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THE VORTICITY EQUATION ON A ROTATING SPHERE AND THE SHALLOW FLUID APPROXIMATION

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Abstract. The material conservation of vorticity in fluid flows confined to a thin layer on the surface of a large rotating sphere, is a central result of geophysical fluid dynamics. In this paper we revisit the conservation of vorticity in the context of global scale flows on a rotating sphere. Starting from the vorticity equation instead of the Euler equation, we examine the kinematical and dynamical assumptions that are necessary to arrive at this result. We argue that, in contrast to the planar case, a two-dimensional velocity field does not lead to a single component vorticity equation on the sphere. The shallow fluid approximation is then used to argue that only one component of the vorticity equation is significant for global scale flows. Spherical coordinates are employed throughout, and no planar approximation is used.

1. **Introduction.** The evolution of the vorticity field in a fluid flow is governed by the dynamical vorticity equation, and understanding the behavior of the vorticity often provides crucial information about the nature of the flow [1, 2]. The material conservation of vorticity [18] plays a crucial role in our understanding of many geophysical flows of interest [6, 3], see [21, 20, 16, 19] for some recent studies in this regard. The vorticity equation is obtained by eliminating the pressure (or geopotential) terms from the Euler equations, which are the governing equations of fluid flow. An alternative formulation is the streamfunction-vorticity formulation of the equations of motion [2], which also forms the basis for numerical approaches to geophysical problems [4]. The theoretical importance of vorticity may be contrasted with the fact that the vorticity is usually calculated from velocity measurements, rather than being measured directly [4]. In this paper, we focus on the vorticity equation itself, for a discussion of vortex solutions in geophysical flows, see [12], for a discussion of vortex motion on a sphere see [8, 10].

In the context of geophysical flows, the earth can be considered to be a sphere with a radius equal to the mean radius of the oblate spheroidal earth, $R \approx 6368$ km [3, 4, 5]. A spherical coordinate system is a natural choice to describe flows on the surface of the earth. For an early study utilising spherical coordinates to study waves on a rotating sphere see [11], and for more recent studies with applications to geophysical flow features such as the Equatorial Undercurrent and the Antarctic Circumpolar Current, see [21, 22]. We are mostly concerned with application of

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our discussion of the vorticity equation to large scale, global flows in the ocean, and for such flows the horizontal velocity scales are taken to be $U = 0.1$ m/s [3, 4]. We can calculate the time scale T from the global length scale R and the velocity scale U to be $T = R/U = 6.368 \times 10^7$ s, or equivalently $T \approx 2.02$ years. This is in agreement with the time scale corresponding to the large scale motions (eg. gyres) in the ocean [5]. The earth rotates with an angular velocity $\Omega = 7.29 \times 10^{-5}$ rad/s which corresponds to the duration of a sidereal day [4]. We evaluate below a few numbers for later use in this paper. This first set of numbers is valid for large scale flows in the ocean:

$$\frac{U^2}{R^2} = 2.46 \times 10^{-16}\ \mathrm{s}^{-2} \quad \text{and} \quad \frac{U\Omega}{R} = 1.14 \times 10^{-12}\ \mathrm{s}^{-2}. \tag{1}$$

If we consider large scale flows in the atmosphere, then the velocity scale changes to $U_{\mathrm{a}} = 10$ m/s [3], and the time scale changes to $T_a = 6.638 \times 10^5$ s ≈ 7.4 days. This time scale is in accordance with the time scale for large scale motions in the atmosphere (eg. cyclones) [4]. The numbers in (1) become

$$\frac{U_a^2}{R^2} = 2.46 \times 10^{-12}\ \mathrm{s}^{-2} \quad \text{and} \quad \frac{U_a\Omega}{R} = 1.14 \times 10^{-10}\ \mathrm{s}^{-2}. \tag{2}$$

The ratios of different length scales in geophysical flows are of central importance in theoretical analyses. Let $H = 10$ km be the typical vertical (i.e. radial) length scale for both oceanic and atmospheric flows. We can use the same length scale in both cases since, the average depth of the oceans is $O(10$ km$)$ (and not $o(10$ km$)$), and the height of the atmospheric layer relevant to weather phenomena is also $O(10$ km$)$ [4, 5]. The length scale H is much smaller than the radius of the earth R. We define the shallow fluid approximation as the case when in an expansion of the small parameter

$$\varepsilon \equiv \frac{H}{R} \approx 1.57 \times 10^{-3}, \tag{3}$$

we only retain terms of the leading order. The large scale flow of a fluid on the surface of a stationary or rotating sphere can effectively be regarded as two-dimensional (2D) flow due to the smallness of ε. The flow of water and air in the earth's oceans and atmosphere is a prototypical example of such flows.

For our purposes, we consider the fluid flow to be incompressible since, the variation of density in the ocean is very small, as is the variation in density in the first 10 km of the atmosphere above the surface of the earth [4, 5]. The effects of viscosity are neglected throughout this paper since the relevant Reynolds number is very large, $O(10^{10})$, and the Ekman number is very small, $O(10^{-15})$.

This paper is organised as follows. We consider first fluid flow on the surface of a stationary sphere in §2. We discuss the general vorticity equation in an inertial reference frame in §2.1 and consider the two-dimensional planar vorticity equation in §2.2. In §2.3, we consider 2D flows on a stationary sphere and obtain the required form of the velocity field and the corresponding vorticity equation. We introduce scaling arguments to neglect the horizontal (i.e. zonal and meridional) components of the vorticity equation. In §3, we discuss fluid flow on the surface of a rotating sphere. The vorticity equation for flow in a non-inertial reference frame is discussed in §3.1 and 2D flow on a rotating sphere is considered in §3.2, once again making use of scaling arguments. The streamfunction-vorticity formulation of the governing equations is derived in §4. An appendix with vector calculus identities and their application to vector calculus algebra in spherical coordinates has been provided at the end. These identities are used throughout the rest of this paper.

2. **Flow on the surface of a stationary sphere.**

2.1. **Vorticity equation for flow in an inertial reference frame.** Consider the three-dimensional (3D) flow of an inviscid fluid in an inertial frame of reference. We use vector notation and take the position vector in the inertial frame to be denoted by $\boldsymbol{r}$, time by t, the density of the fluid by $\rho = \rho(\boldsymbol{r}, t)$, the velocity field by $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r}, t)$, the pressure field by $p = p(\boldsymbol{r}, t)$ and the body forces acting on the fluid (per unit mass of the fluid) by $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}, t)$. The slight abuse of notation committed here by providing the same names to variables and their corresponding functions should not cause any confusion. Conservation of momentum for this fluid flow takes the form of the Euler equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{\boldsymbol{\nabla} p}{\rho} + \boldsymbol{F}, \tag{4}$$

and conservation of mass takes the form of the 'continuity equation'

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0. \tag{5}$$

Here $\boldsymbol{\nabla}$ is the gradient operator in three dimensions. The reader may consult the standard textbooks of fluid dynamics for derivation and discussion of these formulas [1, 2].

We denote the material derivative operator i.e. the derivative following a fluid element, by

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}). \tag{6}$$

The material derivative may act on any scalar or vector quantity of interest. We can rewrite (4) in terms of the material derivative as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{\boldsymbol{\nabla} p}{\rho} + \boldsymbol{F}, \tag{7}$$

with the material derivative acting on the velocity field in this case. We can also rewrite (5) using (6) and (A.1c) as

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{8}$$

with the material derivative acting on the density field in this case.

The vorticity field $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{r}, t)$ for a fluid flow is defined as

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}, \tag{9}$$

and has the interpretation of being twice the local angular velocity in the fluid [2]. An equation of motion for the vorticity field may be derived by taking the curl of (4). Assuming that the derivatives are continuous allows us to apply the curl and the time derivative interchangeably, and we get

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\nabla} \times (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} \times \left(\frac{\boldsymbol{\nabla} p}{\rho}\right) + \boldsymbol{\nabla} \times \boldsymbol{F}, \tag{10}$$

Using the standard vector calculus identities (A.1) we can rewrite some of the terms in this equation as

$$\boldsymbol{\nabla} \times (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla} \times \left[\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u}) + \boldsymbol{\omega} \times \boldsymbol{u}\right] \tag{11a}$$

$$= \boldsymbol{\nabla} \times (\boldsymbol{\omega} \times \boldsymbol{u}) \tag{11b}$$

$$= (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\omega}(\boldsymbol{\nabla} \cdot \boldsymbol{u}), \tag{11c}$$

$$-\boldsymbol{\nabla} \times \left(\frac{\boldsymbol{\nabla} p}{\rho}\right) = \frac{\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} p}{\rho^2}. \tag{11d}$$

Substituting into (10), we obtain the <u>vorticity equation</u> in an inertial frame of reference:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\omega}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = \frac{\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} p}{\rho^2} + \boldsymbol{\nabla} \times \boldsymbol{F}. \tag{12}$$

The basic texts [1, 2, 3] may be consulted for discussion and interpretation of the various terms in the vorticity equation.

We can simplify the vorticity equation if we assume that the fluid is incompressible, barotropic and is acted on only by conservative body forces such as gravity. Since the fluid is incompressible the density of every fluid element is constant i.e. $D\rho/Dt = 0$, and the continuity equation (8) takes the form

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0. \tag{13}$$

As stated in §1, we neglect the effects of density stratification throughout this paper. In this case the continuity equation reduces to the incompressibility condition (13) [3]. The fluid is said to be barotropic [3] if the pressure and density fields satisfy a relationship of the form $p = p(\rho)$ at every point in the flow, in which case we will have

$$\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} p = 0. \tag{14}$$

Fluid flows in which (14) is not satisfied are said to be baroclinic, and are associated with the production of vorticity [5]. In this paper, we are interested in the evolution of vorticity, and so consider the fluid to be barotropic. If the body forces acting on the fluid are conservative (such as gravity and the centrifugal force), then by definition we will have

$$\boldsymbol{\nabla} \times \boldsymbol{F} = 0. \tag{15}$$

Substituting (13), (14) and (15) into (12) we get the vorticity equation for the flow of an incompressible, barotropic fluid, acted on by conservative body forces, in the form [1, 2]

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} = 0. \tag{16}$$

2.2. **Two-dimensional planar flow.** The vorticity equation (16) simplifies further if we consider two-dimensional (2D) flows. Let us first consider a globally planar 2D flow. Let the flow be described by the Cartesian co-ordinate system (x, y, z) with unit vectors $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$. The velocity field for a planar 2D flow can be defined in Cartesian coordinates as

$$\boldsymbol{u} = u(x, y, t)\,\boldsymbol{e}_x + v(x, y, t)\,\boldsymbol{e}_y, \tag{17}$$

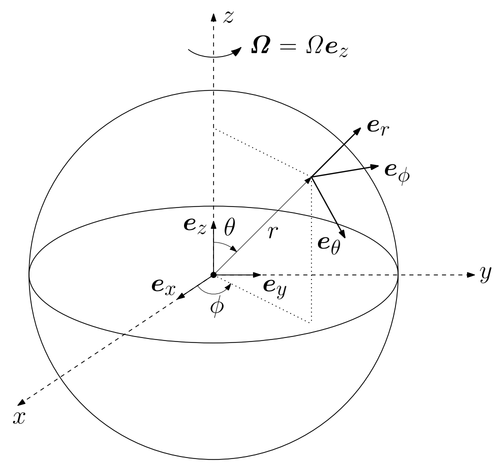

FIGURE 1. A spherical co-ordinate system (r, θ, ϕ), with θ being the polar angle (or colatitude) and ϕ (azimuth) defined with respect to the x-axis of the corresponding Cartesian system (x, y, z). In this paper, we consider a stationary sphere, as well as a rotating sphere with angular velocity $\boldsymbol{\Omega} = \Omega \boldsymbol{e}_z$.

where u and v are respectively the x- and y-components of the velocity. These components are independent of the z-coordinate since the flow is 2D. The corresponding vorticity field can be obtained from (9) as

$$\boldsymbol{\omega} = \zeta(x, y, t)\, \boldsymbol{e}_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \boldsymbol{e}_z, \tag{18}$$

and hence for such a 2D planar flow the vorticity has only one non-zero component viz. $\zeta(x, y, t)$, that is everywhere perpendicular to the velocity field. The vorticity equation (16) in this case has a single non-zero component (the $\boldsymbol{e}_z$ component):

$$\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} = 0, \tag{19}$$

so that the vorticity equation is a transport equation for $\zeta(x, y, t)$. Indeed (19) is equivalent to

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} = 0, \tag{20}$$

which shows the dynamical conservation of vorticity of every fluid particle in planar 2D flow [1, 2].

2.3. **2D flow on a stationary sphere.** If we consider a 2D fluid flow which is not globally planar, such as flow on the surface of a sphere, the vorticity need not have a single non-zero component. It is convenient to work in the spherical coordinate system (r, θ, ϕ) with unit vectors $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$, see Fig. (1). In order to bring out the subtleties and differences in this case when compared to 2D flow on a plane, we start with the spherical coordinate form of (16). Let us define the 3D velocity field

$\boldsymbol{u}$ and vorticity field $\boldsymbol{\omega}$ in component form as

$$\boldsymbol{u} = u_r \boldsymbol{e}_r + u_\theta \boldsymbol{e}_\theta + u_\phi \boldsymbol{e}_\phi \tag{21a}$$

$$\boldsymbol{\omega} = \omega_r \boldsymbol{e}_r + \omega_\theta \boldsymbol{e}_\theta + \omega_\phi \boldsymbol{e}_\phi. \tag{21b}$$

The components of the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ have the expressions in spherical coordinates

$$\omega_r = \frac{1}{r}\left[\frac{\partial u_\phi}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \phi} + u_\phi \cot\theta\right] \tag{22a}$$

$$\omega_\theta = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \tag{22b}$$

$$\omega_\phi = \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\theta}{r}, \tag{22c}$$

which can be derived using the identities (A.8). Substituting (21) into (16) and using the identities (A.9), we obtain the three components of the vorticity equation in spherical coordinates

$$\boldsymbol{e}_r : \frac{\partial \omega_r}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\omega_r - (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})u_r = 0 \tag{23a}$$

$$\boldsymbol{e}_\theta : \frac{\partial \omega_\theta}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\omega_\theta - (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})u_\theta + \frac{\omega_r u_\theta}{r} - \frac{\omega_\theta u_r}{r} = 0 \tag{23b}$$

$$\boldsymbol{e}_\phi : \frac{\partial \omega_\phi}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\omega_\phi - (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})u_\phi + \frac{\omega_r u_\phi}{r} - \frac{\omega_\phi u_r}{r} + \frac{\omega_\theta u_\phi \cot\theta}{r} - \frac{\omega_\phi u_\theta \cot\theta}{r} = 0. \tag{23c}$$

In analogy with 2D fluid flow on a plane, a 2D flow on the surface of a sphere may be defined in spherical coordinates as

$$\boldsymbol{u}(\theta,\phi,t) = u_\theta(\theta,\phi,t)\boldsymbol{e}_\theta + u_\phi(\theta,\phi,t)\boldsymbol{e}_\phi, \tag{24}$$

where the velocity is independent of the radial coordinate and u_θ, u_ϕ are the θ- and ϕ-components respectively of the velocity. The vorticity (22) may then be expected to be purely radial. In contrast to 2D planar flow however, it is seen from (22) that the corresponding vorticity field is not 2D, but instead has non-zero θ- and ϕ-components which are

$$\omega_\theta = -\frac{u_\phi}{r} \qquad \omega_\phi = \frac{u_\theta}{r}. \tag{25a}$$

These components may not in general be neglected arbitrarily for flows on the entire surface of the sphere. Some of the terms in the vorticity originate from the variation of the unit vectors over the surface of the sphere, for example ω_ϕ originates from

$$\boldsymbol{\nabla} \times (u_\theta \boldsymbol{e}_\theta) = u_\theta(\boldsymbol{\nabla} \times \boldsymbol{e}_\theta) + \boldsymbol{\nabla} u_\theta \times \boldsymbol{e}_\phi \tag{26a}$$

$$= \frac{u_\theta}{r}\boldsymbol{e}_\phi + \text{other terms}, \tag{26b}$$

where the identities (A.1) and (A.6) have been used. Further, some of the terms in the different components of the vorticity share a common origin. For example, we have

$$\boldsymbol{\nabla} \times (u_\phi \boldsymbol{e}_\phi) = u_\phi(\boldsymbol{\nabla} \times \boldsymbol{e}_\phi) + \boldsymbol{\nabla} u_\phi \times \boldsymbol{e}_\phi \tag{26c}$$

$$= \frac{u_\phi}{r}\left(\cot\theta\, \boldsymbol{e}_r - \boldsymbol{e}_\theta\right) + \text{other terms}, \tag{26d}$$

where, again, the identities (A.1) and (A.6) have been used. Thus the terms $u_\phi \cot\theta/r$ in ω_r and $-u_\phi/r$ in ω_θ originate from the variation of $\boldsymbol{e}_\phi$ over the surface of the sphere. In the range $0 \leq \theta \leq \pi$, we have $-\infty < \cot\theta < +\infty$, and therefore $|\cot\theta|$ does not dominate over unity at all latitudes.

Lemma 2.1. *We can choose the form of the 2D velocity field $\boldsymbol{u}$ such that the vorticity has zero θ- and ϕ-components.*

It is seen from (22) that this velocity field should obey

$$\frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r} = 0 \quad \text{and} \quad \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} = 0, \tag{27}$$

and that this unique choice is

$$\boldsymbol{u}(r,\theta,\phi,t) = \frac{R}{r}\left[v(\theta,\phi,t)\,\boldsymbol{e}_\theta + u(\theta,\phi,t)\,\boldsymbol{e}_\phi\right]. \tag{28}$$

Here R is a length scale which can conveniently be taken to be the radius of the sphere, v and u are respectively "almost" the θ- and ϕ-components of the velocity. The velocity field in (28) is a locally planar velocity field everywhere on the sphere, but also has a "frozen-in" r-dependence. Calculating the vorticity from (22), we find that it only has a non-zero r-component:

$$\boldsymbol{\omega} = \frac{R}{r^2}\left(\frac{\partial u}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial v}{\partial\phi} + u\cot\theta\right)\boldsymbol{e}_r = \frac{R}{r^2}\,\zeta(\theta,\phi,t)\,\boldsymbol{e}_r, \tag{29}$$

where $\zeta(\theta,\phi,t)$ is part of the radial component of the vorticity, and is defined as

$$\zeta(\theta,\phi,t) \equiv \frac{\partial u}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial v}{\partial\phi} + u\cot\theta. \tag{30}$$

The vorticity equation (23) becomes

$$\boldsymbol{e}_r: \quad \frac{\partial\zeta}{\partial t} + \frac{R}{r^2}\left(v\frac{\partial\zeta}{\partial\theta} + \frac{u}{\sin\theta}\frac{\partial\zeta}{\partial\phi}\right) = 0 \tag{31a}$$

$$\boldsymbol{e}_\theta: \quad \frac{2R^2 v\zeta}{r^4} = 0 \tag{31b}$$

$$\boldsymbol{e}_\phi: \quad \frac{2R^2 u\zeta}{r^4} = 0. \tag{31c}$$

The terms in the θ- and ϕ-components of (31) have two origins, each with an equal contribution. First, the variation of $\boldsymbol{e}_r$ over the surface of the sphere as a fluid element undergoes motion on it, creates the terms

$$\omega_r\frac{\mathrm{D}\boldsymbol{e}_r}{\mathrm{D}t} = \omega_r(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{e}_r = \frac{R^2\zeta}{r^4}(v\boldsymbol{e}_\theta + u\boldsymbol{e}_\phi). \tag{32}$$

Second, the particular form of the velocity that we have chosen in (28) means that there is a contribution to the vorticity equation from the term $(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}$:

$$-(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \frac{R^2\zeta}{r^4}(v\boldsymbol{e}_\theta + u\boldsymbol{e}_\phi). \tag{33}$$

We provide some scaling arguments later in this section to show that the θ- and ϕ-components of the vorticity equation are very small and may be neglected. Without reference to these scaling arguments, it may be thought that the θ- and ϕ-components of the vorticity equation can be neglected since they arise from dynamical effects due to motion on a curved surface. However, this argument is dissatisfactory since the kinematical effects due to motion on a curved surface, such as the $u_\phi\cot\theta$ term in ω_r (22) is not simultaneously neglected.

Theorem 2.2. *Under the shallow water approximation, the vorticity equation (31) reduces to the two-dimensional vorticity equation on the surface of a stationary sphere.*

A common factor of R/r^2 has been dropped from the radial component of (31) since it is a dynamical evolution equation for the vorticity. We can non-dimensionalise the vorticity equation as follows. Let overhead hats denote dimensionless quantities. It is natural to choose the length scale R to be the radius of the sphere. Then, we have

$$u = U\widehat{u}, \quad v = U\widehat{v}, \quad r = R\widehat{r}, \quad t = T\widehat{t}, \tag{34a}$$

$$\zeta = U\widehat{\zeta}. \tag{34b}$$

Here $\widehat{\zeta}$ is as defined in (30), but with u and v replaced by $\widehat{u}$ and $\widehat{v}$ respectively. The θ- and ϕ-components of the vorticity equation become respectively,

$$\frac{2U^2\widehat{v}\widehat{\zeta}}{R^2\widehat{r}^2} = 0 \qquad \text{and} \qquad \frac{2U^2\widehat{u}\widehat{\zeta}}{R^2\widehat{r}^2} = 0. \tag{35}$$

These components therefore scale as U^2/R^2 and we see from (1) that this is a very small number for large scale flows in the ocean. In the case of large scale flows in the atmosphere, we scale by U_a instead of U, but the ratio U_a^2/R^2 is still very small as seen from (2). The θ- and ϕ-components (35) can therefore be neglected. If we choose the time scale as $T = R/U$ (or $T_a = R/U_a$ in the case of large scale atmospheric flows), then the unsteady and advection terms in the radial component of the vorticity equation scale similarly. The non-dimensional form of the radial component is

$$\frac{\partial\widehat{\zeta}}{\partial\widehat{t}} + \frac{1}{\widehat{r}^2}\left(\widehat{v}\frac{\partial\widehat{\zeta}}{\partial\theta} + \frac{\widehat{u}}{\sin\theta}\frac{\partial\widehat{\zeta}}{\partial\phi}\right) = 0. \tag{36}$$

We consider the motion of the fluid to be limited to a shallow layer of height H above the surface of the sphere (see §1). Let us define the height coordinate h in terms of the radial coordinate r as $h = r - R$, and the dimensionless height coordinate $\widehat{h}$ as $h = H\widehat{h}$. Then we have $\widehat{r} = 1 + \varepsilon\widehat{h}$, and since $\varepsilon \ll 1$,

$$\frac{1}{\widehat{r}^2} = 1 - 2\varepsilon\widehat{h} + O(\varepsilon^2). \tag{37}$$

Thus to leading order in ε, we have $1/\widehat{r}^2 \sim O(1)$. It is seen from (35) and (37) that the shallow fluid approximation is consistent with neglecting the θ- and ϕ-components of the vorticity equation. To leading order in the shallow-fluid approximation, the radial vorticity equation is

$$\frac{\partial\widehat{\zeta}}{\partial\widehat{t}} + \widehat{v}\frac{\partial\widehat{\zeta}}{\partial\theta} + \frac{\widehat{u}}{\sin\theta}\frac{\partial\widehat{\zeta}}{\partial\phi} = 0. \tag{38}$$

This equation shows the conservation of vorticity of a fluid element as it moves through the shallow fluid layer. See for ex. [13], where kinematic arguments are presented to define vortices on the surface of a sphere, but the dynamical evolution equation is taken to be (38) without the aid of any scaling arguments.

3. **Flow on the surface of a rotating sphere.**

3.1. **Vorticity equation for flow in a non-inertial reference frame.** We have so far restricted our attention to flow on the surface of a stationary sphere. To include the effects of rotation, we begin with the 3D Euler equation in a non-inertial frame of reference rotating with a steady angular velocity $\boldsymbol{\Omega}$. The Euler equation in this case is obtained by adding to the left hand side of (4) two additional terms, namely the Coriolis term and the centrifugal term [3], respectively

$$2\,\boldsymbol{\Omega}\times\boldsymbol{u} \qquad \text{and} \qquad \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}). \tag{39}$$

Then we get the Euler equation in a non-inertial frame:

$$\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}+2\,\boldsymbol{\Omega}\times\boldsymbol{u}+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})=-\frac{\boldsymbol{\nabla}p}{\rho}+\boldsymbol{F}. \tag{40}$$

The conservation of mass continues to hold in the same form (5) as before [3]. It must be emphasized that the position $\boldsymbol{r}$, velocity $\boldsymbol{u}$ and the fluid particle acceleration $\mathrm{D}\boldsymbol{u}/\,\mathrm{D}t$ are all measured with respect to the non-inertial reference frame.

The vorticity equation in the non-inertial frame can be obtained by taking the curl of (40). We follow the same procedure as in obtaining (12). The additional terms in the vorticity equation due to the Coriolis and centrifugal terms are, respectively,

$$2\,\boldsymbol{\nabla}\times(\boldsymbol{\Omega}\times\boldsymbol{u}) \qquad \text{and} \qquad \boldsymbol{\nabla}\times(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})). \tag{41}$$

Using the identities (A.1) and the fact that $\boldsymbol{\Omega}$ is a constant, we can show that

$$2\,\boldsymbol{\nabla}\times(\boldsymbol{\Omega}\times\boldsymbol{u})=2\,\boldsymbol{\Omega}(\boldsymbol{\nabla}\cdot\boldsymbol{u})-2\,(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}, \tag{42a}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}))=0. \tag{42b}$$

Using these identities we obtain the vorticity equation in a non-inertial reference frame:

$$\frac{\partial \boldsymbol{\omega}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}-(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}+\boldsymbol{\omega}(\boldsymbol{\nabla}\cdot\boldsymbol{u})+2\,\boldsymbol{\Omega}(\boldsymbol{\nabla}\cdot\boldsymbol{u})-2\,(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \frac{\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p}{\rho^2}+\boldsymbol{\nabla}\times\boldsymbol{F}. \tag{43}$$

The form of the non-inertial vorticity equation (43) is not surprising since the fluid vorticity due to a constant background rotation is equal to twice the angular velocity of rotation. The vorticity $\boldsymbol{\omega}$ in (43) is measured with respect to the non-inertial reference frame. It is called the relative vorticity, and is distinguished from the planetary vorticity $2\boldsymbol{\Omega}$. The non-inertial vorticity equation (43) is equivalent to (12), but for the effective vorticity $\boldsymbol{\omega}_{\text{eff}}=\boldsymbol{\omega}+2\,\boldsymbol{\Omega}$.

We can simplify (43) for the flow of an incompressible, barotropic fluid acted on by conservative body forces, using the conditions (13), (14) and (15). The resulting vorticity equation generalizes (16) to include the effects of a non-inertial reference frame:

$$\frac{\partial \boldsymbol{\omega}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}-(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}-2\,(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}=0. \tag{44}$$

3.2. **2D flow on a rotating sphere.** Consider first a 2D, globally planar fluid flow in the form of (17); then the corresponding vorticity field will be in the form of (18). In addition, let us consider a constant background rotation in the form $\boldsymbol{\Omega}=\Omega\boldsymbol{e}_z$. Then the vorticity equation (44) becomes

$$\frac{\partial \omega}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\omega=0 \qquad \text{or} \qquad \frac{\mathrm{D}\omega}{\mathrm{D}t}=0. \tag{45}$$

The absence of an $\mathit{\Omega}$ term in (45) is not surprising since it is a constant. In fact one simple solution of (19) is a constant vorticity field i.e. solid-body rotation.

Consider next a 2D velocity field in the form of (28) on a rotating sphere, and the corresponding vorticity field in the form of (29). Let the constant rotation of the sphere be given by $\boldsymbol{\Omega} = \mathit{\Omega}\boldsymbol{e}_z$; it is then seen that the rotation term does contribute to the vorticity equation. Since (16) and (44) differ only by the presence of the term $-2\,(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}$ in the latter, this term has to be added to (31) to obtain the relevant vorticity equation. The angular velocity of the sphere is (see Fig. (1))

$$\boldsymbol{\Omega} = \mathit{\Omega}\boldsymbol{e}_z = \mathit{\Omega}\cos\theta\,\boldsymbol{e}_r - \mathit{\Omega}\sin\theta\,\boldsymbol{e}_\theta. \tag{46}$$

Using (A.4), we get the operator

$$\boldsymbol{\Omega}\cdot\boldsymbol{\nabla} = \mathit{\Omega}\cos\theta\,\frac{\partial}{\partial r} - \frac{\mathit{\Omega}\sin\theta}{r}\frac{\partial}{\partial\theta}. \tag{47}$$

Using (A.3), we find

$$(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{e}_\theta = \frac{\mathit{\Omega}\sin\theta}{r}\,\boldsymbol{e}_r \qquad \text{and} \qquad (\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{e}_\phi = 0. \tag{48}$$

If $\boldsymbol{u}$ is given by (28), then we have

$$\begin{aligned}(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \frac{R\mathit{\Omega}}{r^2}\Bigg[&v\sin\theta\,\boldsymbol{e}_r - \left(v\cos\theta + \sin\theta\frac{\partial v}{\partial\theta}\right)\boldsymbol{e}_\theta \\ &- \left(u\cos\theta + \sin\theta\frac{\partial u}{\partial\theta}\right)\boldsymbol{e}_\phi\Bigg].\end{aligned} \tag{49}$$

Therefore, for a 2D flow in the form of (28) on a rotating sphere, the vorticity equation (44) takes the form

$$\boldsymbol{e}_r:\quad \frac{\partial\zeta}{\partial t} + \frac{R}{r^2}\left(v\frac{\partial\zeta}{\partial\theta} + \frac{u}{\sin\theta}\frac{\partial\zeta}{\partial\phi}\right) - 2\mathit{\Omega}v\sin\theta = 0 \tag{50a}$$

$$\boldsymbol{e}_\theta:\quad \frac{2R^2v\zeta}{r^4} + \frac{2R\mathit{\Omega}}{r^2}\left(v\cos\theta + \sin\theta\frac{\partial v}{\partial\theta}\right) = 0 \tag{50b}$$

$$\boldsymbol{e}_\phi:\quad \frac{2R^2u\zeta}{r^4} + \frac{2R\mathit{\Omega}}{r^2}\left(u\cos\theta + \sin\theta\frac{\partial u}{\partial\theta}\right) = 0. \tag{50c}$$

Theorem 3.1. *Under the shallow water approximation, the vorticity equation (50) reduces to the two-dimensional vorticity equation on a rotating sphere.*

A common factor of R/r^2 has been dropped from the radial component of (50), but similar terms in the θ- and ϕ-components have been retained. We obtain by substituting (34) into (50)

$$\frac{\partial\widehat{\zeta}}{\partial\widehat{t}} + \frac{1}{\widehat{r}^2}\left(\widehat{v}\frac{\partial\widehat{\zeta}}{\partial\theta} + \frac{\widehat{u}}{\sin\theta}\frac{\partial\widehat{\zeta}}{\partial\phi}\right) - \frac{\widehat{v}\sin\theta}{Ro} = 0, \tag{51}$$

for the radial component of the vorticity equation. Here $Ro = U/2\mathit{\Omega}R$ is a Rossby number. For the θ- and ϕ-components of (50), we get

$$\frac{2U^2\widehat{v}\widehat{\zeta}}{R^2\widehat{r}^2} + \frac{2U\mathit{\Omega}}{R\widehat{r}^2}\left(\widehat{v}\cos\theta + \sin\theta\frac{\partial\widehat{v}}{\partial\theta}\right) = 0 \tag{52a}$$

$$\frac{2U^2\widehat{u}\widehat{\zeta}}{R^2\widehat{r}^2} + \frac{2U\mathit{\Omega}}{R\widehat{r}^2}\left(\widehat{u}\cos\theta + \sin\theta\frac{\partial\widehat{u}}{\partial\theta}\right) = 0. \tag{52b}$$

In the case of large scale flows in the atmosphere, U must be replaced by U_a in (52). Whether in the case of flows in the oceans or the atmosphere, it is seen from (1) and (2) that the terms in (52) are very small, and can therefore be neglected. If we make a shallow-fluid approximation, then by using (37), we find for the radial component of the vorticity equation

$$\frac{\partial \widehat{\zeta}}{\partial \widehat{t}} + \widehat{v}\frac{\partial \widehat{\zeta}}{\partial \theta} + \frac{\widehat{u}}{\sin\theta}\frac{\partial \widehat{\zeta}}{\partial \phi} - \frac{\widehat{v}\sin\theta}{Ro} = 0. \tag{53}$$

This equation shows the conservation of the absolute vorticity $\widehat{\zeta}_a = \widehat{\zeta} + \cos\theta / \mathit{Ro}$ of every fluid element in the flow.

4. **Streamfunction-vorticity formulation.** We have so far considered the conservation of momentum and the vorticity equation in §2 and §3. Let us now turn to the conservation of mass and look at the continuity equation (5). Since the flow is incompressible, the continuity equation takes the simpler form $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, which in spherical coordinates looks like

$$\frac{\partial v}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial u}{\partial \phi} + v\cot\theta = 0 \tag{54}$$

on using (A.4) and (28). It must be noted that the incompressible continuity equation retains its form $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$ in the non-inertial frame of reference [3]. We may rewrite (54) using elementary trigonometry as

$$\frac{\partial}{\partial \theta}(v\sin\theta) + \frac{\partial u}{\partial \phi} = 0. \tag{55}$$

Expanding (55) and dividing by $\sin\theta$ throughout, we get (54). Equation (55) allows us to introduce a streamfunction $\psi(\theta, \phi, t)$ which satisfies

$$u = -\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v = \frac{1}{\sin\theta}\frac{\partial \psi}{\partial \phi}. \tag{56}$$

Substituting (56) into (30), we obtain a relationship between the radial vorticity $\zeta(\theta, \phi, t)$ and the streamfunction:

$$\boldsymbol{\nabla}^2_\Sigma \psi = -\zeta. \tag{57}$$

Here $\boldsymbol{\nabla}^2_\Sigma$ is the Laplace-Beltrami operator on the sphere defined by

$$\boldsymbol{\nabla}^2_\Sigma \equiv \frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2} + \cot\theta\frac{\partial}{\partial \theta}. \tag{58}$$

Defining the gradient operator $\boldsymbol{\nabla}_\Sigma$ as

$$\boldsymbol{\nabla}_\Sigma \equiv \boldsymbol{e}_\theta\frac{\partial}{\partial \theta} + \frac{\boldsymbol{e}_\phi}{\sin\theta}\frac{\partial}{\partial \phi}, \tag{59}$$

and using (A.3), we can find the expression (58) for the Laplace-Beltrami operator $\boldsymbol{\nabla}^2_\Sigma \equiv \boldsymbol{\nabla}_\Sigma\cdot\boldsymbol{\nabla}_\Sigma$.

The relation (57) is a kinematic relationship between the streamfunction and the vorticity. To obtain a dynamical relationship, first consider the case of 2D flow on a stationary sphere discussed in §2.3. Substituting (56) into the radial component of (31), we find the dynamic streamfunction-vorticity relationship:

$$\frac{\partial \zeta}{\partial t} + \frac{R}{r^2\sin\theta}\left[\frac{\partial \psi}{\partial \phi}\frac{\partial \zeta}{\partial \theta} - \frac{\partial \psi}{\partial \theta}\frac{\partial \zeta}{\partial \phi}\right] = 0. \tag{60}$$

When the flow is steady $\partial\zeta/\partial t = 0$, and the above equation simplifies to

$$\frac{\partial\psi}{\partial\phi}\frac{\partial\zeta}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial\phi} = 0, \tag{61}$$

which admits solutions of the form [9]

$$-\zeta = \nabla^2_\Sigma \psi = h(\psi), \tag{62}$$

where h is some arbitrary function. A general discussion of (62) in Euclidean geometry, both 2D and 3D, can be found in [2]. In the case of the rotating sphere discussed in §3.2, we find by substituting (56) into the radial component of (50),

$$\frac{\partial\zeta}{\partial t} + \frac{R}{r^2\sin\theta}\left[\frac{\partial\psi}{\partial\phi}\frac{\partial\zeta}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial\phi}\right] - 2\Omega\frac{\partial\psi}{\partial\phi} = 0. \tag{63}$$

For steady flow on a rotating sphere the above equation can be written as

$$\frac{\partial\psi}{\partial\phi}\left(\frac{\partial\zeta}{\partial\theta} - \frac{2r^2\Omega\sin\theta}{R}\right) - \frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial\phi} = 0, \tag{64}$$

which admits solutions of the form [9]

$$-\zeta = \nabla^2_\Sigma \psi = h(\psi) + \frac{2r^2\Omega\cos\theta}{R}, \tag{65}$$

where h is an arbitrary function. Let us non-dimensionalise the streamfunction using $\widehat{\psi} = U\psi$. If we make a shallow-fluid approximation and use

$$r^2 = R^2\widehat{r}^2 = R^2 + O(\varepsilon), \tag{66}$$

we get

$$-\zeta = \nabla^2_\Sigma \widehat{\psi} = \widehat{h}(\widehat{\psi}) + \frac{\cos\theta}{Ro}, \tag{67}$$

where we have used the fact that h has the same dimensions as the streamfunction, to define $h(\cdot) = U\widehat{h}(\cdot)$. Equation (67) can be rewritten using

$$\nabla^2_\Sigma (\cos\theta) = -2\cos\theta, \tag{68}$$

and defining $\widehat{\Psi} = \widehat{\psi} + \cos\theta/2\,Ro$, as

$$\nabla^2_\Sigma \widehat{\Psi} = \widehat{h}(\widehat{\Psi} - \cos\theta/2\,Ro). \tag{69}$$

Equation (69) is the same equation as derived in [16], where the authors start out with the effectively 2D Euler equation obtained after appropriate scaling. The derivation of the vorticity equation then proceeds in the usual manner, via elimination of the pressure terms. Various exact solutions to (69) are derived and discussed in [11].

5. **Summary and discussion.** We have considered the vorticity equation for 2D fluid flows on the surface of a sphere. We have shown that a general 2D velocity field on the sphere does not lead to a strictly radial vorticity field, unless it has a particular radial dependence. Turning to the dynamics of the vorticity field, we have shown the material conservation of the radial vorticity for dynamics on the entire sphere. At the same time, the meridional and zonal components of the vorticity equation are not strictly zero due to curvature effects on the dynamics. We have then presented scaling arguments for why these latter components can be

neglected for global scale flows. Finally, we have considered the streamfunction-vorticity formulation, and retrieved the vorticity equation derived in [16], using our direct approach.

The difference between fluid flow on a two-dimensional Cartesian plane and flow on the surface of a sphere comes about due to the effects of curvature, both on the kinematics and dynamics of vorticity, as explored in this paper. Scaling arguments arise naturally in the setting of a sphere of finite radius, which can be taken to be a natural length scale, at first approximation. These scaling arguments allow us to estimate the sizes of the different components of the vorticity equation.

The material conservation of vorticity is an old result in geophysical fluid dynamics and in the theoretical literature can be traced back to [14]. For a recent study in the same spirit, see [15]. Although simple physical arguments for the material conservation of vorticity can be made (as found for ex. in [11]), they fail to take full account of the effects of curvature on the dynamics of vorticity. These effects can be especially important in the case of a rotating sphere and lead to non-trivial terms in the zonal and meridional components of the vorticity equation, as in (50). The physical effects of these latter components have been reviewed in [17].

The present paper is limited to discussion of the material conservation of vorticity, but can be extended to include the material conservation of potential vorticity (Ertel's theorem), which is of greater importance to geophysical flows [18]. The condition that the length scale of the flow is on the order of the radius of the earth may be relaxed in order to capture sub-global scale geophysical phenomena, where nevertheless a β-plane approach is too restrictive. An interaction between a global scale analysis of the type presented here, and a β-plane 'local' approach, is currently being pursued and will be reported elsewhere.

Appendix A. **Vector calculus and spherical coordinates.** In this appendix, we record some useful vector calculus identities and apply them to calculus in spherical coordinates. We denote scalar-valued functions by f, g; and vector-valued functions by $\boldsymbol{A}, \boldsymbol{B}$. The following identities below are taken to be well-known and are stated without proof [7].

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0 \tag{A.1a}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0 \tag{A.1b}$$

$$\boldsymbol{\nabla} \cdot (f\boldsymbol{A}) = f\,\boldsymbol{\nabla} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \boldsymbol{\nabla} f \tag{A.1c}$$

$$\boldsymbol{\nabla} \times (f\boldsymbol{A}) = f\,\boldsymbol{\nabla} \times \boldsymbol{A} + \boldsymbol{\nabla} f \times \boldsymbol{A} \tag{A.1d}$$

$$\boldsymbol{\nabla}(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) + \boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) \tag{A.1e}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B} + \boldsymbol{A}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{A}). \tag{A.1f}$$

We now develop some useful identities in the spherical co-ordinate system. Let $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$ be the orthonormal unit vectors in spherical coordinates. It is seen from Fig. 2 that the relationship between the orthonormal unit vectors in the Cartesian and spherical coordinates can be written in matrix form as

$$\begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \\ \boldsymbol{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix}. \tag{A.2}$$

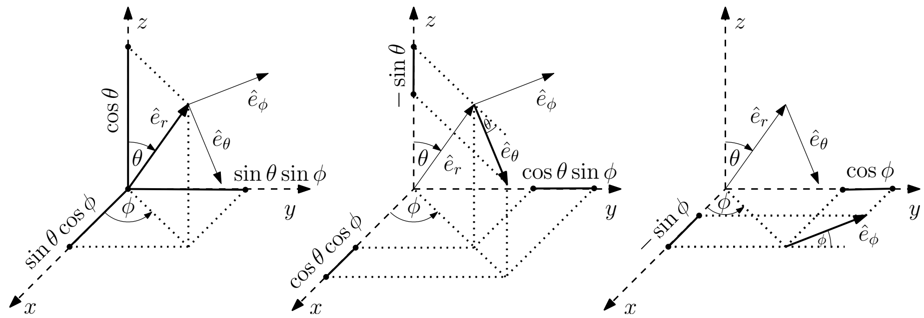

FIGURE 2. Decomposition of the orthonormal unit vectors in the spherical coordinate system into the Cartesian unit vectors.

Differentiating (A.2) row-wise and using the fact that the Cartesian unit vectors are independent of (r, θ, ϕ), we find the matrix of derivatives

$$\frac{\partial(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)}{\partial(r, \theta, \phi)} = \begin{bmatrix} 0 & \boldsymbol{e}_\theta & \sin\theta\, \boldsymbol{e}_\phi \\ 0 & -\boldsymbol{e}_r & \cos\theta\, \boldsymbol{e}_\phi \\ 0 & 0 & -\sin\theta\, \boldsymbol{e}_r - \cos\theta\, \boldsymbol{e}_\theta \end{bmatrix}, \tag{A.3}$$

where the first matrix row corresponds to $(\partial\boldsymbol{e}_r/\partial r, \partial\boldsymbol{e}_r/\partial\theta, \partial\boldsymbol{e}_r/\partial\phi)$, and so on. We note that the unit vectors are independent of r and are functions of θ and ϕ only.

The del operator $\boldsymbol{\nabla}$ in spherical coordinates is given by [1, see Appendix 2]

$$\boldsymbol{\nabla} = \boldsymbol{e}_r \frac{\partial}{\partial r} + \frac{\boldsymbol{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\boldsymbol{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{A.4}$$

Using the del operator together with (A.3), we can evaluate the divergence of each of the unit vectors $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$, and get

$$\boldsymbol{\nabla}\cdot\boldsymbol{e}_r = \frac{2}{r}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{e}_\theta = \frac{\cot\theta}{r}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{e}_\phi = 0. \tag{A.5}$$

For the curl of each of the unit vectors we find

$$\boldsymbol{\nabla}\times\boldsymbol{e}_r = 0, \qquad \boldsymbol{\nabla}\times\boldsymbol{e}_\theta = \frac{\boldsymbol{e}_\phi}{r}, \qquad \boldsymbol{\nabla}\times\boldsymbol{e}_\phi = \frac{\cot\theta}{r}\,\boldsymbol{e}_r - \frac{\boldsymbol{e}_\theta}{r}. \tag{A.6}$$

We now state a few identities used in the text. As earlier, f is any arbitrary scalar valued function. The vector calculus identities (A.1c) and (A.5) are used in their derivation, which is straightforward.

$$\boldsymbol{\nabla}\cdot(f\boldsymbol{e}_r) = \frac{2f}{r} + \frac{\partial f}{\partial r}, \tag{A.7a}$$

$$\boldsymbol{\nabla}\cdot(f\boldsymbol{e}_\theta) = \frac{f\cot\theta}{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}, \tag{A.7b}$$

$$\boldsymbol{\nabla}\cdot(f\boldsymbol{e}_\phi) = \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}. \tag{A.7c}$$

We can similarly use (A.1d) and (A.6) to derive the identities below.

$$\nabla \times (f\boldsymbol{e}_r) = \frac{\boldsymbol{e}_\theta}{r\sin\theta}\frac{\partial f}{\partial\phi} - \frac{\boldsymbol{e}_\phi}{r}\frac{\partial f}{\partial\theta}, \tag{A.8a}$$

$$\nabla \times (f\boldsymbol{e}_\theta) = -\frac{\boldsymbol{e}_r}{r\sin\theta}\frac{\partial f}{\partial\phi} + \boldsymbol{e}_\phi\left(\frac{f}{r} + \frac{\partial f}{\partial r}\right), \tag{A.8b}$$

$$\nabla \times (f\boldsymbol{e}_\phi) = \boldsymbol{e}_r\left(\frac{f\cot\theta}{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\right) - \boldsymbol{e}_\theta\left(\frac{f}{r} + \frac{\partial f}{\partial r}\right). \tag{A.8c}$$

Finally using (A.3), we find for any vector $\boldsymbol{A} = A_r\boldsymbol{e}_r + A_\theta\boldsymbol{e}_\theta + A_\phi\boldsymbol{e}_\phi$,

$$(\boldsymbol{A}\cdot\nabla)\boldsymbol{e}_r = \frac{A_\theta}{r}\boldsymbol{e}_\theta + \frac{A_\phi}{r}\boldsymbol{e}_\phi \tag{A.9a}$$

$$(\boldsymbol{A}\cdot\nabla)\boldsymbol{e}_\theta = -\frac{A_\theta}{r}\boldsymbol{e}_r + \frac{A_\phi\cot\theta}{r}\boldsymbol{e}_\phi \tag{A.9b}$$

$$(\boldsymbol{A}\cdot\nabla)\boldsymbol{e}_\phi = -\frac{A_\phi}{r}\boldsymbol{e}_r - \frac{A_\phi\cot\theta}{r}\boldsymbol{e}_\theta, \tag{A.9c}$$

where A_r does not appear since all the unit vectors are independent of r as seen from (A.3). Since the unit vectors are always "steady," we have

$$\frac{\mathrm{D}\boldsymbol{e}_r}{\mathrm{D}t} = (\boldsymbol{V}\cdot\nabla)\boldsymbol{e}_r, \qquad \frac{\mathrm{D}\boldsymbol{e}_\theta}{\mathrm{D}t} = (\boldsymbol{V}\cdot\nabla)\boldsymbol{e}_\theta, \qquad \frac{\mathrm{D}\boldsymbol{e}_\phi}{\mathrm{D}t} = (\boldsymbol{V}\cdot\nabla)\boldsymbol{e}_\phi, \tag{A.10}$$

where $\boldsymbol{V}$ is the 3D velocity field.

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