_i = &\{x \in L_i \ : \ x \text{ appears after } x_2^i \text{ in } \sigma_i, f(x) \in \{0,1\}\} \\ &\cap \ \{x \in L_i \ : \ x \text{ appears after } y_2^i \text{ in } \tau_i, f(x) \in \{0,1\}\}. \end{aligned} \tag{1}$$

The flexibility in coloring these indifferent vertices allow us to only focus on a limited number of Type 4 and Type 5 colorings.

Type 4: One of the colors 1 or 2 appears at least twice while the other color appears exactly once. The remaining vertices are assigned the color 0. Here it is sufficient to just consider only the two left important vertices, the two right important vertices from L_i, and the special vertex in L_i. The number of representative colorings to be considered is upper bounded by $2|L_i|^5$.

Type 5: Exactly one of the colors 1 or 2 appears at least twice and all the other vertices are assigned the color 0. Similar to the above case, it is sufficient to choose two left important vertices and two right important vertices from L_i. The number of representative colorings is upper bounded by $2|L_i|^4$.

Like in the proof of Theorem 17, we now construct a layered companion hypergraph $\mathcal{G} = (V', \mathcal{E})$ with vertices in $p+1$ layers. Each layer T_i of $\mathcal{G}$ corresponds to the layer L_i of G where $i \in [p] \cup \{0\}$. Each vertex in layer T_i of $\mathcal{G}$ corresponds to a Type 1, 2, or 3 coloring of the vertices in L_i of G, or one of the Type 4 or 5 representatives. We thus have the following claim.

Claim 22. *The number of vertices in each layer T_i of $\mathcal{G}$ is at most $1+2|L_i|+|L_i|^2+2|L_i|^5+2|L_i|^4$, which is loosely upper bounded by $8|L_i|^5$.*

We now explain how the hyperedges $\mathcal{E}$ of $\mathcal{G}$ are determined.

Like in Theorem 17, for $1 \leq i \leq p-1$, the vertices $x \in T_{i-1}$, $y \in T_i$, $z \in T_{i+1}$ form a hyperedge $\{x, y, z\}$ if the corresponding colorings, when assigned to L_{i-1}, L_i and L_{i+1} respectively, ensures that every vertex in L_i has a uniquely colored neighbor. We also have hyperedges of size 2 corresponding to the layers L_0 and L_p.

The main task that remains is to determine the hyperedges $\mathcal{E}$. For this, we need to check whether $\{x, y, z\}$ (when assigned respectively to L_{i-1}, L_i, L_{i+1}) ensures a uniquely colored neighbor for every vertex in L_i. This task is somewhat harder than the analogous task in Theorem 17. This is because x, y, z could be representative colorings and provide flexibility in coloring. If y is a coloring of Type 1, 2 or 3, then all the colors of L_i are fixed. The colors of those vertices from L_{i-1} and L_{i+1} that matter to L_i are also fixed by x and z (even if x, z are representatives). If y is of Type 4 or 5, what remains to be determined is if there is a coloring y' of L_i, consistent with the representative coloring y, such that all the vertices of L_i see a uniquely colored neighbor. The point to note is that the layers L_i of G need not be independent sets.

We will explain the case assuming that y is a Type 4 coloring with at least two vertices of L_i colored 1, and exactly one vertex of L_i colored 2. Let X_i be the set of indifferent vertices (as defined in Equation (1)) in L_i. This means that any recoloring of the vertices in X_i using the colors $\{0, 1\}$ do not impact the sets L_{i-1} and L_{i+1}. To begin with, we consider all the vertices in X_i as not assigned any color. Given the colorings $\{x, y, z\}$, we could classify the vertices in L_i as follows:

Class a: Vertices v which have a uniquely colored neighbor and $N(v) \cap X_i = \emptyset$.

Class b: Vertices v which see a uniquely colored neighbor which is colored 2.

Class c: Vertices v which see a uniquely colored neighbor which is colored 1, sees 0 or multiple neighbors colored 2, and $N(v) \cap X_i \neq \emptyset$.

Class d: Vertices v which see no neighbors colored 1, see 0 or multiple neighbors colored 2, and $N(v) \cap X_i \neq \emptyset$.

Class e: Vertices v which see multiple neighbors colored 1, see 0 or multiple neighbors colored 2, and $N(v) \cap X_i \neq \emptyset$.

Class f: Vertices v which do not see a uniquely colored neighbor and $N(v) \cap X_i = \emptyset$.

Vertices that belong to classes a and b, regardless of which way we assign colors from $\{0, 1\}$ to the vertices of X_i, will continue to see a uniquely colored neighbor. Vertices in classes e and f cannot have a uniquely colored neighbor, regardless of how we color X_i. If we have vertices in classes e or f, we can conclude that the triplet $\{x, y, z\} \notin \mathcal{E}$. Now we consider vertices in class c. If a neighbor (from X_i) of a class c vertex is assigned the color 1, then it will cause the class c vertex to see at least two vertices colored 1. Hence we assign color 0 to those neighbors and update X_i as follows:

$$X_i = X_i \setminus \left(\bigcup_{v:v \text{ in class c}} N(v) \right).$$

As a consequence of coloring some vertices 0 and updating X_i, some vertices in class d may now have no neighbors in the updated set X_i. If there are such vertices, we can conclude that the triplet $\{x, y, z\} \notin \mathcal{E}$. If not, we consider the vertices in class d. Let $D_i \subseteq L_i$ denote those vertices of L_i that are in class d.

We may now focus on the graph $H = G[D_i \cup X_i]$. We retain the colors of the vertices in D_i which have already been assigned a color (note that D_i may intersect with X_i). Notice that H is an induced subgraph of an interval graph G and hence H is also an interval graph. The goal is to assign $\{0, 1\}$ colors to the vertices in X_i so that all the vertices in D_i see a unique vertex with the color 1. We can use a procedure similar to that in the proof of Theorem 17 to determine this. The main differences are:

- Instead of checking all the valid colorings, we will only check those valid colorings that are consistent with the colors already assigned. For instance, there may be a vertex in H that is already colored 2.
- While considering a coloring, instead of checking whether all the vertices in H see a uniquely colored neighbor, we only need to check if the vertices in D_i see a uniquely colored neighbor.

The above points imply that we only need to check a subset of possibilities. Hence the running time of this process will be at most $O(|D_i \cup X_i|^5)$ which is upper bounded by $O(n^5)$.

A similar approach works in the color-inverted setting, and when y is a Type 5 coloring. The time taken to check if $\{x, y, z\}$ is a valid triplet of colorings is calculated as follows: For each vertex in L_i, we need $O(n)$ time to check its neighbors in the graph G. This requires $O(n^2)$ time. If y is Type 4 or 5, then we need to run Theorem 17, which takes $O(n^5)$ time.

The number of possible colorings to check for each layer L_i was upper bounded by $8|L_i|^5 \leq O(n^5)$ (see Claim 22). Hence we may have to consider $O(n^{15})$ possible triplets. Since it takes $O(n^5)$ time to check if $\{x, y, z\} \in \mathcal{E}$, the running time for constructing the layered hypergraph $\mathcal{G}$ is at most $O(n^{20})$. Once we have $\mathcal{G}$, we can apply Lemma 18 with $\alpha = O(n^5)$ and $\beta = n$, obtaining a running time of $O(n^{16})$. The construction of $\mathcal{G}$ dominates the running time, and thus we have an $O(n^{20})$ time algorithm to check if there is a CFON* coloring of the entire graph G. □

Using Theorems 17 and 19, we can now infer Theorem 3.

Remark: Recently, the work of Gonzalez and Mann [22] (done simultaneously and independently from ours) on mim-width showed that the CFON* coloring problem is polynomial-time solvable on graph classes for which a branch decomposition of constant mim-width can be computed in polynomial time. This includes the class of interval graphs. We note that our work gives a more explicit algorithm without having to go through the machinery of mim-width. We also note that the mim-width algorithm, as presented in [22], requires a running time in excess of $\Omega(n^{300})$. Hence our algorithm is better in this regard as well.

5 Biconvex Graphs

It is known that there exists a family of bipartite graphs G for which $\chi^*_{ON}(G) = \Theta(\sqrt{n})$, where n is the number of vertices of G. As discussed in Section 1.1, CFON* coloring is NP-hard on bipartite graphs. In this section, we study CFON* coloring on biconvex graphs, which is a subclass of bipartite graphs. We show that CFON* coloring is polynomial time solvable on biconvex graphs.

Definition 23 (Biconvex Graph)**.** *We say that an ordering σ of X in a bipartite graph $B = (X, Y, E)$ satisfies the* consecutive adjacency property *(with respect to Y) if for every vertex $y \in Y$, the neighborhood $N(y)$ is a set of vertices that are consecutive in the ordering σ of X. A bipartite graph (X, Y, E) is* biconvex *if there are orderings of X (with respect to Y) and Y (with respect to X) that fulfill the consecutive adjacency property.*

We first observe the following on chain graphs, previously defined in Definition 11. It is known that a biconvex graph admits multi-chain ordering [9, 13, 15].

Observation 24. *If G is a chain graph, then $\chi^*_{ON}(G) = 1$.*

Proof. Let $G = (A, B)$ be a chain graph. Without loss of generality, we may assume G is connected. If not, we can CFON* color each connected component using one color. Then there exist two vertices $u \in A$ and $v \in B$ such that $N(u) = B$ and $N(v) = A$. This follows by an argument similar to what we saw in Observation 15. We assign the color 1 to u and v, and the remaining vertices are assigned the color 0. It is easy to see that u and v are the uniquely colored neighbors for every vertex in B and A respectively. □

Lemma 25. *If G is a biconvex graph, then $\chi^*_{ON}(G) \leq 2$.*

Proof. Let $L_0, L_1, \ldots, L_p$ be the distance layers that form a multichain ordering of G. Since G is bipartite, each distance layer L_i is an independent set. For each layer L_i, where $0 \le i \le p-1$, let $r_i \in L_i$ be the vertex such that $N(r_i) \supseteq L_{i+1}$. as As the subgraph induced on $L_i \cup L_{i+1}$ forms a chain graph, such a vertex r_i exists. Let $f : V(G) \to \{0,1,2\}$ be a function that assigns colors to $V(G)$. We assign $f(r_0) = 1$, $f(r_1) = 1$. For each $i \ge 2$, we assign $f(r_i) = 2$ if $(f(r_{i-1}), f(r_{i-2})) = (1,1)$ or $(f(r_{i-1}), f(r_{i-2})) = (2,1)$ and $f(r_i) = 1$ otherwise. The remaining uncolored vertices are assigned the color 0.

Each vertex in L_i, where $1 \le i \le q$, has the vertex r_{i-1} as the uniquely colored neighbor, and the vertex $r_0 \in L_0$ has the vertex r_1 as its uniquely colored neighbor. ☐

Proof of Theorem 4. Let G be a biconvex graph. From Lemma 25, we get that $\chi^*_{ON}(G) \le 2$.

We characterize graphs that require one color by using Theorem 17 in time $O(n^5)$. This is possible because the key property used by Theorem 17 is the multi-chain ordering of the interval graph. Biconvex graphs too admit multi-chain ordering, with the added property that the graph induced on each distance layer L_i is an independent set. This possibly simplifies the algorithm. We omit the details for brevity. ☐

6 Conclusion

In this paper, we study CFCN* coloring on chordal graphs and show that it is NP-complete. We show that CFON* coloring is polynomial time solvable on interval graphs and biconvex graphs by using the multi-chain ordering property of these graphs. We believe that this property may be useful in designing polynomial time algorithms for other problems on these graph classes. We also believe that a similar adaptation of the results to the full coloring variant of the problem (that requires each vertex to be assigned a color) is polynomial time solvable on these graph classes. One obvious research direction is to improve the running time of the algorithm for interval graphs, given in Theorem 3. It may be of interest to study the CFON* problem on other subclasses of bipartite graphs, such as convex bipartite graphs, chordal bipartite graphs and tree-convex bipartite graphs. It may also be interesting to settle the complexity of the problems on AT-free graphs and permutation graphs.

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