Conflict-Free Coloring: Graphs of Bounded Clique-Width and Intersection Graphs*

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Abstract. A conflict-free coloring of a graph G is a (partial) coloring of its vertices such that every vertex u has a neighbor whose assigned color is unique in the neighborhood of u. There are two variants of this coloring, one defined using the open neighborhood and one using the closed neighborhood. For both variants, we study the problem of deciding whether the conflict-free coloring of a given graph G is at most a given number k.

In this work, we investigate the relation of clique-width and minimum number of colors needed (for both variants) and show that these parameters do not bound one another. Moreover, we consider specific graph classes, particularly graphs of bounded clique-width and types of intersection graphs, such as distance hereditary graphs, interval graphs and unit square and disk graphs. We also consider Kneser graphs and split graphs. We give (often tight) upper and lower bounds and determine the complexity of the decision problem on these graph classes, which improve some of the results from the literature. Particularly, we settle the number of colors needed for an interval graph to be conflict-free colored under the open neighborhood model, which was posed as an open problem.

1 Introduction

Graph coloring is one of the most fundamental problems in graph theory. A *proper coloring* of a given undirected graph G is an assignment of colors to the vertices of G such that no two adjacent vertices have the same color. The minimum number of colors for which a proper coloring of G exists is called the *chromatic number* of G. There have been extensive studies on this parameter, both algorithmically (e.g., determining or approximating the chromatic number)

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and structurally (e.g., worst-case bounds on the chromatic number of a given graph class, notably planar graphs); see [25] for example for an overview.

Besides the classical coloring above, there have been many variants. One such variant is introduced in 2002 by Even, Lotker, Ron and Smorodinsky [14], motivated by the frequency assignment problem in cellular networks, where base stations and clients communicate with one another. To avoid interference, it is required that for each client, among the base stations that it connects to, there exists one with a unique frequency. This is formalized as a *conflict-free coloring*. In the below definition, the open neighborhood of a vertex is the set of its adjacent vertices.

Definition 1 (Conflict-Free Coloring). *A* partial conflict-free open-neighborhood coloring *(CFON* coloring) of a graph G, $G = (V, E)$, using k colors is an assignment $C : V \to \{0\} \cup \{1, 2, \ldots, k\}$ such that for every vertex $v \in V$, there is a vertex with a unique non-zero color in the open neighborhood of v.*

The smallest k for which there is a CFON coloring of G is called the* CFON* chromatic number *of G, denoted by $\chi^*_{ON}(G)$. Given a graph G and a natural number k, the* CFON* coloring problem *asks whether $\chi^*_{ON}(G)$ is at most k.*

Similarly, we can define a coloring variant for closed neighborhoods, where a closed neighborhood of a vertex contains the vertex itself and its adjacent vertices. We call this a *partial conflict-free closed-neighborhood coloring* or *CFCN* coloring*. The CFCN* chromatic number $\chi^*_{CN}(G)$ and the CFCN* coloring problem are defined analogously. Collectively, we refer to the chromatic numbers of these two variants as conflict-free chromatic numbers. The CFON* and CFCN* colorings are referred to as "partial colorings" because the vertices colored 0 are treated as "uncolored".

Conflict-free coloring has been well studied for nearly 20 years (e.g., see the survey by Smorodinsky [34]) and also found applications in the area of sensor networks [17,31] and coding theory [27]. Similar to the classic setting, these works explored various combinatorial and algorithmic questions on conflict-free coloring. What are worst-case bounds on any of the conflict-free chromatic numbers? What is the computational complexity of the conflict-free coloring problems? For what kind of graph classes can we get better bounds and complexity for the questions above? While many papers have addressed these questions, there are still many gaps, which we bridge in this paper.

1.1 Results and Discussion

This paper is the extended version of the preliminary version [5] published in the proceedings of the IWOCA 2021 conference. In this paper, we obtain many new results and provide the full proofs that were omitted in the preliminary version. In the following, we briefly present the results of the current paper and highlight the changes from the preliminary paper. A summary of the results for CFON* and CFCN* colorings are also presented in Tables 1 and 2, respectively.

- In Section 3, we discuss the conflict-free chromatic numbers in relation with the parameter clique-width. In the preliminary version [5], we presented fixed parameter tractable (FPT) algorithms for all the conflict-free coloring problems with respect to the number of colors and clique-width, with full correctness proof. If the CFON* and CFCN* chromatic numbers are bounded by a function of clique-width, the result above will translate to an FPT algorithm with respect to only the clique-width. As a new result, we show that the conflict-free chromatic numbers cannot be bounded by a function of clique-width. Towards this end, we show the existence of graphs with clique-width three and conflict-free chromatic numbers $\Omega(\log n)$. The existence of an FPT algorithm with respect to the clique-width remains open.

– In Section 4, we discuss certain graphs with bounded clique-width. In particular, for distance-hereditary graphs, we show that the CFCN* chromatic number is at most three. Consequently, we can obtain a polynomial time algorithm for this graph class, by applying any FPT or XP algorithm with respect to the clique-width and the number of colors [4, 5, 21]. The CFON* chromatic number for this graph class, however, is unbounded. Still, we show that it is bounded for two subclasses, cographs and block graphs, and hence the CFON* coloring problem is polynomial time solvable on them. The results related to these two subclasses have been announced in the preliminary version [5], and here we provide the full proof. The results for distance-hereditary graphs are new.
– In Section 5, we show that for an interval graph G, $\chi^*_{ON}(G) \leq 3$ and that this bound is tight. This result answers an open question posed in [33]. Moreover, we show that two colors are sufficient to CFON* color proper interval graphs. All these results were announced in the preliminary version [5], but the full proof was only provided for the upper bound on proper interval graphs.
– In Section 6, we provide the full proof for the upper bounds of the CFON* chromatic numbers of 27 and 54 for unit square and unit disk intersection graphs, respectively.
  Further, in Section 7, we show a new NP-completeness result for the CFON* coloring problem on unit square and unit disk intersection graphs. These results complement the corresponding bounds and complexity for the closed-neighborhood variant studied previously by Fekete and Keldenich [15].
– In the last two sections, we provide the full proofs related to Kneser graphs and split graphs, as announced in [5]. In particular, in Section 8, for the Kneser graph $K(n, \kappa)$, $\kappa + 1$ colors are sufficient when $n \geq 2\kappa + 1$ and are also necessary when $n \geq 2\kappa^2 + \kappa$. For CFCN* coloring of $K(n, \kappa)$, we show an upper bound of κ colors for $n \geq 2\kappa + 1$.
  In Section 9, we prove that for split graphs, the CFON* coloring problem is NP-complete and that the CFCN* coloring problem is polynomial time solvable.

1.2 Related works

Here, we briefly review a few results related to the materials of this paper. We will elaborate the relevant results further at each subsequent section. For a more

Graph Class	Upper Bound	Lower Bound	Complexity
Distance hereditary graphs	-	$\Omega(\log n)$ (Cor. 15)	-
Block graphs	3	3 (Fig. 4)	P
Cographs	2	2 (K_3)	P
Interval graphs	3	3 (Fig. 5)	P [6,21]
Proper interval graphs	2	2 (K_3)	P [6,21]
Unit square graphs	27	3 (Fig. 6)	NP-hard
Unit disk graphs	54	3 (Fig. 6)	NP-hard
Kneser graphs $K(n, \kappa)$	$\kappa + 1$	$\kappa + 1$ (Lem. 37)	-
Split graphs	-	-	NP-hard

Table 1. Bounds and algorithmic status on various graph classes for the CFON* coloring problem. The results that were previously known are indicated by providing citations to the papers. The absence of a citation indicates that the result is shown in this paper. Here a "Lower Bound" of ℓ indicates the existence of a graph G such that $\chi^*_{ON}(G) = \ell$. Such graphs are indicated in parenthesis. If the bounds or the algorithmic status (whether P or NP-hard) for a graph class is unknown, we indicate it by "-".

Graph Class	Upper Bound	Complexity
Distance Hereditary Graphs	3	P
Block graphs	2	P
Cographs	2	P
Interval graphs	2 [15]	P [5,21]
Proper interval graphs	2 [15]	P [5,21]
Unit square graphs	4 [15]	NP-hard [15]
Unit disk graphs	6 [15]	NP-hard [15]
Kneser graphs $K(n, \kappa)$	κ	-
Split graphs	2	P

Table 2. Bounds and algorithmic status on various graph classes for the CFCN* coloring problem. The results that were previously known are indicated by providing citations to the papers. The absence of a citation indicates that the result is shown in this paper. If the algorithmic status (whether P or NP-hard) for a graph class is unknown, we indicate it by "-".

general overview on conflict-free coloring, we refer the reader to the survey by Smorodinsky [34].

Many papers in the literature of conflict-free coloring considered the variants where the partial coloring is a full coloring. We remove the asterisks to denote these cases. That is, we denote by CFON coloring the *full conflict-free open-neighborhood coloring*, and denote by CFCN coloring for the closed neighborhood variant. The corresponding chromatic numbers and decision problems are defined analogously.

In terms of asymptotic worst-case bounds, Pach and Tárdos [32] showed a bound of $O(\log n)$ for a general n-vertex graph for any of the four variants of conflict-free coloring. Glebov, Szabó, and Tardos [19] proved that this bound is tight, using a randomized construction. This bound can be improved for special graph classes, such as random graphs $G(n, p)$ for $p \in \omega(1/n)$ [19] and graphs with bounded degrees [22, 32].

From an algorithmic perspective, Gargano and Rescigno [18] showed that the CFON and CFCN coloring problems are NP-complete, and the corresponding chromatic numbers are hard to approximate within a factor less than 3/2. Abel et al. [1] later showed the NP-completeness for the CFON* and CFCN* coloring problems, where the former problem is NP-complete even for planar bipartite graphs. Given these complexity results, two natural approaches for further investigation are to study the parameterized complexity of these problems or to restrict the classes of graphs for which the problems may be efficiently solved.

The parameterized complexity of conflict-free coloring problems has captured the interest of the research community recently. The CFON and CFCN coloring problems are FPT[6] when parameterized by tree-width [2, 7], distance to cluster (distance to disjoint union of cliques) [33], neighborhood diversity [18].

Our preliminary paper [5] and a recent paper by Bergougnoux, Dreier, and Jaffke [4] showed that the four variants of conflict-free coloring problems are FPT with respect to the combined parameters clique-width and the number of colors. Clique-width is more general than the parameters mentioned in the preceding paragraph. In other words, if these parameters are bounded, then the clique-width is also bounded. More recently, Gonzalez and Mann [21] showed that the problems are polynomial time solvable when the mim-width and the number of colors are constant. Although this parameter mim-width is more general than clique-width, the algorithms by Gonzalez and Mann are not FPT algorithms.

In the direction of restricted classes of graphs, many geometric graph classes have been given attention specially, due to the original motivation from the frequency assignment problem. The original paper [14] considered graphs induced by coverage of point sets on the plane by convex regions. Abel et al. [1] presented many combinatorial and algorithmic results for the planar graphs. The latest paper for these graphs by Huang, Guo, and Yuan [24] gives a tight bound of 4 for the CFON* chromatic number. Intersection graphs are also natural classes of geometric graphs to consider. Fekete and Keldenich [15] studied CFCN* coloring on common intersection graphs such as interval graphs, unit disk graphs and unit square graphs. (See the references therein for further related works on intersection graphs.) This paper poses an open question on the existence of a polynomial time algorithm for the CFON* problem on interval graphs. This was recently proved affirmatively, independently by Bhyravarapu, Kalyanasundaram and Mathew [6], and Gonzalez and Mann [21]. Beside these intersection graphs, several others have been considered, such as string graphs and circle graphs [26].

[6] For the formal definition of FPT and more details on parameterized complexity, we refer the reader to [10, 13].

2 Preliminaries

Throughout the paper, we assume that the input graph $G = (V, E)$ is connected. We also assume that G does not contain any isolated vertices as the CFON* problem is not defined for an isolated vertex. All the results of this paper hold for disconnected graphs without isolated vertices by the application of the respective theorems on each connected component. We use $[k]$ to denote the set $\{1, 2, \ldots, k\}$ and $C : V \to \{0\} \cup [k]$ to denote the coloring function. A *universal vertex* is a vertex that is adjacent to all other vertices of the graph. In some of our algorithms and proofs, it is convenient to distinguish between vertices that are intentionally left uncolored, and the vertices that are yet to be assigned any color. The assignment of color 0 is used to denote that a vertex is left intentionally uncolored by the coloring function.

To simplify the notation and for ease of readability, we use the shorthand notation vw to denote the edge $\{v, w\}$. The open neighborhood of a vertex $v \in V$ is the set of vertices $\{w : vw \in E\}$ and is denoted by $N(v)$. Given a conflict-free coloring C, a vertex $w \in N(v)$ is called a *uniquely colored neighbor* of v if $C(w) \neq 0$ and $\forall x \in N(v) \setminus \{w\}, C(w) \neq C(x)$. The closed neighborhood of v is the set $N(v) \cup \{v\}$, denoted by $N[v]$. The notion of a uniquely colored neighbor in the closed neighborhood variant is analogous to the open neighborhood variant, and is obtained by replacing $N(v)$ by $N[v]$. Given a vertex set $V' \subseteq V$, we define $C(V')$ as follows: $C(V') = \bigcup_{v \in V'} \{C(v)\}$.

In many of the sections, in order to establish bounds on the CFON* and CFCN* chromatic numbers, we use the full coloring conflict-free coloring variants, defined as follows.

Definition 2 (Conflict-Free Coloring – Full Coloring Variant). *A* CFON coloring *of a graph G, $G = (V, E)$, using k colors is an assignment $C : V \to \{1, 2, \ldots, k\}$ such that for every $v \in V$, there exists an $i \in \{1, 2, \ldots, k\}$ such that $|N(v) \cap C^{-1}(i)| = 1$. The smallest number of colors required for a CFON coloring of G is called the CFON chromatic number of G, denoted by $\chi_{ON}(G)$.*

The corresponding closed neighborhood variant is denoted by CFCN coloring, and the CFCN chromatic number is denoted by $\chi_{CN}(G)$.

Remark 3. A full conflict-free coloring, where all the vertices are colored with a non-zero color, is also a partial conflict-free coloring (as defined in Definition 1), while the converse is not necessarily true. One extra color suffices to obtain a full conflict-free coloring from a partial conflict-free coloring. However, it is not always clear if the extra color is actually necessary.

Recall the proper coloring defined in Section 1. For a graph G, denote by $\chi(G)$ the chromatic number of G. Observe that such a proper coloring gives a CFCN coloring, but in general, the CFCN chromatic number is much lower than the chromatic number. For example, $\chi(K_n) = n$ but $\chi_{CN}(K_n) = 2$ where K_n is a clique on n vertices.

3 Clique-width

In this section, we study conflict-free coloring on graphs of bounded clique-width.

Definition 4 (Clique-width [9]). *Let $w \in \mathbb{N}$. A w-expression Φ defines a graph G_Φ where each vertex receives a label from $[w]$. The graph consisting of a solitary vertex v with label i has the w-expression $v(i)$. Graphs that contain two or more vertices are defined inductively using the three operations described below. Let $G_{\Phi'}$ and $G_{\Phi''}$ be graphs given by the w-expressions Φ' and Φ'' respectively.*

1. *Disjoint union: The graph G_Φ which is the disjoint union of $G_{\Phi'}$ and $G_{\Phi''}$ is given by the w-expression $\Phi = \Phi' \oplus \Phi''$.*
2. *Relabel: Let the graph G_Φ be $G_{\Phi'}$ where each vertex labeled i in $G_{\Phi'}$ is relabeled with the label j. The graph G_Φ is given by the w-expression $\Phi = \rho_{i\to j}(\Phi')$.*
3. *Join: Let the graph G_Φ obtained from $G_{\Phi'}$ by adding edges between all the vertex pairs (u, v), where u has label i and v has label j. The graph G_Φ is given by the w-expression $\Phi = \eta_{i,j}(\Phi')$.*

The clique-width *of a graph G denoted by* cw(G) *is the minimum number w such that there is a w-expression Φ that defines G.*

Given a graph $G = (V, E)$ and its w-expression it was shown in [5] that all the four variants of the conflict-free coloring problem (CFON, CFCN, CFON* and CFCN*) can be solved in time $2^{O(w3^k)}n^{O(1)}$, where w is the clique-width of the graph, k is the number of colors, and n is the number of vertices of G. Recently, Bergougnoux, Dreier and Jaffke in [4] introduced a logic called *distance neighborhood* (DN) logic which extends existential MSO_1. It was shown in the same paper that the CFON and CFCN coloring problems can be expressed in DN logic. Using similar ideas, the CFON* and CFCN* coloring problems can also be expressed in DN logic. By applying Theorem 1.1 in [4], we obtain an algorithm that runs in time $2^{O(wk^2)}n^{O(1)}$. Thus, all the variants of conflict-free coloring are fixed-parameter tractable when parameterized by the clique-width of the graph and the number of colors. As a consequence, we have the following.

Theorem 5. *Given a w-expression of a graph G, all the four variants of the conflict-free coloring problem (CFON, CFCN, CFON* and CFCN*) can be solved in time $2^{O(wk^2)}n^{O(1)}$ where k is the number of colors and n is the number of vertices of G.*

3.1 Graphs of bounded clique-width and unbounded χ_{CN} and χ_{ON}

Since the CFCN and CFON coloring problems are FPT when parameterized by clique-width and the number of colors, an open question is then whether there exists an FPT algorithm with respect to only the clique-width. One solution to this question would be to bound the CFON and CFCN chromatic numbers by a function of the clique-width. However, this turns out to be impossible, even for graphs of clique-width three. We construct graphs $G_2, G_3, \ldots, G_k$ of clique-width at most 3 such that a conflict-free coloring of G_i requires at least i colors. Interestingly, graphs of clique-width at most 2, i.e., cographs (see [9] for a reference), have bounded CFON and CFCN chromatic numbers, as shown in Theorems 13 and 16 in the next section. In the following, we consider the full coloring variant. Let us first consider CFCN colorings.

Theorem 6. *For any given integer $k \geq 2$, there exists a graph G_k of clique-width at most 3 with $\chi_{CN}(G_k) \geq k$.*

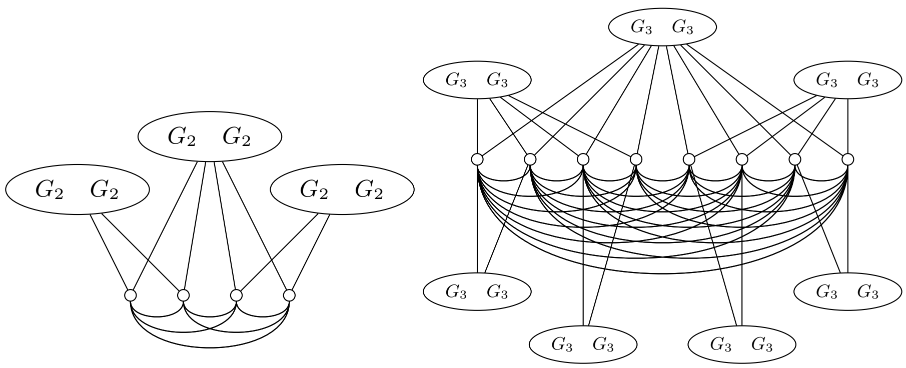

Fig. 1. G_3 (left) and G_4 (right) have clique-width 3 but cannot be CFCN colored with 2 and 3 colors, respectively. Each G_i,$i \geq 2$ stands for a copy of the graph G_i. Every vertex in an ellipse is adjacent to every vertex that is connected to that ellipse.

Proof. We construct graphs G_i, $i \geq 2$ inductively. The graph G_{k+1} is such that it cannot be CFCN colored with k colors. Thus at least $k+1$ colors are required.

- Let G_2 be the graph isomorphic to K_2.
- The graph G_{k+1}, for $k \geq 2$, is constructed as follows. It consists of 2^k *bottom vertices* $B = \{b_0, \ldots, b_{2^k-1}\}$ and $2(2^k - 1)$ copies of G_k. The vertices of B form a clique. To describe the edges between the vertices in the copies of G_k and those in B, it will be simpler to consider an imaginary binary tree T. Let T be the full binary tree with k levels and with leaves B. That is, T consists of $k+1$ levels $L_0, \ldots, L_k$, where level L_i contains 2^{k-i} vertices $b^i_0, \ldots, b^i_{2^{k-i}-1}$ for $0 \leq i \leq k$. Each vertex b^i_j has children b^{i-1}_{2j} and b^{i-1}_{2j+1} for $1 \leq i \leq k$ and $0 \leq j < 2^{k-i}$. Then we identify the bottom vertices B with the leaves L_0, which is $b^0_j = b_j$ for $0 \leq j < 2^k$. For a non-leaf x of T, let $B(x) \subseteq B$ be the set of descendants of x among the leaves B. Let $\mathcal{B} = \{B(x) \mid x \in V(T) \setminus L_0\}$ be the family of such sets. For every set $S \in \mathcal{B}$, introduce two disjoint copies of G_k and make them adjacent to S, i.e., all the vertices in the two copies of G_k are adjacent to all the vertices in S. See Fig. 1 for illustrations of G_3 and G_4.

Inductively we show that G_k has clique-width at most 3. That is, there is a 3-expression Φ_k where G_{Φ_k} equals G when ignoring the labels. We will use the labels $\{\alpha, \beta, \gamma\}$ instead of numbers, since numbers are already used for colors.

- Graph G_2, a single edge, can be constructed using 2 labels.
- Consider the graph G_{k+1}. By the induction hypothesis, there is a 3-expression Φ_k that describes G_k. We may assume that every vertex of G_{Φ_k} has label β since we can apply relabelling operations at the end. Let vertex sets B and T with levels $L_0, \dots, L_k$ be as in the construction of G_{k+1}. We show the following properties for every node $x \in L_i$ of T by induction on the level $i = 0, \dots, k$:
  (*) There is a 3-expression $\Phi_{k+1,x}$ where $G_{\Phi_{k+1,x}}$ equals the induced subgraph of G_{k+1} that contains $B(x)$ and the copies of G_k whose neighborhoods are subsets of $B(x)$; and
  (**) $B(x)$ has label α and the copies of G_k have label γ.

  Then $\Phi_{k+1,r}$, where r is the root of T, is the desired 3-expression.
  - For the induction basis, let $i = 0$ and $x \in L_0$. Hence x is a leaf b_i: Simply introduce the single vertex of label α.
  - For the induction step, let $i \geq 1$ and $x \in L_i$. The vertex x has two children, say y and z, in level L_{i-1}. Thus by induction hypothesis there are 3-expressions $\Phi_{k+1,y}$ and $\Phi_{k+1,z}$ with the properties (*) and (**) described above.

    We construct $\Phi_{k+1,x}$: We first need to add all the edges between $B(y)$ and $B(z)$ by combining the respective 3-expressions. Towards this end, we do (i) a relabel operation $\rho_{\alpha \to \beta}(\Phi_{k+1,y})$, (ii) disjoint union of $\rho_{\alpha \to \beta}(\Phi_{k+1,y})$ and $\Phi_{k+1,z}$, (iii) a join operation $\eta_{\alpha,\beta}$ on the graph obtained, and (iv) relabel all the vertices assigned the label β to α.

    We now need to introduce two copies of G_k and add edges between the introduced copies of G_k and the vertices of $B(x) = B(y) \cup B(z)$. Towards this end, we (i) inductively construct two copies of G_k, (ii) relabel the vertices of these copies of G_k to β, (iii) take a disjoint union of these copies of G_k and the graph constructed above on $B(x)$, and (iv) use a join operation $\eta_{\alpha,\beta}$ on the resulting graph. Finally, we relabel all the vertices assigned the label β to γ to maintain the property (**).

Lastly, we show by induction that G_{k+1} has no CFCN coloring with only k colors, for every $k \geq 1$. For the induction basis, consider G_2, a single edge. There, a 1-coloring is not possible.

For the induction step, $G_k \rightsquigarrow G_{k+1}$, suppose for a contradiction that there is a CFCN coloring $c : V(G_{k+1}) \to \{1, \dots, k\}$.

We first show that each set $S \in \mathcal{B}$ contains a uniquely colored vertex $f(S)$. To be precise, the mapping $f : \mathcal{B} \to B$ such that for each set $S \in \mathcal{B}$ there is a vertex $f(S) = v \in S$ and $c(v) \neq c(v')$ for every other vertex $v' \in S \setminus \{v\}$.

Recall that G_{k+1} contains two copies C_1, C_2 of G_k where each $C_i, i \in \{1, 2\}$ has $N[C_i] \setminus C_i = S$. Now suppose for a contradiction that S contains no uniquely colored vertex. Let c_i be the coloring c restricted to vertices $V(C_i)$, for $i \in \{1, 2\}$. Then c_i is a CFCN coloring of graph C_i, for $i \in \{1, 2\}$. Indeed by induction hypothesis, the restricted coloring c_i is surjective. Hence in $V(C_1) \cup V(C_2)$ each of the k colors occurs twice. Then every vertex in $u \in S$ has every color from $\{1, \ldots, k\}$ at least twice in its neighborhood. This contradicts the claim that u has a uniquely colored neighbor. Therefore, each set $S \in \mathcal{B}$ contains a uniquely colored vertex $f(S)$.

Now, without loss of generality we may assume that the uniquely colored element of the set B is $f(B) = b_{2^k-1}$ and that b_{2^k-1} is colored with color k. Then the subset $\{b_0, \ldots, b_{2^{k-1}-1}\} \in \mathcal{B}$ consists only of vertices of color $1, \ldots, k-1$. Again without loss of generality, we may assume that $f(\{b_0, \ldots, b_{2^{k-1}-1}\}) = b_{2^{k-1}-1}$ and that the vertex $b_{2^{k-1}-1}$ is colored with $k-1$. By repeating this argument, we eventually obtain that b_0 and b_1 must take the color 1. This contradicts the claim that $\{b_0, b_1\} \in \mathcal{B}$ has a uniquely colored element. Therefore, G_{k+1} cannot be colored with just k colors. □

To show that the CFON coloring number is also unbounded for graphs with clique-width three, we use an analogous approach. We define a sequence of graphs $G'_2, G'_3, \ldots$, such that each graph G'_{k+1} for $k \geq 2$ has clique-width at most three and cannot be CFON colored with k colors. Let G'_2 be a copy of K_3, which cannot be CFON colored with one color and which has clique-width 2. We use the same inductive process to construct G'_{k+1} from the copies of G'_k. Inductively it follows that G'_{k+1} has clique-width at most 3. Again, by the same induction step as before, it follows that G'_{k+1} cannot be CFON colored with k colors. We also provide an alternative construction in Lemma 14.

Theorem 7. *For any given integer $k \geq 2$, there exists a graph G_k of clique-width at most 3 with $\chi_{ON}(G_k) \geq k$.*

4 Graph classes of bounded clique-width

One consequence of Theorem 5 is that if both the clique-width and the CFON* (or CFCN*) chromatic number of the input graph is bounded, then there exists a polynomial time algorithm to solve the CFON* (or CFCN*, respectively) coloring problem. Theorems 6 and 7 show that even when the clique-width is at most 3, the CFON* and CFCN* chromatic numbers can be unbounded. Hence, this section explores some graph classes with clique-width at most 3, where the CFON* or CFCN* chromatic number is bounded.

Firstly, we consider the graphs with clique-width at most 2, which are exactly the cographs [9].

Definition 8 (Cograph [8]). *A graph G is a* cograph *if it can be constructed recursively by the following rules. An isolated vertex is a cograph, the disjoint union of two cographs is a cograph and the complement of a cograph is a cograph.*

We will show that cographs have CFCN* and CFON* chromatic numbers at most 2 (Lemmas 13 and 16).

These graphs are a special case of distance hereditary graphs, whose clique-width is at most 3 [20].

Definition 9 (Distance hereditary graph [23]). *A graph G is* distance hereditary *if for every connected induced subgraph H of G, the distance (i.e., the length of a shortest path) between any pair of vertices in H is the same as that in G.*

Bandelt and Mulder [3] gave the following alternative definition of connected distance hereditary graphs. For a given ordering of the vertices $(v_1, v_2, \ldots, v_n)$ of $V(G)$, let $G[i]$ be the induced subgraph of G on $\{v_1, \ldots, v_i\}$. The sequence $(v_1, v_2, \ldots, v_n)$ is a *one-vertex extension sequence* if $G[2] = K_2$, and for every $i \geq 3$, $G[i]$ can be formed by adding v_i to $G[i-1]$ and edges incident to v_i such that for some $j < i$, one of the following holds:

- v_i is adjacent to v_j and no other vertex (we say v_i is a *pendant* of v_j);
- v_i is adjacent to all the neighbors of v_j (we say v_i is a *false twin* of v_j); or
- v_i is adjacent to v_j and all the neighbors of v_j (we say v_i is a *true twin* of v_j).

Then a connected graph is distance hereditary if and only if there exists a one-vertex extension sequence $(v_1, v_2, \ldots, v_n)$.

Note that if the pendant operation is absent, then we obtain exactly the cographs. In other words, cographs are exactly the distance hereditary graphs that can be constructed from a single vertex by the true twin and false twin operations [3]. If the true twin operation is absent, then we obtain bipartite distance hereditary graphs. Lastly, if the false twin operation is missing, we obtain a graph class that contains block graphs as a subclass [29].

Definition 10 (Block Graph [12]). *A* block graph *is a graph in which every 2-connected component (i.e., a maximal subgraph which cannot be disconnected by the deletion of one vertex) is a clique.*

4.1 CFCN* chromatic number

We first show an upper bound for the CFCN* chromatic number of distance hereditary graphs.

Lemma 11. *If G is a distance hereditary graph, then $\chi^*_{CN}(G) \leq 3$.*

Proof. Suppose $(v_1, v_2, \ldots, v_n)$ is a one-vertex extension sequence of G. We will give an iterative algorithm to provide a CFCN* coloring with colors 0, 1, 2, 3.

We use $N_i(v)$ and $N_i[v]$ to refer to the open and closed neighborhoods of a vertex v in the graph $G[i]$, respectively, where $i \in [n]$ is the current iteration of the algorithm.

For each vertex v, we specify a tuple $\mathcal{C}(v) = (a, b)$ with $a \in \{0, 1, 2, 3\}$ as the color of v, and $b \in \{1, 2, 3\}$ as the color of the uniquely colored neighbor of v. We maintain the following two invariants for each iteration $i \in [n]$ of the coloring algorithm:

- **Invariant 1**: For every vertex $v \in G[i]$, with $\mathcal{C}(v) = (a, b)$, possibly $a = b$, there is a uniquely colored neighbor of v in $G[i]$ of color b.

– **Invariant 2**: For every vertex $v \in G[i]$, if $\mathcal{C}(v) = (a, a)$, then condition (*) or condition (**) is true.
  - Condition (*): There is a color $d \in \{1, 2, 3\} \setminus \{a\}$, such that every vertex $w \in N_i(v)$ has $\mathcal{C}(w) = (0, d)$.
  - Condition (**): There is a color $y \in \{1, 2, 3\} \setminus \{a\}$ that appears exactly once in $N_i(v)$.

We are now ready to describe the coloring scheme. Recall that v_1 and v_2 are adjacent to one another. We assign $\mathcal{C}(v_1) = (1, 2)$ and $\mathcal{C}(v_2) = (2, 1)$. Clearly, the Invariants 1 and 2 hold. For $i \geq 3$, consider j such that v_i is either a pendant, false or true twin of v_j. Let $\mathcal{C}(v_j) = (a, b)$ for some $a \in \{0, 1, 2, 3\}$ and $b \in \{1, 2, 3\}$. We distinguish the following cases:

– **Case 1a:** v_i **is a pendant of** v_j **and** $a = b$**.**
  - **Case 1a′:** $\mathcal{C}(w) = (0, d)$ **for all** $w \in N_{i-1}(v_j)$ **and** $d \neq a$**.** That is, condition (*) holds for v_j in $G[i-1]$.
    We assign $\mathcal{C}(v_i) = (x, a)$, where x is the color in $\{1, 2, 3\} \setminus \{a, d\}$.
  - **Case 1a′′: otherwise.** That is, condition (**) holds for v_j in $G[i-1]$.
    We assign $\mathcal{C}(v_i) = (0, a)$.

– **Case 1b:** v_i **is a pendant of** v_j **and** $a \neq b$**.**
  - **Case 1b′:** $a \neq 0$**.**
    We assign $\mathcal{C}(v_i) = (0, a)$.
  - **Case 1b′′:** $a = 0$**.**
    We assign $\mathcal{C}(v_i) = (x, x)$, for an arbitrary color x in $\{1, 2, 3\} \setminus \{b\}$.

– **Case 2a:** v_i **is a true twin of** v_j **and** $a = b$**.**
  - **Case 2a′:** $\mathcal{C}(w) = (0, d)$ **for all** w **in** $N_{i-1}(v_j)$ **and** $d \neq a$**.** That is, condition (*) holds for v_j in $G[i-1]$.
    We assign $\mathcal{C}(v_i) = (x, a)$, where x is the color in $\{1, 2, 3\} \setminus \{a, d\}$.
  - **Case 2a′′: otherwise.** That is, condition (**) holds for v_j in $G[i-1]$.
    We assign $\mathcal{C}(v_i) = (0, a)$.

– **Case 2b:** v_i **is a true twin of** v_j **and** $a \neq b$**.**
  We assign $\mathcal{C}(v_i) = (0, b)$.

– **Case 3a:** v_i **is a false twin of** v_j **and** $a = b$**.**
  - **Case 3a′:** $\mathcal{C}(w) = (0, d)$ **for all** w **in** $N_{i-1}(v_j)$ **and** $d \neq a$**.** That is, condition (*) holds for v_j in $G[i-1]$.
    We assign $\mathcal{C}(v_i) = (a, a)$.
  - **Case 3a′′: otherwise.** That is, condition (**) holds for v_j in $G[i-1]$. That is, there is a vertex $w \in N_{i-1}(v_j)$ with a unique color $y \in \{1, 2, 3\} \setminus \{a\}$ among the vertices in $N_{i-1}(v_j)$. We assign $\mathcal{C}(v_i) = (0, y)$.

– **Case 3b:** v_i **is a false twin of** v_j**, and** $a \neq b$**.**
  We assign $\mathcal{C}(v_i) = (0, b)$.

We prove the invariants by induction. Invariant 1 for iteration $i = n$ implies that the coloring above is a CFCN* coloring for $G = G[n]$.

These invariants are trivially true for the base case of $i = 2$. For the inductive step, observe that for any vertex $u \notin N_i[v_i]$, there is no change in the closed neighborhood of u, and hence the invariants hold for u by the inductive hypothesis. For v_j and $N_i[v_i]$ we show that Invariants 1 and 2 are satisfied in $G[i]$.

Case 1a$'$. Vertex v_i: Invariant 1 holds, since v_j with color a is the uniquely colored neighbor of v_i. Invariant 2 is vacuously true.

Vertex v_j: Since v_i has a different color than a, the uniquely colored neighbor of v_j remains unchanged (in fact, it is v_j itself), i.e., Invariant 1 holds. Invariant 2 holds by condition (**), because v_i becomes the only vertex in $N_i(v_j)$ with color x assigned to it.

Case 1a$''$. Vertex v_i: Invariant 1 is satisfied since v_i has v_j with color a as neighbor. Invariant 2 is vacuously true.

Vertex v_j: Invariant 1 is satisfied, since v_j still has itself with color a as its uniquely colored neighbor. By the case assumption, there exists a vertex $w \in N_{i-1}(v_j)$ that is assigned a color from $\{1,2,3\} \setminus \{a\}$ and w is uniquely colored among the vertices in $N_{i-1}(v_j)$. The vertex w continues to be uniquely colored in $N_i(v_j)$ since v_i is colored 0. Hence Invariant 2 holds by condition (**) for v_j in $G[i]$.

Case 1b$'$. Vertex v_i: Invariant 1 is satisfied, since v_i has v_j of color $a \neq 0$ as its neighbor. Invariant 2 is vacuously true.

Vertex v_j: The vertex v_j retains its uniquely colored neighbor, since v_i is colored with 0. Thus Invariant 1 holds. Invariant 2 is vacuously true.

Case 1b$''$. Vertex v_i: Invariant 1 is satisfied, since v_i has itself as its uniquely colored neighbor. Further, condition (*) for Invariant 2 holds since $b \neq x$.

Vertex v_j: Invariant 1 is satisfied, since v_i is not colored b. Invariant 2 is vacuously true.

Case 2a$'$. Vertex v_i: Invariant 1 is satisfied for v_i, since v_j is its only neighbor with color a. Invariant 2 is vacuously true.

Vertex v_j: Invariants 1 and 2 (due to condition (**)) are true. The arguments are the same as those in the Case 1a$'$.

Vertices in $N_i(v_j) \setminus \{v_i\}$: Invariant 1 remains true for these vertices, since v_i is colored with x, and $x \neq d$. Invariant 2 is vacuously true.

Case 2a$''$. Vertex v_i: Invariant 1 holds since v_j has color a and v_j is uniquely colored among the vertices in $N_{i-1}[v_j]$. Invariant 2 is vacuously true.

Vertex v_j: Invariants 1 and 2 (due to condition (**)) are true. The arguments are the same as those in the Case 1a$''$.

Vertices in $N_i(v_j) \setminus \{v_i\}$: Invariant 1 holds since v_i is colored 0. For $w \in N_i(v_j) \setminus \{v_i\}$, if $\mathcal{C}(w) = (x, x')$, where $x \neq x'$, the Invariant 2 is vacuously true. For $w \in N_i(v_j) \setminus \{v_i\}$, if $\mathcal{C}(w) = (x, x)$ for some $x \in \{1,2,3\}$, the Invariant 2 held by condition (**) in $G[i-1]$ (since $\mathcal{C}(v_j) = (a, a)$). Invariant 2 continues to hold in $G[i]$ by condition (**) since v_i is colored 0.

Case 2b. Vertex v_i: Invariant 1 holds since $N_i[v_j] = N_i[v_i]$ and since v_i is colored 0. Invariant 2 is vacuously true.

Vertex v_j: Invariant 1 holds since since v_i is colored 0. Invariant 2 is vacuously true.

Vertices in $N_i(v_j) \setminus \{v_i\}$: Invariant 1 holds since since v_i is colored 0. For $w \in N_i(v_j) \setminus \{v_i\}$, if $\mathcal{C}(w) = (x, x')$, where $x \neq x'$, the Invariant 2 is vacuously true. Suppose that for $w \in N_i(v_j) \setminus \{v_i\}$, if $\mathcal{C}(w) = (x, x)$ for some $x \in \{1,2,3\}$.

If Invariant 2 held by condition (*) for w in $G[i-1]$, it means that for all $w' \in N_{i-1}(w)$, we have $\mathcal{C}(w') = (0, b)$ (since $\mathcal{C}(v_j) = (a, b)$ by the case assumption). Invariant 2 continues to hold by condition (*) for w in $G[i-1]$ since $\mathcal{C}(v_i) = (0, b)$.

If Invariant 2 held by condition (**) for w in $G[i-1]$, then it continues to hold by condition (**) in $G[i]$ since v_i is colored 0.

Case 3a′. Vertex v_i: Invariants 1 and 2 hold for v_i since $N_i(v_i) = N_i(v_j) = N_{i-1}(v_j)$.

Vertex v_j: Invariants 1 and 2 continue to hold as the neighborhood of v_j is unaffected by the false twin operation.

Vertices in $N_i(v_j)$: Invariant 1 is true for $w \in N_i(v_j)$ since $a \neq d$. Invariant 2 is vacuously true.

Case 3a″. Vertex v_i: By the case assumption, $w \in N_{i-1}(v_j) = N_i(v_i)$ is the unique vertex in $N_i(v_i)$ that is colored y. Hence Invariant 1 holds. Invariant 2 is vacuously true.

Vertex v_j: Invariants 1 and 2 continue to hold as the neighborhood of v_j is unaffected by the false twin operation.

Vertices in $N_i(v_j)$: Invariants 1 and 2 are true. The arguments are identical to those in the Case 2a″.

Case 3b. Vertex v_i: Note that by the case assumption, there is a unique vertex $w \in N_{i-1}(v_j)$ that is colored b. Since $N_i(v_i) = N_{i-1}(v_j)$ and since v_i is colored 0, Invariant 1 holds. Invariant 2 is vacuously true.

Vertex v_j: Invariants 1 and 2 continue to hold as the neighborhood of v_j is unaffected by the false twin operation.

Vertices in $N_i(v_j)$: Invariants 1 and 2 are true. The arguments are identical to those in the Case 2b.

□

Since the CFCN* chromatic number of distance hereditary graphs is at most 3, its CFCN chromatic number is at most 4. Hence the algorithm of Theorem 5 combined with these bounds provides the following result:

Corollary 12. *For distance hereditary graphs, the CFCN* and CFCN coloring problems are polynomial time solvable.*

In the following lemma, we show that we need fewer colors when we restrict the operations used to construct the distance hereditary graphs.

Lemma 13. *Let G be distance hereditary graph defined by a one-vertex extension sequence which only uses two of the three operations: adding a pendant vertex, adding a false twin and adding a true twin. Then $\chi^*_{CN}(G) \leq 2$. In particular, this holds for cographs and block graphs.*

Proof. In principle, we use the same construction and case distinction as in Lemma 11.

If the pendant operation is absent (i.e., G is a cograph), observe that Cases 1a and 1b do not occur. The only remaining case that assigns $\mathcal{C}(v_i) = (x, x)$

for some color $x \in \{1,2,3\}$ is Case 3a′. However, the prerequisite of this case is that $\mathcal{C}(v_j) = (a,a)$ for some $a \in \{1,2,3\}$. Hence, starting with $\mathcal{C}(v_1) = (1,2)$ and $\mathcal{C}(v_2) = (2,1)$, by induction, Case 3a never occurs and we never assign $\mathcal{C}(u) = (x,x)$ for some color $x \in \{1,2,3\}$ to any vertex u. This means that only Cases 2b and 3b occur, and therefore all vertices other than v_1 and v_2 are colored 0. Hence, 2 colors suffice for a CFCN* coloring. Notice that v_1 and v_2 have each other as their uniquely colored neighbor. Further, all other vertices have v_1 or v_2 as their uniquely colored neighbor. Thus, this CFCN* coloring is also a CFON* coloring.

If the true twin operation is absent, the graph is bipartite. We can color one part of the bipartition with color 1 and the other part with color 2. Since all vertices with the same color are not adjacent to each other, each vertex is its own uniquely colored neighbor.

If the false twin operation is absent (this subclass includes the block graphs), we modify the coloring scheme as follows. Recall that v_1 and v_2 are adjacent to one another. We assign $\mathcal{C}(v_1) = (1,2)$ and $\mathcal{C}(v_2) = (2,1)$.

For $i \geq 3$, we consider two cases, where we assume $\mathcal{C}(v_j) = (a,b)$ for some $a \in \{0,1,2\}$ and $b \in \{1,2\}$:

- **Case 1: v_i is a pendant of v_j.**
  - **Case 1′: $a \neq 0$.**
    We assign $\mathcal{C}(v_i) = (0,a)$.
  - **Case 1″: $a = 0$.**
    We assign $\mathcal{C}(v_i) = (x,x)$, for an arbitrary color x in $\{1,2\} \setminus \{b\}$.
- **Case 2: v_i is a true twin of v_j.**
  We assign $\mathcal{C}(v_i) = (0,b)$.

Note that the color assignments above are similar to those in Case 1b and Case 2b in the proof of Lemma 11.

We prove by induction that at the end of every iteration $i \in [n]$, every vertex has a uniquely colored neighbor in $G[i]$. This holds for the base case $i = 2$. For the inductive step, it is easy to see that if v_i has color 0, then we only need to show the claim for v_i, and otherwise, we have to show the claim also for all vertices in $N_i[v_i]$ (recall that this refers to the neighborhood of v_i in $G[i]$).

Case 1′. The vertex v_j is the uniquely colored neighbor of v_i.
Case 1″. The vertex v_i is its own uniquely colored neighbor.
It remains to consider v_j, the only other vertex in $N_i[v_i]$. As $\mathcal{C}(v_j) = (0,b)$ and $b \neq x$, v_j retains its uniquely colored neighbor from $G[i-1]$.
Case 2. In this case, $N_i[v_i] = N_i[v_j]$. Hence v_i and v_j share the same uniquely colored neighbor whose color is b.

□

From Lemma 13, we have that $\chi^*_{CN}(G) \leq 2$, when G is a block graph. This bound is tight when G is a bull graph, illustrated in Figure 2. To see that there is no CFCN* coloring that uses only one color, observe that the degree 2 vertex in G necessitates that exactly one of the vertices of the K_3 subgraph is colored. A simple case analysis completes the proof.

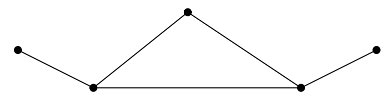

Fig. 2. Bull graph G with $\chi^*_{CN}(G) = 2$.

4.2 CFON* chromatic number

In contrast to the closed neighborhood setting, the class of distance hereditary graphs has unbounded CFON chromatic number and consequently also unbounded CFON* chromatic number.

Lemma 14. *For any $k \geq 1$, there exists a bipartite distance hereditary graph G such that $\chi_{ON}(G) \geq k$.*

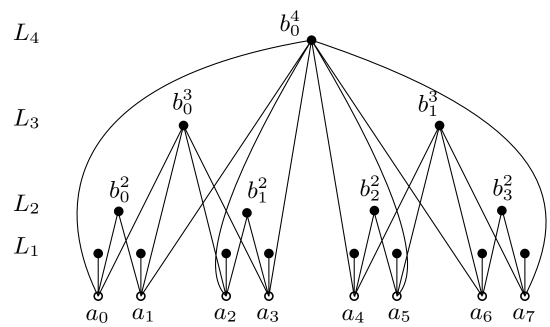

Fig. 3. A bipartite distance hereditary graph G_4 with $\chi_{ON}(G_4) \geq 4$.

Proof. We define a family of graphs $G_2, G_3, \ldots$ as follows. Each graph G_k, for $k \geq 2$, is bipartite with the vertex sets A and B that satisfy the following:

- Set A consists of 2^{k-1} vertices $a_0, \ldots, a_{2^{k-1}-1}$.
- Set B consists of vertices in k levels $L_1, \ldots, L_k$. Level L_i contains 2^{k-i} vertices $b_0^i, \ldots, b_{2^{k-i}-1}^i$, for $i \in [k]$.
- There are 2^{k-1} edges between each level L_i and A in a binary fashion. To be precise, the vertex b_j^i is connected with vertices a_t for $t = 2^{i-1}j, \ldots, 2^{i-1}(j+1) - 1$.

Figure 3 illustrates the graph G_4. We construct G_k recursively, starting from the graph of only one vertex called the root. Our construction satisfies the property ($\star$): Every vertex of $A \setminus \{a_0\}$ is indirectly a false twin of a_0; which means that it is created by a sequence of false twin operations on some vertices $u_0, u_1, \ldots, u_z$, $z \geq 1$ where $u_0 = a_0$, and each u_i, $i \geq 1$, is introduced as a false twin of u_{i-1}. The construction is as follows:

- For $k = 2$, we add a pendant to the root, i.e., G_2 is isomorphic to K_2.

– For $k \geq 3$, as mentioned above, we call a_0 the root. We add b_0^k as a pendant of a_0. Next, we add a false twin of a_0, called $a_{2^{k-2}}$. Then, recursively construct a G_{k-1} with root a_0 and another G_{k-1} with root $a_{2^{k-2}}$. The first G_{k-1} introduces $a_1, \ldots, a_{2^{k-2}-1}$, the second G_{k-1} introduces $a_{2^{k-2}+1}, \ldots, a_{2^{k-1}-1}$. The property ($\star$) holds for these copies of G_{k-1}. That is, every vertex of $A \setminus \{a_0, a_{2^{k-2}}\}$ is created indirectly as a false twin of a_0 or $a_{2^{k-2}}$. Since $a_{2^{k-2}}$ is created as a false twin of a_0, property ($\star$) holds for G_k. Further, because ($\star$) is true, b_0^k is adjacent to all of A. Thus, we have constructed G_k.

We will show that the CFON chromatic number of G_k is at least k. This holds trivially for $k = 2$. Consider the case where $k \geq 3$. Observe that b_0^k needs to have a neighbor with a unique color. Without loss of generality, we color $a_{2^{k-1}-1}$ with the color c_k. Next, b_0^{k-1} also needs a neighbor with a unique color. Note that this color must be different than c_k, because all neighbors of b_0^{k-1} are neighbors of b_0^k, while $a_{2^{k-1}-1}$ is not a neighbor of b_0^{k-1}. Without loss of generality, we color $a_{2^{k-2}-1}$ with the color c_{k-1}. Repeating the above argument, we can see that we need at least k colors. □

Notice that the number of vertices in the graph G_k constructed above is $\Theta(2^k)$. Since $\chi_{ON}(G_k) \geq k$, we have the following corollary.

Corollary 15. *There exists a distance hereditary graph G on n vertices for which $\chi_{ON}(G) = \Omega(\log n)$.*

Although in general, a distance hereditary graph can have arbitrarily large CFON* chromatic number, we show that this number is bounded for two subclasses, as in the following two lemmas.

Lemma 16. *If G is a cograph, then $\chi^*_{ON}(G) \leq 2$.*

Proof. As observed in the proof of Lemma 13, the coloring scheme there gives a CFON* coloring with the colors $\{0, 1, 2\}$. In this coloring, v_1 has color 1, v_2 color 2, and all other vertices color 0. □

Lemma 17. *If G is a block graph, $\chi_{ON}(G) \leq 3$, hence $\chi^*_{ON}(G) \leq 3$.*

Proof. Our proof is by induction on $|V|$. Trivially, if $|V| \leq 3$, then $\chi_{ON}(G) \leq 3$. For the inductive step, if G is 2-connected, then by definition of a block graph, G is a clique. We can color two vertices with two different colors and all other vertices with the third color. It is easy to see that this is a CFON coloring.

Now suppose G is not 2-connected. Then there exists a vertex v whose removal disconnects the graph, and a connected component C satisfies that $V(C) \cup \{v\}$ induces a 2-connected component in G, i.e., a clique. (This component is sometimes called a leaf block, for example, in [35].)

Consider the induced subgraph G' of G obtained by removing $V(C)$ from G. It is easy to see that G' is also a block graph. Hence, applying the inductive hypothesis, we can obtain a CFON coloring of G' with 3 colors. Let c_1 be the the color of v and c_2 be the color of its uniquely colored neighbor. We apply the same coloring of G' to the vertices in G, where we additionally color all vertices in C with the color other than c_1 and c_2. Certainly, this does not invalidate the uniquely colored neighbor of v. No other vertex in G' is adjacent to a vertex of C in G. Further, all vertices in C have v as their uniquely colored neighbor. Hence, this is a CFON coloring of G with 3 colors. □

We show that the above result is tight.

Lemma 18. *There is a block graph G with $\chi^*_{ON}(G) = 3$.*

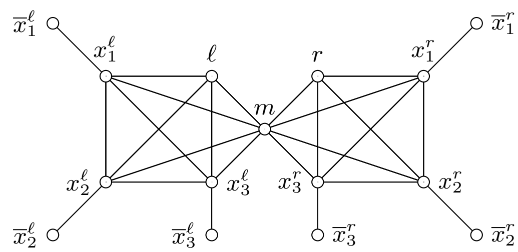

Fig. 4. A block graph G with $\chi^*_{ON}(G) = 3$.

Proof. Let G have vertex set $V = \{\ell, m, r\} \cup \bigcup_{i \in \{1,2,3\}} \{x_i^\ell, \overline{x}_i^\ell, x_i^r, \overline{x}_i^r\}$, see also Fig. 4. Let the edge set be defined by the set of maximal cliques $\{x_1^s, x_2^s, x_3^s, s, m\}$ and $\{x_i^s, \overline{x}_i^s\}$ for every $s \in \{\ell, r\}$ and $i \in \{1, 2, 3\}$. It is easy to see that G is a block graph. To prove that $\chi^*_{ON}(G) > 2$, suppose for a contradiction that there is a χ^*_{ON} coloring $C : V \to \{0, 1, 2\}$. Then there is a mapping h on V that assigns each vertex $v \in V$ its uniquely colored neighbor $w \in N(v)$. Note that x_i^s, for $s \in \{\ell, r\}$ and $i \in \{1, 2, 3\}$, has to be colored 1 or 2, since it is the only neighbor of $\overline{x}_i^s$. Without loss of generality, we may assume that $h(m) \in \{\ell, x_1^\ell\}$. Further, we may assume that $C(h(m)) = 2$.

First suppose that $h(m) = \ell$ and hence $C(\ell) = 2$. Then $(x_i^s) = 1$ for every $s \in \{\ell, r\}$ and $i \in \{1, 2, 3\}$. It follows that $h(\ell) = m$ and hence $C(m) = 2$. Then the vertex x_1^ℓ contains two neighbors colored 1 and two neighbors colored 2, which is a contradiction.

Thus it remains to consider that $h(m) = x_1^\ell$ and hence $C(x_1^\ell) = 2$. Then $C(w) = 1$ for every vertex $w \in \{x_2^\ell, x_3^\ell, x_1^r, x_2^r, x_3^r\}$. It follows that $h(r) = m$ and $C(m) = 2$. Then the vertex ℓ has two neighbors colored 1 and two neighbors colored 2, which is a contradiction.

Since both cases lead to a contradiction, it must be that $\chi^*_{ON}(G) > 2$. □

Together with the fact that distance hereditary graphs have clique-width at most 3, Theorem 5, Lemma 16 and Lemma 17 imply the following corollary.

Corollary 19. *For cographs and block graphs, the CFON* and CFON coloring problems are polynomial time computable.*

5 Interval Graphs

In this section, we consider interval graphs. We prove that three colors are sufficient and sometimes necessary to CFON* color an interval graph. For *proper interval graphs*, we show that two colors are sufficient.

Definition 20 (Interval Graph). *A graph G, $G = (V, E)$, is an* interval graph *if there exists a set $\mathcal{I}$ of closed intervals on the real line such that there is a bijection $f : V \to \mathcal{I}$ satisfying the following: $v_1v_2 \in E$ if and only if $f(v_1) \cap f(v_2) \neq \emptyset$.*

For an interval graph G, we refer to the set of intervals $\mathcal{I}$ as an *interval representation* of G. An interval graph G is a *proper interval graph* if it has an interval representation $\mathcal{I}$ such that no interval in $\mathcal{I}$ is properly contained in any other interval of $\mathcal{I}$. An interval graph G is a *unit interval graph* if it has an interval representation $\mathcal{I}$ where all the intervals are of unit length. It is known that the class of proper interval graphs and unit interval graphs are identical [16].

For each interval $I \in \mathcal{I}$, we use $L(I)$ and $R(I)$ to denote its left endpoint and right endpoint respectively. Throughout this section, we assume that no two intervals share the same endpoint (either left or right). If there exists two intervals that share an endpoint, we can carefully adjust them such that they do not share the same endpoint. We use the terms "vertex" and "interval" interchangeably.

It was shown in [15] that $\chi^*_{CN}(G) \leq 2$, when G is an interval graph. We use similar ideas to show the bound for $\chi^*_{ON}(G)$.

Lemma 21. *If G is an interval graph, then $\chi^*_{ON}(G) \leq 3$.*

Proof. Let G be an interval graph and $\mathcal{I}$ be an interval representation of G. We use the function $C : \mathcal{I} \to \{1, 2, 3, 0\}$ to assign colors. We assign the colors 1, 2 and 3 cyclically. We start with the interval I_1 for which $R(I_1)$ is the least and assign $C(I_1) = 1$. Then choose the interval I_2 such that $I_2 \in N(I_1)$ and $R(I_2) > R(I), \forall I \in N(I_1)$ and assign $C(I_2) = 2$. For $j \geq 3$, we do the following: choose the interval I_j such that $I_j \in N(I_{j-1})$ and $R(I_j) > R(I), \forall I \in N(I_{j-1})$ and assign the color $\{1, 2, 3\} \setminus \{C(I_{j-1}), C(I_{j-2})\}$ to the interval I_j. The procedure terminates at the value j for which $R(I_j)$ maximizes $R(I)$ amongst all $I \in \mathcal{I}$. We refer to this value of j as ℓ. Now we assign 0 to all the uncolored intervals.

Since G is connected and because of the coloring procedure, the graph induced on the intervals $I_1, I_2, \ldots, I_\ell$ is a path. For $1 \leq j \leq \ell - 1$, the interval I_{j+1} will be a uniquely colored neighbor for the interval I_j. The interval $I_{\ell-1}$ will be a uniquely colored neighbor for I_ℓ.

Consider an interval I assigned the color 0. Recall that the intervals $I_1, I_2, \ldots, I_\ell$ induce a path. This implies that I is adjacent to an interval I_j, where $1 \leq j \leq \ell$ and $R(I_j) > R(I)$. We claim that I_j will be a uniquely colored neighbor of I. Assume for the sake of contradiction that I is adjacent to I_{j-3}, the vertex that was assigned the color $C(I_j)$ immediately before I_j. This implies that I is adjacent to I_{j-2} and I_{j-1} as well. Since the graph induced by $I_1, I_2, \ldots, I_\ell$ is a path, I_{j-2} is not adjacent to I_j. Since $I \in N(I_j)$, it follows that $R(I) > R(I_{j-2})$. This contradicts the coloring procedure as we must have chosen I in place of I_{j-2}. Thus I is not adjacent to I_{j-3}. By a similar argument, we can see that I is not adjacent to I_{j+3} as well. □

The bound of $\chi^*_{ON}(G) \leq 3$ for interval graphs is tight. In particular, there is an interval graph G (see Figure 5) that cannot be CFON colored with three colors. This implies that $\chi^*_{ON}(G) > 2$.

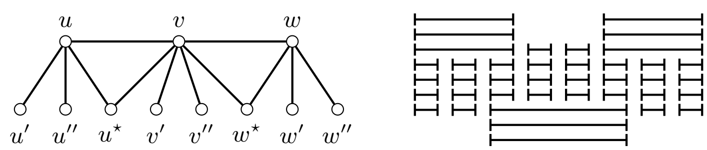

Fig. 5. On the left hand side, we have the graph G', and on the right hand side we have an interval representation of G, a graph in which $\chi_{ON}(G) > 3$. The graph G is obtained by adding two true twins each to the vertices $u, v, w, u^\star, v^\star$ of G' and adding three true twins each to the vertices $u', u'', v', v'', w', w''$ of G'.

Lemma 22. *There is an interval graph G such that $\chi_{ON}(G) > 3$ (and thus $\chi^*_{ON}(G) \geq 3$).*

Proof. We define the graph $G = (V, E)$, an interval representation seen in Figure 5, with the help of a preliminary graph $G' = (V', E')$. V' consists of vertices u, v, w and $u', u'', u^\star, v', v'', w^\star, w', w''$. Let E' be the edges which form the maximal cliques $\{u', u\}, \{u'', u\}, \{u^\star, u, v\}, \{v, v'\}, \{v, v''\}, \{w^\star, v, w\}$, $\{w, w'\}, \{w, w''\}$. By this ordering of maximal cliques, we observe that G' is an interval graph.

For a vertex z, recall that a vertex z' is said to be a true twin of z if z' is adjacent to z and all the vertices in $N(z)$. The graph G is obtained by adding two true twins each to the vertices $u, v, w, u^\star, w^\star$ of G' and adding three true twins each to the vertices $u', u'', v', v'', w', w''$ of G'. In other words, $V = \bigcup_{x \in V'} \{x_1, x_2, x_3\} \cup \{u'_4, u''_4, v'_4, v''_4, w'_4, w''_4\}$ and $E = \bigcup_{pq \in E',\, i,j \in [4]} p_i q_j$ (for those where vertices p_i and q_j exist). Since G is an interval graph, G' is also an interval graph.

Now we show that G cannot be CFON colored with 3 colors. Suppose there is a CFON coloring $C : V \to \{1, 2, 3\}$. Let h map each vertex $x \in V$ to a uniquely colored neighbor $y \in N(x)$.

Claim. $|C(\{u_1, u_2, u_3\})| = |C(\{v_1, v_2, v_3\})| = |C(\{w_1, w_2, w_3\})| = 2.$

Proof. Suppose for a contradiction that $|C(\{u_1, u_2, u_3\})| = 1$. Without loss of generality, we may assume $C(u_1) = C(u_2) = C(u_3) = 1$. Note that the neighborhood $N(\{u'_1, u'_2, u'_3, u'_4\}) = \{u_1, u_2, u_3, u'_1, u'_2, u'_3, u'_4\}$. It follows that the uniquely colored neighbors for each vertex in $\{u'_1, u'_2, u'_3, u'_4\}$ belong to the same set $\{u_1', u_2', u_3', u_4'\}$. This implies further that a vertex amongst u_1', u_2', u_3', u_4' is colored 2, and another one is colored 3. Analogously it follows that there are a vertex colored 2 and one colored 3 amongst $u_1'', u_2'', u_3'', u_4''$. Then each vertex in $\{u_1, u_2, u_3\}$ has two neighbors each of the colors from $\{1, 2, 3\}$, which is a contradiction to the fact that C is a CFON coloring using three colors. By symmetry we can also show that $|C(\{v_1, v_2, v_3\})| \neq 1$ and $|C(\{w_1, w_2, w_3\})| \neq 1$.

It remains to show that $|C(\{u_1, u_2, u_3\})| \neq 3$, $|C(\{v_1, v_2, v_3\})| \neq 3$ and $|C(\{w_1, w_2, w_3\})| \neq 3$. For the sake of contradiction, assume without loss of generality that $C(u_1) = 1, C(u_2) = 2$, and $C(u_3) = 3$. These vertices are adjacent to $\{v_1, v_2, v_3\}$. As shown in the previous paragraph, we have $|C(\{v_1, v_2, v_3\})| \geq 2$. If $|C(\{v_1, v_2, v_3\})| = 3$, then each of the colors $\{1, 2, 3\}$ appear twice in the neighborhood of the vertex $u_1^\star$. Hence, $u_1^\star$ does not have a uniquely colored neighbor. If $|C(\{v_1, v_2, v_3\})| = 2$, then without loss of generality, we assume $C(v_1) = 1, C(v_2) = 1, C(v_3) = 2$. Then each of the colors $\{1, 2\}$ appear twice in the neighborhood of u_3 and thus $C(h(u_3)) = 3$. However, since $h(u_3) \in N(u_1)$, the vertex u_1 has two neighbors each of the colors from $\{1, 2, 3\}$ and u_1 cannot have a uniquely colored neighbor. By symmetry, we can show that $|C(\{v_1, v_2, v_3\})| \neq 3$ and $|C(\{w_1, w_2, w_3\})| \neq 3$. $\square$

Without loss of generality, we may now assume that $C(v_1) = 1, C(v_2) = 2, C(v_3) = 2$. If $3 \notin C(\{u_1, u_2, u_3\})$, then $C(h(u_1^\star)) = 3$ and $h(u_1^\star) \in \{u_2^\star, u_3^\star\}$. Without loss of generality, let $h(u_1^\star) = u_2^\star$. This means $C(u_2^\star) = 3$ and $C(u_3^\star) \in \{1, 2\}$. By a similar reasoning $C(h(u_2^\star)) = 3$. This forces $h(u_2^\star) = u_1^\star$ and $C(u_1^\star) = 3$. However now $u_3^\star$ has at least two neighbors from each of the colors in $\{1, 2, 3\}$. Therefore, $u_3^\star$ does not have a uniquely colored neighbor. Hence, $3 \in C(\{u_1, u_2, u_3\})$, and analogously, $3 \in C(\{w_1, w_2, w_3\})$.

However, v_1 is now adjacent to at least two vertices of color 3 and two of color 2. Hence, v_1 must be adjacent to exactly one vertex with color 1. This implies either $1 \notin C(\{u_1, u_2, u_3\})$ or $1 \notin C(\{w_1, w_2, w_3\})$. Without loss of generality, suppose $1 \notin C(\{u_1, u_2, u_3\})$. By the claim above, $C(\{u_1, u_2, u_3\}) = \{2, 3\}$.

However, v_2 is then adjacent to two vertices with color 1 (i.e., v_1 and $h(v_1)$), two vertices of color 2 (i.e., v_3 and at least one in $\{u_1, u_2, u_3\}$), two vertices of color 3 (i.e., a vertex in $\{u_1, u_2, u_3\}$ and one in $\{w_1, w_2, w_3\}$). That means v_2 does not have a uniquely colored neighbor, a contradiction. Therefore, G cannot be CFON colored with 3 colors. $\square$

Lemma 23. *If G is a proper interval graph, then $\chi^*_{ON}(G) \leq 2$.*

Proof. Let G be a proper interval graph and $\mathcal{I}$ be a unit interval representation of G. We use the function $C : \mathcal{I} \to \{1, 2, 0\}$ to assign colors as follows.

At each step $i \geq 1$, we pick two intervals $I_1^i, I_2^i \in \mathcal{I}$. The interval I_1^i is chosen such that $L(I_1^i)$ is the least among intervals for which C has not yet been assigned. The choice of I_2^i depends on the following two cases.

- **Case 1:** *I_1^i has a neighbor for which C is unassigned.*
  We choose I_2^i such that $R(I_2^i)$ is the largest amongst the intervals in $N(I_1^i)$ for which C is yet to be assigned. Notice that $L(I_1^i) < L(I_2^i)$ and hence

$R(I_1^i) < R(I_2^i)$. We assign $C(I_1^i) = 1$ and $C(I_2^i) = 2$. We assign the color 0 to all the other intervals adjacent to either I_1^i or I_2^i.

– **Case 2:** *C is already assigned for all the neighbors of I_1^i.*
This cannot happen for $i = 1$, because otherwise the graph has an isolated vertex. Let $\widehat{I}$ be an interval in $N(I_2^{i-1}) \cap N(I_1^i)$. Such an $\widehat{I}$ exists, because otherwise G is disconnected. We reassign $C(I_1^{i-1}) = 0$, $C(I_2^{i-1}) = 1$, $C(\widehat{I}) = 2$ and assign $C(I_1^i) = 0$.

By the choice of I_1^i, all intervals whose left endpoints are smaller than $L(I_1^i)$ have been assigned a color (which may be the color 0). Therefore, Case 2 can only occur at the last step. Let the last step be the j-th step of the coloring process.

We prove by induction on i that C is a CFON* coloring for the induced subgraph containing $N[I_1^i \cup I_2^i]$ and all the intervals whose left endpoints are less than $L(I_1^i)$. For the base case $i = 1$, the subgraph only contains I_1^1, I_2^1, and their neighbors. The claim then holds by construction. Since Case 1 applies for the base case, the intervals I_1^1 and I_2^1 see each other as their uniquely colored neighbors, and the vertices colored 0 see I_2^1 as their uniquely colored neighbor.

For the inductive step for $i > 1$, we first consider the situation when Case 1 applies at step i. Note that the intervals I_1^i and I_1^{i-1} have the same color, and so do I_2^i and I_2^{i-1}. However, because of the unit length of the intervals and the choice of the two intervals in each step, it is easy to see that no interval intersects both I_1^i and I_1^{i-1}. We have the following cases based on whether $N(I_2^i) \cap N(I_2^{i-1})$ is empty.

– $N(I_2^i) \cap N(I_2^{i-1}) = \emptyset$.
All intervals colored in the previous steps (till $i - 1$) retain their uniquely colored neighbors.

– There is an interval $I \in N(I_2^i) \cap N(I_2^{i-1})$.
Notice that by construction, I_2^{i-1} and I_2^i are disjoint, and $L(I_2^{i-1}) < L(I_2^i)$. Hence, $L(I_2^{i-1}) < L(I)$. Further, $I \notin N(I_1^{i-1})$, because otherwise we would have chosen the interval I in place of I_2^{i-1}.
Moreover, $I \in N(I_1^i)$. We have I_1^i as the uniquely colored neighbor for the interval I. This argument holds for all the intervals in $N(I_2^i) \cap N(I_2^{i-1})$. For all the other intervals colored in the previous steps (till $i - 1$), the uniquely colored neighbors remain the same.

Further, the intervals I_1^i and I_2^i act as the uniquely colored neighbors for each other. Lastly, as every interval has unit length, all neighbors of I_1^i that are assigned 0 in step i are also neighbors of I_2^i. Therefore, I_2^i is the uniquely colored neighbor of all intervals that are assigned 0 in this step.

Now suppose that Case 2 applies to step i, i.e., we are at the last step $i = j$. That is, there is no I_2^j. In the j-th step, we reassign $C(I_1^{j-1}) = 0$, $C(I_2^{j-1}) = 1$ and $C(\widehat{I}) = 2$. As argued above, before the reassignment in this step, the set of intervals that relied on I_1^{j-1} for their uniquely colored neighbor is $\{I_2^{j-1}\} \cup \{I \mid I \in N(I_2^{j-1}) \cap N(I_2^{j-2})\}$. After the reassignment, the set of intervals $\{I \mid I \in$

$N(I_2^{j-1}) \cap N(I_2^{j-2})\}$ depend on I_2^{j-1} (which is reassigned the color 1) for the uniquely colored neighbor. This is fine as $I \notin N(I_1^{j-2})$. The intervals I_2^{j-1} and I_1^j rely on $\widehat{I}$ while all other intervals that relied on I_2^{j-1} previously will continue to rely on I_2^{j-1}. $\square$

5.1 Algorithmic Status of Conflict-free Coloring on Interval Graphs

Fekete and Keldenich [15] studied CFCN* coloring on intersection graphs. They showed that for an interval graph G, $\chi^*_{CN}(G) \leq 2$. The CFCN* coloring problem was shown to be polynomial time solvable on interval graphs in [5].

From Lemma 21, we have that $\chi^*_{ON}(G) \leq 3$. Bhyravarapu, Kalyanasundaram and Mathew in [6] showed that CFON* coloring problem is solvable in time $O(n^{20})$ using the structural properties of interval graphs. Independently, Gonzalez and Mann in [21] showed that all the four variants of conflict-free coloring can be solved in time $n^{O(w)}$, where w is the *mim-width* of the graph. Interval graphs have mim-width one. The algorithms resulting from the formulation in [21] result in a running time of $O(n^{300})$ on interval graphs. Thus the complexity status of the problem on interval graphs is settled.

Corollary 24. *CFCN* and CFON* coloring problems are polynomial time solvable on interval graphs.*

6 Unit Square and Unit Disk Intersection Graphs

Unit square (or unit disk) intersection graphs are intersection graphs of closed unit sized axis-aligned squares (or disks, respectively) in the Euclidean plane. Figure 6 is a unit square and unit disk graph. It is shown in [15] that $\chi^*_{CN}(G) \leq 4$ for a unit square intersection graph G. They also showed that $\chi^*_{CN}(G) \leq 6$ for a unit disk intersection graph G. We study the CFON* coloring problem on these graphs and obtain constant upper bounds. To the best of our knowledge, no upper bound for CFON* chromatic number was previously known on unit square and unit disk graphs.

6.1 Unit Square Intersection Graphs

We first discuss the unit square intersection graphs. Consider a unit square representation of such a graph. Each square is identified by its center, which is the intersection point of its diagonals. By unit square, we mean that the distance between its center and its sides is one, i.e., the length of each side is two. Sometimes we interchangeably use the term “vertex” for unit square. Throughout, we denote the X-coordinate and the Y-coordinate of a vertex v with v_x and v_y respectively. A *stripe* is the region between two horizontal lines, and the *height* of the stripe is the distance between these two lines. We consider a unit square as *belonging* to a stripe if its center is contained in the stripe. For a unit square whose center lies on a horizontal line, we consider it belonging to the

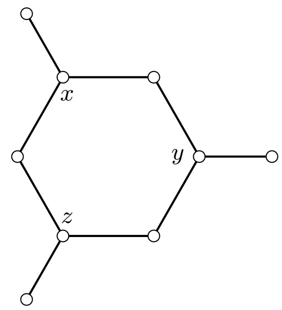

Fig. 6. A unit square graph G for which $\chi^*_{ON}(G) \geq 3$. The vertices x, y, z have to be assigned distinct non-zero colors. Note that G is also a unit disk graph.

stripe that is immediately below the horizontal line. We say that a unit square intersection graph has *height* h, if the centers of all the squares lie in a stripe of height h.

Lemma 25. *If G is a unit square intersection graph of height 2, then $\chi^*_{ON}(G) \leq 2$.*

Proof. Let G be a unit square intersection graph of height 2. Note that vertices u and v are adjacent if and only if $|u_x - v_x| \leq 2$. We may represent G as a unit interval graph (with each interval of length 2) by mapping every vertex v to an interval from $v_x - 1$ to $v_x + 1$. It is easy to note that two vertices in G are adjacent if and only if the corresponding vertices in the interval representation are adjacent. By Lemma 23, we obtain that $\chi^*_{ON}(G) \leq 2$. □

Theorem 26. *If G is a unit square intersection graph, then $\chi^*_{ON}(G) \leq 27$.*

Proof. We assign colors for the vertices of G, $G = (V, E)$, in two phases — in phase 1, we color all the vertices and in phase 2, we modify the coloring to ensure that all the vertices have a uniquely colored neighbor. In phase 1, we use 6 colors $C : V \to \{0\} \cup \{c_0^i, c_1^i \mid i \in \{0, 1, 2\}\}$. Without loss of generality, we assume that the centers of all the squares have positive Y-coordinates. We partition the plane into stripes S_ℓ for $\ell \in \mathbb{N}$ where each stripe is of height 2. We assign vertex v with Y-coordinate v_y to S_ℓ if $2(\ell - 1) < v_y \leq 2\ell$. Let $G[S_\ell]$ be the graph induced by the vertices belonging to the stripe S_ℓ. Then $G[S_\ell]$ has height 2. Notice that $G[S_\ell]$ may be disconnected. We apply Lemma 25 on each of the connected components, and color vertices in S_ℓ using colors c_0^i and c_1^i where $i = \ell \bmod 3$. Then every vertex $u \in S_\ell$ that is not isolated in $G[S_\ell]$ has a uniquely colored neighbor v in $G[S_\ell]$. The isolated vertices in $G[S_\ell]$ are assigned the color 0. Every $w \notin S_\ell$ with color $C(w) = C(v)$ must be in a stripe $S_{\ell^\star}$ with $|\ell - \ell^\star| \geq 3$. Thus $w \notin N(u)$ and hence v is also a uniquely colored neighbor of u in G. It remains to determine uniquely colored neighbors for the vertices $u \in S_\ell$ which are isolated in $G[S_\ell]$. Let I be the set of all such vertices, which belong to all the stripes in the graph.

In phase 2, we reassign colors to some of the vertices of G to ensure a uniquely colored neighbor for each vertex in I. For each vertex $v \in I$, choose an arbitrary

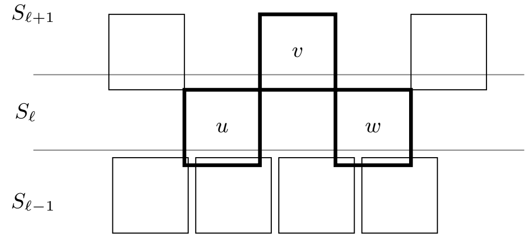

Fig. 7. The vertex $v \in S_{\ell+1}$ is adjacent to two vertices u and w in S_ℓ, which are representative vertices for some isolated vertices. In the worst case, $|u_x - w_x| = 4$. The picture describes the positions of the isolated vertices whose representative vertex r is such that $u_x \leq r_x \leq w_x$.

representative vertex $r(v) \in N(v)$. Let $R = \{r(v) \mid v \in I\} \subseteq V$ be the set of representative vertices. We update the coloring C by recoloring the vertices in R using the colors $\{c_j^i \mid i \in \{0,1,2\}, j \in \{2,3,\ldots,8\}\}$. Consider a stripe S_ℓ for $\ell \in \mathbb{N}$. We order the vertices $S_\ell \cap R$ non-decreasingly by their X-coordinate and sequentially color them with $c_2^i, \ldots, c_8^i$ where $i = \ell \bmod 3$.

Total number of colors used: The numbers of colors used in phase 1 and phase 2 are 6 and 21 respectively, giving a total of 27.

Correctness: We now prove that the assigned coloring is a valid CFON* coloring. For this we need to prove the following,

- Each vertex in I has a uniquely colored neighbor.
- The coloring in phase 2 does not upset the uniquely colored neighbors (determined in phase 1) of the vertices in $V \setminus I$.

We first prove the following claim.

Claim. For each vertex $v \in V$, all vertices in $N(v) \cap R$ are assigned distinct colors in phase 2.

Proof (Claim's proof). Let $v \in S_{\ell+1}$ (see Figure 7). Suppose for a contradiction that there are two vertices $u, w \in N(v) \cap R$ such that $C(u) = C(w)$. Then u and w have to be from the same stripe that neighbors $S_{\ell+1}$. Without loss of generality, we may assume that $u, w \in S_\ell$, $\ell = 0 \bmod 3$ and $u_x \leq w_x$. We may further assume that $C(u) = C(w) = c_2^0$. Then there are eight vertices (including u and w), $R' \subseteq R \cap S_\ell$, that are assigned the colors $c_2^0, c_3^0, \ldots, c_8^0, c_2^0$ and have their X-coordinates between u_x and w_x. Note that $|u_x - v_x| \leq 2$ and $|w_x - v_x| \leq 2$. Vertices R' are the representative vertices of some eight vertices $I' \subseteq I$. By definition, $I' \subseteq S_{\ell+1} \cup S_{\ell-1}$.

First, let us consider $I' \cap S_{\ell+1}$. We claim that there is at most one vertex $u' \in I' \cap S_{\ell+1}$ such that $u'_x < v_x$. Indeed any such vertex $u' \in I'$ must be adjacent to some representative $r \in R'$ with $|r_x - v_x| \leq 2$. Thus the distance between u'_x and v_x is at most 4 and hence there is at most one vertex in $I' \cap S_{\ell+1}$ with lower

X-coordinate than v. Analogously, there is at most one vertex $w' \in I' \cap S_{\ell+1}$ such that $w'_x > v_x$. Considering the possibility that $v \in I'$, we have $|I' \cap S_{\ell+1}| \leq 3$.

Now, consider the vertices in $I' \cap S_{\ell-1}$. Again any vertex in $I' \cap S_{\ell-1}$ must be adjacent to some representative $r \in R'$ with $|r_x - v_x| \leq 2$. Thus the X-coordinates of the vertices in $I' \cap S_{\ell-1}$ differ by at most 8. Since the vertices in $I' \cap S_{\ell-1}$ are non-adjacent, we have that $|I' \cap S_{\ell-1}| \leq 4$. This contradicts the assumption that $|I'| = 8$. Thus all vertices $N(v) \cap R$ are assigned distinct colors. □

We now proceed to the proof of correctness.

- **Every vertex $v \in I$ has a uniquely colored neighbor.**
  Let $v \in S_{\ell+1} \cap I$. By the above claim, no two vertices in $N(v) \cap R$ are assigned the same color in phase 2. Since v is not isolated in G, we have that $|N(v) \cap R| \geq 1$ and v has a uniquely colored neighbor.
- **The coloring in phase 2 does not upset the uniquely colored neighbors of vertices in $V \setminus I$.**
  Let $v \in V \setminus I$ and u be its uniquely colored neighbor after the phase 1 coloring. If u is no longer the uniquely colored neighbor of v after phase 2, it has to be the case that u was recolored in phase 2, and v had another vertex $w \in N(v)$ which was assigned the same color as u in phase 2. This implies that both u and w are representative vertices for some vertices in I and they are recolored in phase 2. This contradicts the above claim. □

6.2 Unit Disk Intersection Graphs

In this section, we prove an upper bound for the CFON* chromatic number on unit disk intersection graphs. Consider a unit disk representation of such a graph. Each disk is identified by its center. By unit disk, we mean that its radius is 1. Sometimes we interchangeably use the term “vertex” for unit disk. We consider a unit disk as belonging to a stripe if its center is contained in the stripe. If a unit disk has its center on the horizontal line that separates two stripes then it is considered in the stripe below the line.

We say that a unit disk intersection graph has *height* h, if the centers of all the disks lie in a horizontal stripe of height h. The approach is to divide the graph into horizontal stripes of height $\sqrt{3}$ and color the vertices in two phases. Throughout, we denote the X-coordinate and the Y-coordinate of a vertex v with v_x and v_y respectively.

Theorem 27. *If G is a unit disk intersection graph, then $\chi^*_{ON}(G) \leq 54$.*

The proof of this theorem is similar to the proof of Theorem 26, but different in the following three aspects:

- In Theorem 26, we used the result that unit square graphs of height 2 are CFON* 2-colorable. In this theorem, we will use the result that unit disk intersection graphs of height $\sqrt{3}$ are CFCN* 2-colorable, and not CFON* 2-colorable.

– In Theorem 26, the set I for which we needed to identify the uniquely colored neighbor was the set of isolated vertices in the respective stripe. In this theorem, the set I will be the set of vertices colored in phase 1.
– In Theorem 26, the phase 2 coloring involved considering the representative vertices in the order of their X-coordinate. For the phase 2 coloring of this theorem, we consider the vertices in I in the order of their X-coordinate and then color their representative vertices.

We will use the following lemma from [15].

Lemma 28 (Theorem 5 in [15]). *If G is a unit disk intersection graphs of height $\sqrt{3}$, then $\chi^*_{CN}(G) \leq 2$. Further, the horizontal distance between two colored vertices is greater than 1.*

Note that the above lemma pertains to CFCN* coloring and not CFON* coloring. The second sentence in the above lemma is not stated in the statement of Theorem 5 in [15], but rather in its proof. We will use the CFCN* coloring used in the lemma stated above to obtain a CFON* coloring for unit disk intersection graphs. Below we reproduce the coloring process used in the proof of the above lemma in [15].

Coloring process used in the proof of Lemma 28: Let $G = (V, E)$ be a unit disk intersection graph such that the centers of all the disks in G lie in a stripe of height $\sqrt{3}$. The vertices in V are colored in the order of their non-decreasing X-coordinates. A vertex v is *covered* if and only if it is colored or has a colored neighbor. In each step of the algorithm, we choose a vertex v whose v_x is the maximum and that covers all uncovered vertices to its left. We assign the color 1 (or 2) to v if the previously colored vertex was assigned the color 2 (or 1). At the end, each uncolored vertex is assigned the color 0. It follows from the algorithm that the horizontal distance between any two colored vertices is greater than 1.

Proof (Proof of Theorem 27). We assign color $C(v)$ to each unit disk v of G in two phases. In phase 1, we use 6 non-zero colors to color the vertices of G, i.e., $C : V \to \{0\} \cup \{c_0^i, c_1^i \mid i \in \{0, 1, 2\}\}$. WLOG we assume that the centers of all the disks have positive Y-coordinates. We partition the plane into horizontal stripes S_ℓ for $\ell \in \mathbb{N}$ where each stripe is of height $\sqrt{3}$. We assign vertex v with Y-coordinate v_y to S_ℓ if $\sqrt{3}(\ell - 1) < v_y \leq \sqrt{3}\ell$. Let $G[S_\ell]$ be the graph induced by the vertices belonging to the stripe S_ℓ. Then $G[S_\ell]$ has height $\sqrt{3}$. We CFCN* color vertices in S_ℓ accordingly using (nonzero) colors c_0^i, c_1^i where $i = \ell \bmod 3$, according to Lemma 28. Let I be the set of all colored vertices after this phase. Our goal is to CFON* color all the vertices. After phase 1, each vertex not in I has a uniquely colored neighbor that is not itself. Hence we only need to identify uniquely colored neighbors for vertices in I.

In phase 2, we modify the colors assigned to some vertices of G to ensure a uniquely colored neighbor for each vertex in I. For each vertex $v \in I$, choose an arbitrary *representative vertex* $r(v) \in N(v)$. Note that two vertices in I may share the same representative vertex. Let $R = \{r(v) \mid v \in I\}$ be the set of representative vertices. We use the coloring function $D : R \to \{d^i_j \mid i \in \{0,1,2,3,4,5\}, j \in \{0,1,\ldots,7\}\}$ to assign colors to vertices in R. Consider a stripe S_ℓ for $\ell \in \mathbb{N}$. We order the vertices $S_\ell \cap I$ non-decreasingly by their X-coordinate. We consider the vertices sequentially in that order. If the representative vertex of the current vertex has not yet been colored in phase 2, we color it with a color in $\{d^i_0, \ldots, d^i_7 \mid i \equiv \ell \bmod 6\}$ in a cyclic manner (i.e., the first vertex to be colored will take color d^i_0, and the next d^i_1, and so on).

Total number of colors used: The number of colors used in phase 1 and phase 2 are 6 and 48 respectively, giving a total of 54.

Correctness: We now prove that the assigned coloring is a CFON* coloring, by showing that every vertex has a uniquely colored neighbor.

Firstly, we consider a vertex v in $G[S_\ell]$, for some ℓ, and not in I. By definition of the set I, v is adjacent to its uniquely colored neighbor u colored by C, after phase 1. Suppose that u is not recolored in phase 2. Then since C is a CFCN* coloring of $G[S_\ell]$, v is adjacent to a uniquely colored neighbor $u \neq v$ in $G[S_\ell]$. By the coloring, the distance between v and other vertices in another stripe with the same color as u is at least $2\sqrt{3} > 2$. Hence, u is the uniquely colored neighbor of v in G.

Now suppose u is recolored in phase 2 to some color d^i_j. For a contradiction, suppose that v is also adjacent to another vertex w with the same color d^i_j. Then u and w must be the representative vertices of two vertices a and b in I that are in stripes $S_{\ell'}$ and $S_{\ell''}$, respectively, such that $\ell' \equiv \ell'' \bmod 6$. Since (a, u, v, w, b) forms a path in G, and since two adjacent vertices in G have Euclidean distance at most 2, we conclude that a and b are at the distance of at most 8. If $\ell' \neq \ell''$, then $|\ell' - \ell''| \geq 6$. This implies $|a_y - b_y| \geq 5\sqrt{3} > 8$, a contradiction. Hence, a and b are in the same stripe. Because u and w have the same color, there must be 7 other vertices in I between a and b in terms of the X-coordinate. By Lemma 28, this implies $|a_x - b_x| > 8$, another contradiction. Hence, v cannot be adjacent to two vertices of the same color.

Lastly, we consider a vertex v in I. Then the representative u of v is colored by D. For v to not have a uniquely colored neighbor, there should exist another representative vertex w such that $D(u) = D(w)$. As in the above paragraph, we can use distance arguments to note that two neighbors of v cannot be assigned the same color in phase 2. Thus we can conclude that v is not adjacent to any other vertex with the same color as $D(u)$. □

7 NP-completeness on Unit Square and Unit Disk Intersection Graphs

In this section, we show that the CFON* coloring problem is NP-hard for unit disk and unit square intersection graphs. The idea of the proofs is similar to the NP-completeness proofs in [7, 15] for the CFCN* coloring problem.

Theorem 29. *It is NP-complete to determine if a unit disk intersection graph can be CFON* colored using one color.*

Proof. Given a unit disk intersection graph $G = (V, E)$, and a partial vertex-coloring using one color, we can verify in polynomial time whether the coloring is a CFON* coloring. To show that the problem is NP-hard, we give a reduction from Positive Planar 1-in-3-SAT. The input to Positive Planar 1-in-3-SAT is a Boolean CNF formula where each clause has exactly three literals with each literal being positive, and the clause-variable incidence graph is planar. The objective is to check if there exists an assignment of Boolean values to the variables such that each clause has exactly one true literal. This problem is known to be NP-hard, see Mulzer and Rote [30] for more details.

Given ϕ, an instance of Positive Planar 1-in-3-SAT, we construct a unit disk intersection graph $G(\phi)$ as follows.

Construction of the graph $G(\phi)$: Let $\{x_1, x_2, \ldots, x_n\}$ be the variables and $\{t_1, t_2, \ldots, t_k\}$ be the clauses of the formula ϕ. For each variable x_j, $1 \leq j \leq n$, we introduce a variable gadget which is isomorphic to a cycle of length $8k$, where k is the number of clauses. We start with an arbitrary vertex and name the vertex as $a_{1,j}$. The next three consecutive vertices (in anti-clockwise direction) are called $b_{1,j}$, $c_{1,j}$ and $d_{1,j}$. The vertices are named in sets of 4. After $a_{1,j}$, $b_{1,j}$, $c_{1,j}$ and $d_{1,j}$, the next four vertices are named $a_{2,j}$, $b_{2,j}$, $c_{2,j}$ and $d_{2,j}$, then $a_{3,j}$, $b_{3,j}$, $c_{3,j}$ and $d_{3,j}$, and so on till $a_{2k,j}$, $b_{2k,j}$, $c_{2k,j}$ and $d_{2k,j}$. See Figure 8 (right) for an illustration.

The clause gadget is illustrated in Figure 8 (left). Each clause t_ℓ, $1 \leq \ell \leq k$ is represented by a *clause vertex* c_ℓ. The vertex c_ℓ is adjacent to a tree of five vertices, depicted below c_ℓ in the figure. Additionally, there are three paths connecting the clause vertex c_ℓ with the three corresponding variable gadgets; for each variable x_j of the clause t_ℓ, a path connects c_ℓ with the vertex $a_{y,j}$ of the corresponding gadget, for a suitable $y \in [2k]$. While choosing y, we ensure that a vertex $a_{y,j}$ connects to at most one clause gadget. The length of each path (defined as the number of vertices excluding c_ℓ and the vertex $a_{y,j}$ in the variable gadget) is a multiple of 4. For illustration, we show such a path in Figure 9.

We now argue that the graph $G(\phi)$ is a unit disk intersection graph and can be constructed in polynomial time. The arguments are similar to those in [15]. We start with a planar embedding of the clause-variable incidence graph of ϕ. We transform all the curved edges in the embedding into straight line segments with vertices placed on an $O(n + k) \times O(n + k)$ grid. Fraysseix, Pach and Pollack [11] showed that such a straight line segment embedding can be obtained in polynomial time. We spread out the vertices in this embedding to ensure that the clause and variable gadgets can be accommodated with adequate distance between them. The clause vertex in the embedding is replaced by the clause gadget and the variable vertex is replaced with variable gadget.

The edges between variables and clauses are replaced by paths whose lengths are multiples of 4. We perform some local shifting (we move the vertices of the path by a small distance, retaining the adjacencies) to ensure that the path lengths are multiples of 4. When connecting a clause gadget to a variable gadget x_j, we choose an $a_{y,j}$ that is not already connected to any clause gadget. Note that we may have to bend some paths while trying to make the connections, and

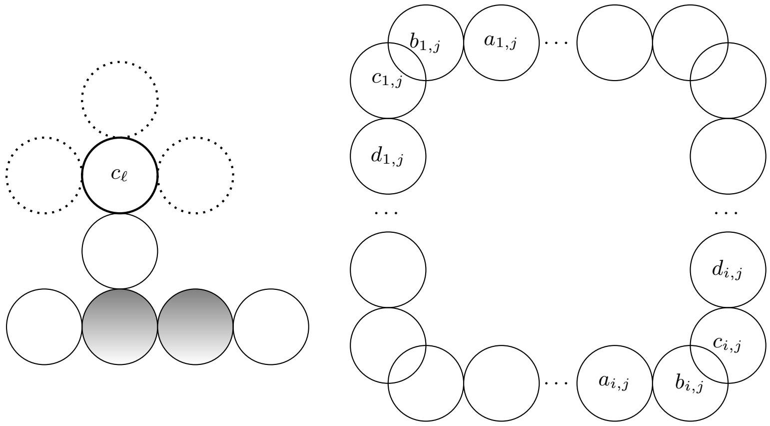

Fig. 8. The clause gadget is on the left. The dotted disks around the clause vertex c_ℓ indicate the connection with the variable gadgets. The shaded vertices force the clause vertex to not draw its uniquely colored neighbor from within the clause gadget.
On the right side, we have the variable gadget for x_j.

ensure that the connecting paths between clause gadgets and variable gadgets do not intersect.

Below, we show that $G(\phi)$ is CFON* colorable using one color if and only if ϕ is 1-in-3-satisfiable.

Observation 30 *In any CFON* coloring of $G(\phi)$ using one color, each clause vertex c_ℓ, where $1 \leq \ell \leq k$, remains uncolored. Further, the uniquely colored neighbor of c_ℓ is not from the tree adjacent to it.*

Proof. Consider the tree of 5 vertices adjacent to c_ℓ. In any CFON* coloring, the shaded vertices of the tree (See Figure 8) are forced to be assigned the non-zero color because of its pendant neighbors. This forces the remaining three vertices of the tree and c_ℓ to remain uncolored. Since the neighbor of c_ℓ in the tree is uncolored, its uniquely colored neighbor does not belong to the tree. $\square$

Lemma 31. *If $G(\phi)$ is CFON* colorable using one color, then ϕ is 1-in-3-satisfiable.*

Proof. Let $G(\phi)$ have a CFON* coloring using one color. We first consider the clause gadgets. Let $m_\ell^1, m_\ell^2, m_\ell^3 \in N(c_\ell)$ be the first vertices of the paths that connect the clause vertex c_ℓ to each of the variable gadgets. Let the paths from c_ℓ to the variable gadgets terminate at the vertices a_{y_1,j_1}, a_{y_2,j_2} and a_{y_3,j_3} respectively. As noted in Observation 30, one of these vertices has to be the uniquely colored neighbor of c_ℓ. Without loss of generality, let m_ℓ^1 be the uniquely colored neighbor of c_ℓ. This implies that m_ℓ^2 and m_ℓ^3 are not colored. Let g_ℓ^1 be the last vertex in the path that connects c_ℓ to the variable gadget. Recall that the length

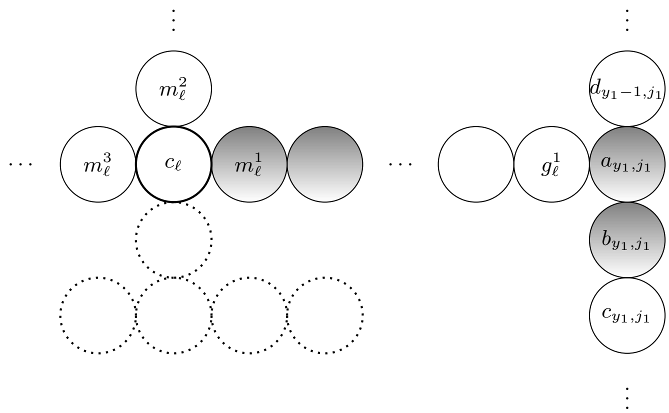

Fig. 9. Illustration of a CFON* coloring of the path connecting the clause vertex c_ℓ to the vertex a_{y_1,j_1} in the variable gadget of x_{j_1}. The dotted vertices are a part of the clause gadget.

of the path from m_ℓ^1 to g_ℓ^1 connecting c_ℓ to a_{y_1,j_1}, in the corresponding variable gadget, is a multiple of four. Consider the path starting from the colored vertex m_ℓ^1 to g_ℓ^1. Since m_ℓ^1 is colored, it follows that in any CFON* coloring using one color, the first two vertices have to be colored, the next two vertices have to be uncolored, the next two have to be colored and so on. Consequently, the vertex g_ℓ^1 is uncolored and a_{y_1,j_1} is colored (to be the uniquely colored neighbor of g_ℓ^1). An illustration is given in Figure 9.

Now consider the path from m_ℓ^2 (resp. m_ℓ^3) to its corresponding variable gadget. The starting vertex m_ℓ^2 (resp. m_ℓ^3) is not colored. The next two vertices have to be colored, followed by two uncolored vertices, then followed by two colored vertices and so on. The last vertex g_ℓ^2 (resp. g_ℓ^3) in the path will be uncolored. This implies that the vertices a_{y_2,j_2} and a_{y_3,j_3} in the corresponding variable gadgets are uncolored.

Notice that for $i \in \{1,2,3\}$ the vertex g_ℓ^i is uncolored. Hence the vertex $a_{y_i j_i}$, for $i \in \{1,2,3\}$, has its uniquely colored neighbor within the variable gadget of x_{j_i}. Because of this, we also have the same coloring pattern along the cycle of any variable gadget, i.e. a pair of colored vertices followed by a pair of uncolored vertices repeating. This implies that in any CFON* coloring using one color, for each variable gadget of a variable x_j, the vertices $a_{y,j}$, for $1 \leq y \leq 2k$, are either all colored or all uncolored. In addition, as argued above, if $a_{y,j}$ is connected to a clause vertex c_ℓ, then $a_{y,j}$ is colored if and only if the corresponding adjacent vertex m_ℓ^i of c_ℓ is colored.

To obtain the desired 1-in-3-satisfying assignment of ϕ, we consider the vertices $a_{y,j}$ in the clause gadget corresponding to x_j. If the vertices $a_{y,j}$ for $1 \leq y \leq 2k$ are colored in the CFON* coloring, we set x_j to true. Else we set x_j to false. The arguments above imply that for each clause t_ℓ, exactly one of the variables x_j in the clause will be set to true. This implies that ϕ is 1-in-3-satisfiable. □

Lemma 32. *If ϕ is 1-in-3-satisfiable, then $G(\phi)$ is CFON* colorable using one color.*

Proof. Consider a 1-in-3-satisfying assignment of ϕ. For each variable x_j in ϕ that is set to true, color all vertices $a_{y,j}$ and $b_{y,j}$ for each $1 \leq y \leq 2k$. Else, color all vertices $b_{y,j}$ and $c_{y,j}$ for each $1 \leq y \leq 2k$. In either case, the remaining vertices in the gadget are left uncolored. Such a coloring ensures that every vertex in the variable gadget associated with x_j has a uniquely colored neighbor from the gadget. Figure 9 illustrates the case when all vertices $a_{y,j}$ and $b_{y,j}$ are colored.

Consider the case when all the vertices $a_{y,j}$ are colored for $1 \leq y \leq 2k$. Suppose a vertex $a_{y',j}$ is adjacent to a vertex g_ℓ^i along the path to a clause gadget representing the clause t_ℓ. We leave g_ℓ^i uncolored along with its other neighbor in $N(g_\ell^i) \setminus \{a_{y',j}\}$ on the path. We now color the next two vertices on the path, leave the next two vertices uncolored and so on till we reach m_ℓ^i (which is the vertex adjacent to c_ℓ). Since the length of the path is a multiple of four, the vertices m_ℓ^i and $N(m_\ell^i) \setminus \{c_\ell\}$ will be colored.

The other case is that all $a_{y,j}$ are left uncolored in the variable gadget of x_j. Consider the connecting paths to the clause gadgets starting from a vertex say g_ℓ^i (adjacent to $a_{y',j}$) and ending at m_ℓ^i, where m_ℓ^i is adjacent to the vertex c_ℓ from the clause gadget. In this case we leave g_ℓ^i uncolored, coloring the next two vertices in the path, leaving the next two vertices uncolored and so on. The vertex m_ℓ^i is hence uncolored.

Since ϕ is positive planar 1-in-3-satisfiable, the assignment assigns true to exactly one variable of each clause t_ℓ. This ensures that there is exactly one colored neighbor of c_ℓ. The rest of the vertices in the clause gadget can easily be CFON* colored. We have a CFON* coloring using one color according to the above rules. □

Lemmas 31 and 32 imply that $G(\phi)$ is CFON* colorable using one color if and only if ϕ is 1-in-3-satisfiable. □

Theorem 33. *It is* NP*-complete to determine if a unit square intersection graph can be CFON* colored using one color.*

Proof. The reduction is from Positive Planar 1-in-3 Sat, and similar to the reduction in the proof of Theorem 29. The graphs corresponding to the clause and the variable gadgets are the same as the ones used in the proof of Theorem 29. The clause and variable gadgets can be realised as unit square graphs. For instance, see Figure 10 for an illustration of the clause gadget. □

8 Kneser Graphs

In this section, we study CFON* and CFCN* colorings of Kneser graphs. We shall use the words *κ-set* or *κ-subset* to refer to a set of size κ. We shall sometimes

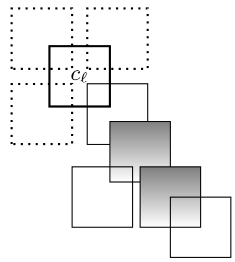

Fig. 10. The clause gadget corresponding to the clause t_ℓ. The dotted squares around the clause vertex c_ℓ indicate the connection with the variable gadgets. The shaded vertices force c_ℓ to not draw its uniquely colored neighbor from within the clause gadget.

refer to the κ-subsets of $[n]$ and the vertices of $K(n,\kappa)$ in an interchangeable manner. We also use the symbol $\binom{S}{\kappa}$ to denote the set of all κ-subsets of a set S.

Definition 34 (Kneser graph). *The Kneser graph $K(n,\kappa)$ is the graph whose vertices are $\binom{[n]}{\kappa}$, the κ-sized subsets of $[n]$, and the vertices x and y are adjacent if and only if $x \cap y = \emptyset$ (when x and y are viewed as sets).*

Observe that for $n < 2\kappa$, $K(n,\kappa)$ has no edges, and for $n = 2\kappa$, $K(n,\kappa)$ is a perfect matching. Since we are only interested in connected graphs, we assume $n \geq 2\kappa + 1$. For this range of values of n, we show that $\chi^*_{ON}(K(n,\kappa)) \leq \kappa + 1$. Further, we prove that this bound is tight for $n \geq 2\kappa^2 + \kappa$. We conjecture that this bound is tight for all $n \geq 2\kappa + 1$. In addition, we also show an upper bound for $\chi^*_{CN}(K(n,\kappa))$.

Theorem 35. *For $n \geq 2\kappa^2 + \kappa$, $\chi^*_{ON}(K(n,\kappa)) = \kappa + 1$.*

The above theorem is an immediate corollary of the two lemmas below.

Lemma 36. *$\kappa + 1$ colors are sufficient to CFON* color $K(n,\kappa)$ for $n \geq 2\kappa + 1$.*

Proof. Consider the following assignment of colors to the vertices of $K(n,\kappa)$:

- For any vertex (κ-set) v that is a subset of $\{1, 2, \ldots, 2\kappa\}$, we assign $C(v) = \max_{\ell \in v} \ell - (\kappa - 1)$.
- All the remaining vertices are assigned the color 0.

For example, for the Kneser graph $K(n,3)$, we assign the color 1 to the vertex $\{1,2,3\}$, color 2 to the vertices $\{1,2,4\}, \{1,3,4\}, \{2,3,4\}$, color 3 to the vertices $\{1,2,5\}, \{1,3,5\}, \{1,4,5\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}$, color 4 to the vertices $\{1,2,6\}, \{1,3,6\}, \{1,4,6\}, \{1,5,6\}, \{2,3,6\}, \{2,4,6\}$, $\{2,5,6\}, \{3,4,6\}$, $\{3,5,6\}$, $\{4,5,6\}$, and color 0 to all the remaining vertices.

Now, we prove that the above coloring is a CFON* coloring. Let C_i be the set of all vertices assigned the color i. Notice that $C_1 \cup C_2 \cup \cdots \cup C_{\kappa+1} = \binom{[2\kappa]}{\kappa}$.

In other words, all the colored vertices induce a $K(2\kappa, \kappa)$, which, as observed at the beginning of this section, is a perfect matching. Thus each colored vertex has exactly one colored vertex as its neighbor, which serves as its uniquely colored neighbor.

Now we have to show the presence of uniquely colored neighbors for vertices that have some elements not contained in $[2\kappa]$. Let v be such a vertex. That is, $v \cap [2\kappa] \neq v$. Let t be the smallest nonnegative integer such that $|[\kappa + t] \setminus v| = \kappa$. Since v has at least one element not contained in $[2\kappa]$, t is at most $\kappa - 1$.

By construction, the vertex $u = [\kappa+t] \setminus v$ has color $t+1$ and is adjacent to v. Also by construction, $[\kappa+t]$ contains exactly κ elements not in v and all these κ elements are in u. Hence, for another vertex with color $t+1$, all of its κ elements are in $[\kappa + t]$ and at least one of them is contained in v. This implies that no other neighbors of v have color $t+1$, and u is the uniquely colored neighbor of v. □

Lemma 37. *$\kappa + 1$ colors are necessary to CFON* color $K(n, \kappa)$ when $n \geq 2\kappa^2 + \kappa$.*

Proof. We prove this by contradiction. Suppose that $n \geq 2\kappa^2 + \kappa$ and $K(n, \kappa)$ can be colored using the κ colors $1, 2, 3, \ldots, \kappa$, besides the color 0. For each i, $1 \leq i \leq \kappa$, let C_i denote the set of all vertices colored with the color i.

We will show that there exists a vertex x that does not have a uniquely colored neighbor, i.e., $|N(x) \cap C_i| \neq 1$, for all i, $1 \leq i \leq \kappa$. We construct the vertex (κ-set) x, by choosing elements in it as follows. Suppose that there are C_i's that are singleton, i.e., $|C_i| = 1$. For every i, $1 \leq i \leq \kappa$ such that $|C_i| = 1$ add to x an arbitrary element from the lone vertex in C_i. In other words, we choose elements in x so as to ensure that x intersects with the vertices in all the singleton C_i's. This partially constructed x may also intersect with vertices in other color classes. Some of the other C_i's might become "effectively singleton", that is x may intersect with all the vertices in those C_i's except one. We now choose further elements in x so that x intersects these effectively singleton C_i's too. Finally, we terminate this process when all the remaining C_i's are not singleton.

At this stage, if x has exactly κ elements, then it must be the case that x intersects with all the vertices in all the C_i's. Hence no colored vertices are adjacent to x.

Otherwise, the number of elements in x is $t < \kappa$. There are two possible subcases. The first subcase is when x intersects with all the colored vertices. In this case, we add $\kappa - t$ arbitrary elements to x from $[n] \setminus x$. This vertex x is not adjacent to any colored vertex. The second subcase is when there are color classes that do not become effectively singleton. This is because each of these color classes contain at least two vertices that do not intersect with x. For each of these color class(es) C_j, we choose two distinct vertices, say $y_j, y'_j \in C_j$. We choose the remaining elements of x so that $x \cap y_j = \emptyset$ and $x \cap y'_j = \emptyset$. The number of such sets C_j is $\kappa - t$. So for choosing the remaining $\kappa - t$ elements of x, we have at least $n - t - 2\kappa(\kappa - t)$ choices. The t elements already present in x cannot be used again. There could be a maximum of $\kappa - t$ color classes C_j which do not become effectively singleton. In each of these color classes, we want to avoid intersecting two vertices each, which forbids a maximum of $2\kappa(\kappa - t)$ elements. Because $n \geq 2\kappa^2 + \kappa$, it follows that the available $n - t - 2\kappa(\kappa - t)$ choices suffice to fill up the remaining $\kappa - t$ elements in x. Thus in this subcase, by construction, we ensure that x has no neighbors in the color classes that become effectively singleton, and has at least two neighbors in the remaining color classes. □

Next, we consider CFCN* coloring of Kneser graphs. Observe that since the chromatic number of $K(n, \kappa)$ is $n - 2\kappa + 2$ [28], we have that $\chi_{CN}(K(n, \kappa)) \leq n - 2\kappa + 2$. We show the following:

Theorem 38. *When* $2\kappa + 1 \leq n \leq 3\kappa - 1$*, we have* $\chi^*_{CN}(K(n, \kappa)) \leq n - 2\kappa + 1$*. When* $n \geq 3\kappa$*, we have* $\chi^*_{CN}(K(n, \kappa)) \leq \kappa$*.*

Lemma 39. *When* $n \geq 2\kappa + 1$*, we have* $\chi^*_{CN}(K(n, \kappa)) \leq \kappa$*.*

Proof. We assign the following coloring to the vertices of $K(n, \kappa)$:

– For any vertex (κ-set) v that is a subset of $\{1, 2, \ldots, 2\kappa - 1\}$, we assign $C(v) = \max_{\ell \in v} \ell - (\kappa - 1)$.
– All the remaining vertices are assigned the color 0.

For $1 \leq i \leq \kappa$, let C_i be the color class of the color i. Notice that $C_1 \cup C_2 \cup \cdots \cup C_\kappa = \binom{[2\kappa-1]}{\kappa}$. Since any two κ-subsets of $\{1, 2, \ldots, 2\kappa - 1\}$ intersect, it follows that $\binom{[2\kappa-1]}{\kappa}$ is an independent set. Hence each of the color classes $C_1, C_2, \ldots, C_\kappa$ are independent sets, and each colored vertex serves as its own uniquely colored neighbor.

If v is assigned the color 0, then $v \not\subset [2\kappa - 1]$. That is, v has some elements not contained in $[2\kappa - 1]$. Let t be the smallest nonnegative integer such that $|[\kappa + t] \setminus v| = \kappa$. Since v has at least one element not contained in $[2\kappa - 1]$, t is at most $\kappa - 1$. We claim that the vertex $w = [\kappa + t] \setminus v$ is the only neighbor of v with color $t + 1$.

First note that $\kappa + t \notin v$, because otherwise, the minimality of t would not hold. It follows that the vertex w is colored $t + 1$. To show that w is the only neighbor of v with color $t + 1$, assume the contrary. Let w' be another neighbor of v that is colored $t + 1$. By the coloring used, $w' \subseteq [\kappa + t]$. Since $w \neq w'$, it follows that $|w \cup w'| \geq \kappa + 1$, and hence $|[\kappa + t] \setminus v| \geq \kappa + 1$. This again contradicts the choice of t. Thus w is a uniquely colored neighbor of v. □

Lemma 40. $\chi_{CN}(K(2\kappa + 1, \kappa)) = 2$*, for all* $\kappa \geq 1$*.*

Proof. Consider a vertex v of $K(2\kappa + 1, \kappa)$. If $v \cap \{1, 2\} \neq \emptyset$, we assign color 1 to v. Otherwise, we assign color 2 to v.

Let C_1 and C_2 be the sets of vertices colored 1 and 2 respectively. Below, we discuss the unique colors for every vertex of $K(n, \kappa)$.

– If $v \in C_1$ and $\{1, 2\} \subseteq v$, then v is the uniquely colored neighbor of itself. This is because all the vertices in C_1 contain either 1 or 2 and hence v has no neighbors in C_1.

– Let $v \in C_1$ and $|v \cap \{1,2\}| = 1$. WLOG, let $1 \in v$ and $2 \notin v$. In this case, v has a uniquely colored neighbor $w \in C_2$. This vertex w is the κ-set $w = [2\kappa+1] \setminus (v \cup \{2\})$.
– If $w \in C_2$, w is the uniquely colored neighbor of itself. This is because C_2 is an independent set. For two vertices $w, w' \in C_2$ to be adjacent, we need $|w \cup w'| = 2\kappa$, but vertices in C_2 are subsets of $\{3, 4, 5, \ldots, 2\kappa+1\}$, which has cardinality $2\kappa - 1$.

□

Lemma 41. *$\chi_{CN}(K(2\kappa+d, \kappa)) \leq d+1$, for all $\kappa \geq 1$.*

Proof. We prove this by induction on d. The base case of $d = 1$ is true by Lemma 40. Suppose $K(2\kappa+d, \kappa)$ has a CFCN coloring that uses $d+1$ colors. Let us consider $K(2\kappa+d+1, \kappa)$. For all the vertices of $K(2\kappa+d+1, \kappa)$ that appear in $K(2\kappa+d, \kappa)$ we use the same assignment as in $K(2\kappa+d, \kappa)$. The new vertices (the vertices that contain $2\kappa+d+1$) are assigned the new color $d+2$. As all the new vertices contain $2\kappa+d+1$, they form an independent set. Hence each of the new vertices serve as their own uniquely colored neighbor.

The vertices of $K(2\kappa+d+1, \kappa)$ already present in $K(2\kappa+d, \kappa)$ get new neighbors, but all the new neighbors are colored with the new color $d+2$. Hence the unique colors of the existing vertices are retained. □

Lemma 41 implies that $\chi^*_{CN}(K(n, \kappa)) \leq \chi_{CN}(K(n, \kappa)) \leq n - 2\kappa + 1$, when $n \geq 2\kappa+1$. So, from Lemma 39 and Lemma 41 we get Theorem 38.

$$\chi^*_{CN}(K(n, \kappa)) \leq \left\{ \begin{array}{ll} n - 2\kappa + 1, & \text{for } 2\kappa+1 \leq n \leq 3\kappa - 1 \\ \kappa, & \text{for } n \geq 3\kappa \end{array} \right\}.$$

9 Split Graphs

In this section, we study CFON* and CFCN* colorings of split graphs. We show that the CFON* coloring problem is NP-complete and the CFCN* coloring problem is polynomial time solvable.

Definition 42 (Split Graph). *A graph G, $G = (V, E)$, is a split graph if there exists a partition of its vertex set $V = K \cup I$ such that the graph induced by K is a clique and the graph induced by I is an independent set.*

Theorem 43. *The CFON* coloring problem is NP-complete on split graphs.*

Proof. We give a reduction from the classical Graph Coloring problem. Given an instance $(G = (V, E), k)$ of Graph Coloring, we construct an auxiliary graph G_1, $G_1 = (V_1, E_1)$ from G such that $V_1 = V \cup \{x, y\}$ and $E_1 = E \cup \{xy\} \cup \bigcup_{v \in V} \{xv, yv\}$. Note that $N(x) = V \cup \{y\}$ and $N(y) = V \cup \{x\}$. Now we construct the graph G_2, $G_2 = (V_2, E_2)$, from G_1 such that

$$V_2 = V_1 \cup \{I_{uv} \mid uv \in E_1\} \cup \{I_v \mid v \in V_1\}, \text{ and}$$

$$E_2 = \{uv \mid u, v \in V_1\} \cup \{uI_{uv}, vI_{uv} \mid uv \in E_1\} \cup \{uI_u \mid u \in V_1\}.$$

Note that G_2 is a split graph (K, I) with the clique $K = V_1$ and $I = V_2 \setminus V_1$. See Figure 11 for an illustration. The construction of the graph G_2 from G can be done in polynomial time. Let $I = I_1 \cup I_2$ where I_1 and I_2 represent the set of degree one vertices and the set of degree two vertices in I respectively.

Now, we argue that $\chi(G) \leq k$ if and only if $\chi^*_{ON}(G_2) \leq k{+}2$, where $k \geq 3$. We first prove the forward direction. Given a k-coloring C_G of G, we extend C_G to the coloring C_{G_2} for G_2 using $k{+}2$ colors. For all vertices $v \in V$, $C_{G_2}(v) = C_G(v)$. We assign $C_{G_2}(x) = k{+}1$, $C_{G_2}(y) = k{+}2$. All vertices in $I_1 \cup I_2$ are left uncolored. Every vertex $v \in K \setminus \{x\}$ has x as its uniquely colored neighbor whereas the vertex y is the uniquely colored neighbor for x. For each vertex $I_{uv} \in I_2$, we have $N(I_{uv}) = \{u, v\}$ and $C_{G_2}(u) \neq C_{G_2}(v)$. Hence the vertices u and v act as the uniquely colored neighbors for I_{uv}. Each vertex $I_u \in I_1$ will have the vertex u as its uniquely colored neighbor.

Now, we prove the converse. Given a CFON* $(k+2)$-coloring C_{G_2} of G_2, we show that the restriction of C_{G_2} to the vertices of G gives a proper k-coloring C_G of G. Observe that each vertex in K receives a non-zero color in any CFON* coloring of G_2, because it is adjacent to a degree-one vertex in I_1. For every edge $uv \in E_1$, we have $C_{G_2}(u) \neq C_{G_2}(v)$ as $N(I_{uv}) = \{u, v\}$. This implies that x and y do not share the same color with each other nor with other vertices in V. It also implies that for every edge $uv \in E$, we have $C_{G_2}(u) \neq C_{G_2}(v)$. Hence, the coloring C_{G_2} when restricted to the set $K \setminus \{x, y\} = V$ is a k-coloring of G. $\square$

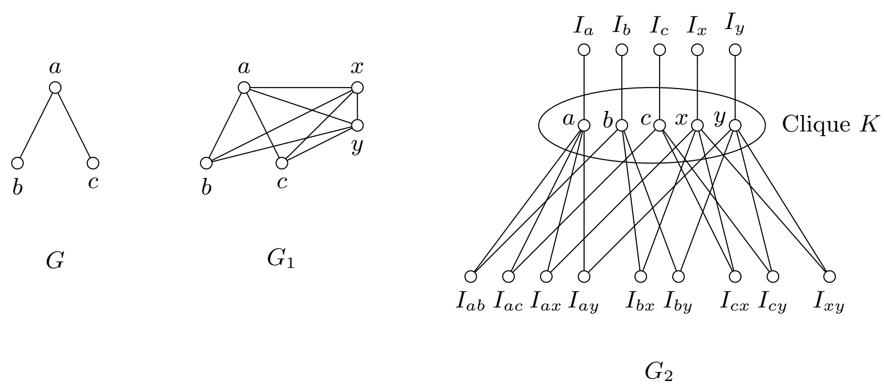

Fig. 11. Illustration of the graphs G (on the left), G_1 (in the middle) and G_2 (on the right). The vertices $\{a, b, c, x, y\}$ of G_2 drawn inside the ellipse form the clique K.

Theorem 44. *The CFCN* coloring problem is polynomial time solvable on split graphs.*

The proof of Theorem 44 is through a characterization. We first show that for split graphs G, $\chi^*_{CN}(G) \leq 2$. Then we characterize split graphs G for which $\chi^*_{CN}(G) = 1$ thereby proving Theorem 44.

Lemma 45. *If G is a split graph, then $\chi^*_{CN}(G) \leq 2$.*

Proof. Let $V = K \cup I$ be a partition of vertices into a clique K and an independent set I. We use $C : V \to \{1, 2, 0\}$ to assign colors to the vertices of V. Choose an arbitrary vertex $u \in K$ and assign $C(u) = 2$. The remaining vertices (if any) in $K \setminus \{u\}$ are assigned the color 0. For every vertex $v \in I$, we assign $C(v) = 1$. Each vertex in I will have itself as the uniquely colored neighbor and every vertex in K will have the vertex u as the uniquely colored neighbor. □

We now characterize all the split graphs that are CFCN* colorable using one color.

Lemma 46. *Let G be a split graph with $V = K \cup I$, where K and I are the clique and the independent set respectively. We have $\chi^*_{CN}(G) = 1$ if and only if at least one of the following is true: (i) G has a universal vertex, or (ii) $\forall v \in K, |N(v) \cap I| = 1$.*

Proof. We first prove the "if" statement. If there exists a universal vertex $u \in V$, then we assign the color 1 to u and assign the color 0 to all the other vertices. This is a CFCN* coloring.

Suppose that for each vertex $v \in K$, $|N(v) \cap I| = 1$. (Note that K cannot be empty because we assume G to be connected.) We assign the color 1 to each vertex in I and color 0 to the vertices in K. Each vertex in I acts as the uniquely colored neighbor for itself and for its neighbor(s) in K.

For showing the "only if" statement, let $C : V \to \{1, 0\}$ be a CFCN* coloring of G. We further assume that there exists $y \in K$ such that $|N(y) \cap I| \neq 1$ and show that there exists a universal vertex. We assume that $|K| \geq 2$ and $|I| \geq 1$ (if either assumption is violated, G has a universal vertex). We first prove the following claim.

Claim. Exactly one vertex in K is assigned the color 1.

Proof. Suppose that there are two vertices $v, v' \in K$ such that $C(v) = C(v') = 1$. Then none of the vertices in K have a uniquely colored neighbor.

Suppose that all vertices in K are assigned the color 0. For vertices in I to have a uniquely colored neighbor, each vertex in I has to be assigned the color 1. By assumption, there is a vertex $y \in K$ such that $|N(y) \cap I| \neq 1$. This means that y does not have a uniquely colored neighbor.

In either case, it follows that C is not a CFCN* coloring of G, which is a contradiction. □

By the above claim, there is a unique vertex $v \in K$ such that $C(v) = 1$. We will show that v is a universal vertex. If not, there is a $w' \in I$ such that $w' \notin N(v) \cap I$. For w' to have a uniquely colored neighbor, either w' or one of its neighbors in K has to be assigned the color 1. The latter is not possible because v is the lone vertex in K that is colored 1. If $C(w') = 1$, then its neighbor(s) in K does not have a uniquely colored neighbor because of the vertices w' and v. Hence, v is a universal vertex. □

By Lemmas 45, 46, and the fact that conditions in the latter lemma can be checked in polynomial time, we obtain Theorem 44.

10 Conclusion

In the preliminary version of our paper [5], we had shown that the conflict-free coloring problem is FPT when parameterized by combined parameters clique-width w and number of colors k. Since the problem is NP-hard for any $k \geq 3$, the problem is not FPT when parameterized by k unless $\mathsf{P} = \mathsf{NP}$. As we have shown in Theorems 6 and 7, the conflict-free chromatic numbers are not bounded by a function of the clique-width. Therefore it remains an open question if there exists an FPT algorithm with only clique-width as a parameter.

Recently, Gonzalez and Mann [21] showed that both open neighborhood and closed neighborhood variants are polynomial time solvable when mim-width and the number of colors are bounded. In particular, they design XP algorithms in terms of mim-width and k. Since mim-width generalizes clique-width, it is interesting to see if there exists an FPT algorithm parameterized by mim-width and k.

Further, we presented an upper bound of conflict-free chromatic numbers for several graph classes. For most of them we established graph classes that match or almost match the upper bounds for their respective conflict-free chromatic numbers. For unit square and square disk graphs there is still a wide gap, and it would be interesting to improve those bounds.

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