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Evolving geometry of a vortex triangle

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The motion of three interacting point vortices in the plane can be thought of as the motion of three geometrical points endowed with a dynamics. This motion can therefore be reformulated in terms of dynamically evolving geometric quantities, viz., the circle that circumscribes the vortex triangle and the angles of the vortex triangle. In this study, we develop the equations of motion for the center, Z, and radius, R, of this circumcircle; and for the angles of the vortex triangle, A, B, and C; and for the triangle orientation given by φ_1. The equations of motion for R, A, B, and C form an autonomous dynamical system. A number of known results in the three-vortex problem follow readily from the equations, giving an alternate geometrical perspective on the problem.

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I. INTRODUCTION

Interacting regions of concentrated vorticity play a central role in the dynamics of a vast many fluid systems. A classic example of such interactions is the vortex tripole [1], in which three vortices move around each other for an extended period of time. A collection of three vortices is arguably the most fundamental of the vortex configurations, as it contains the smallest number of vortices capable of exhibiting relative motion. For modeling purposes, three point vortices in the plane, with positions (x_α, y_α) and circulations Γ_α ($\neq 0$, for $\alpha = 1,2,3$), provide the simplest reduced-order representation of a three-vortex system. A number of fundamental observations regarding vortex motion can be ascertained by considering the dynamics of three point vortices in the plane [2–5].

The motion of three interacting point vortices in the plane was solved by Gröbli in his thesis of 1877 [2]. More than 70 years later his solutions were classified in terms of the vortex circulations (or strengths) by Synge [3] using a geometrical approach. This work showed that the various possible regimes of motion are largely determined by the signs of the three symmetric functions of the circulations, $\Gamma_1, \Gamma_2, \Gamma_3$, of the three interacting vortices, namely,

$$\gamma_1 = \Gamma_1 + \Gamma_2 + \Gamma_3, \quad \gamma_2 = \Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1, \quad \gamma_3 = \Gamma_1\Gamma_2\Gamma_3; \tag{1a}$$

the relation

$$\frac{\gamma_2}{\gamma_3} = \frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} \tag{1b}$$

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is useful to note. While Gröbli's approach leads to detailed solutions of the initial value problem, typically in terms of elliptic or hyperelliptic functions, Synge's approach is more in line with the qualitative, geometrical methods of modern dynamical systems theory. The two approaches intersect for the case of identical vortices, where Gröbli discusses a geometrical solution that is closely related to Synge's method for the general case. Both the work of Gröbli and of Synge lay dormant until Novikov revived interest in the three-vortex problem in 1975 [4]. Novikov rediscovered, independently and yet almost word for word, the geometrical solution to the problem of three identical vortices that Gröbli had given almost a century before. Generalizing Novikov's work, Aref rediscovered Synge's geometrical approach for general vortex strengths, albeit in a slightly different form [5]. The contorted history of rediscovery of solutions, and some background on the little known career of Gröbli, has been recounted elsewhere [6].

In each of these previous geometrical solutions, the focus has been on describing the evolution of the vortex triangle in terms of the lengths of the sides, s_1, s_2, s_3, and the area, Δ. Alternatively, the geometry of the vortex triangle can be given in terms of the interior angles and the properties of the circumcircle that passes through the vortex locations [7]. In Ref. [7], this alternative geometrical description was used merely to arrive at a concise derivation of the existing equations for the lengths of the triangle sides. In Secs. III A and III B we develop and explore the autonomous dynamical system given by the evolution of the circumcircle and the interior angles. In Sec. III C, we determine an evolution equation for the orientation of the triangle in terms of the present geometric variables. In Sec. III D we discuss how the integrals of motion, including the Hamiltonian for the three-vortex system, can be written purely in terms of the geometric variables. In Sec. III E we consider the relationship between the present formulation and previous work. In Sec. III F we derive and discuss simple equations relating the center and radius of the circumcircle through the constants of motion, which are valid throughout the dynamical evolution of the system. In Sec. IV we retrieve some of the well-known results in three-vortex motion through simple application of the equations of motion for the geometrical variables.

II. BASIC EQUATIONS

The motion of three vortices in the plane is given by six coupled, nonlinear, first-order ordinary differential equations (ODEs), two for the Cartesian coordinates of each of the vortices, (x_α, y_α), $\alpha = 1,2,3$. If we concatenate the coordinates into complex positions $z_\alpha = x_\alpha + i y_\alpha$, we have the equations of motion of these complex positions as

$$\overline{\frac{dz_1}{dt}} = \frac{1}{2\pi i}\left(\frac{\Gamma_2}{z_1 - z_2} + \frac{\Gamma_3}{z_1 - z_3}\right), \tag{2a}$$

$$\overline{\frac{dz_2}{dt}} = \frac{1}{2\pi i}\left(\frac{\Gamma_1}{z_2 - z_1} + \frac{\Gamma_3}{z_2 - z_3}\right), \tag{2b}$$

$$\overline{\frac{dz_3}{dt}} = \frac{1}{2\pi i}\left(\frac{\Gamma_1}{z_3 - z_1} + \frac{\Gamma_2}{z_3 - z_2}\right), \tag{2c}$$

where the overbar denotes complex conjugation. We shall assume the basic equations (2) to be known. For background on the information presented in this section we refer the reader to the textbook and monograph literature [8–11].

Equations (2) have a number of well-known integrals. Two of these are the components of linear impulse,

$$Q = \Gamma_1 x_1 + \Gamma_2 x_2 + \Gamma_3 x_3, \quad P = \Gamma_1 y_1 + \Gamma_2 y_2 + \Gamma_3 y_3, \tag{3}$$

which pertain to the absolute positions of the vortices. The linear impulse determines the center of vorticity,

$$z_{\mathrm{cv}} = \frac{Q + iP}{\gamma_1}. \tag{4}$$

For $\gamma_1 \neq 0$, the origin may be shifted to z_{cv} under the coordinate transformation $z_\alpha = z_{cv} + z'_\alpha$. The linear impulse with respect to the center of vorticity is clearly zero, i.e.,

$$\Gamma_1 z'_1 + \Gamma_2 z'_2 + \Gamma_3 z'_3 = 0.$$

Another integral of (2) is the angular impulse,

$$I_0 = \Gamma_1\left(x_1^2 + y_1^2\right) + \Gamma_2\left(x_2^2 + y_2^2\right) + \Gamma_3\left(x_3^2 + y_3^2\right), \tag{5a}$$

which also by definition requires the absolute positions of the vortices. The subscript 0 signifies that I_0 is computed relative to the chosen origin of coordinates; the angular impulse with respect to an arbitrary point z can be written as

$$I_z = \sum_{\alpha=1}^{3} \Gamma_\alpha |z_\alpha - z|^2. \tag{5b}$$

One can also employ the above coordinate transformation to arrive at a *parallel axis theorem* for the angular impulse,

$$I_z = I_{cv} + \gamma_1 |z - z_{cv}|^2, \tag{6}$$

where I_{cv} is the angular impulse calculated relative to the center of vorticity (4).

The angular impulse is related to the quantity

$$L = \Gamma_1\Gamma_2 s_3^2 + \Gamma_2\Gamma_3 s_1^2 + \Gamma_3\Gamma_1 s_2^2 = \gamma_1 I_0 - Q^2 - P^2. \tag{7}$$

If we define the angular impulse with respect to z_{cv}, we have $L = \gamma_1 I_{cv}$.

Gröbli discovered that one can isolate within this system of six real first-order ODEs a subsystem of three ODEs for the sides of the vortex triangle, s_α, defined by

$$\begin{aligned} s_1^2 &= |z_2 - z_3|^2 = (x_2 - x_3)^2 + (y_2 - y_3)^2, \\ s_2^2 &= |z_3 - z_1|^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2, \\ s_3^2 &= |z_1 - z_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2. \end{aligned} \tag{8}$$

These equations are [2]

$$\frac{ds_1^2}{dt} = \frac{2\Delta}{\pi}\Gamma_1\frac{s_3^2 - s_2^2}{s_2^2 s_3^2}, \quad \frac{ds_2^2}{dt} = \frac{2\Delta}{\pi}\Gamma_2\frac{s_1^2 - s_3^2}{s_3^2 s_1^2}, \quad \frac{ds_3^2}{dt} = \frac{2\Delta}{\pi}\Gamma_3\frac{s_2^2 - s_1^2}{s_1^2 s_2^2}, \tag{9a}$$

where the area of the vortex triangle, Δ, is given by *Heron's formula* [12],

$$16\Delta^2 = 2s_2^2 s_3^2 + 2s_3^2 s_1^2 + 2s_1^2 s_2^2 - s_1^4 - s_2^4 - s_3^4. \tag{9b}$$

The sign of Δ needed in (9a) is indeterminate from (9b) since a triangle and its mirror image have the same absolute area. One needs to "step outside" the reduced system (9a) whenever the vortices become collinear and consider the triangle area with orientation,

$$\Delta = \tfrac{1}{2}(x_1 y_2 + x_2 y_3 + x_3 y_1 - x_1 y_3 - x_3 y_2 - x_2 y_1). \tag{9b$'$}$$

This definition makes $\Delta > 0$ when vortices 123 appear counterclockwise and $\Delta < 0$ when they appear clockwise. The evolution of Δ may now be traced via (2). The subsystem (9a),(9b), then, is "closed" except for instants when the vortices become collinear. In order to know how to continue the motion through such instants, one needs to either appeal to the full equations of motion (2) or develop a fourth equation of motion for Δ itself in terms of s_1,s_2,s_3. This equation is [13]

$$\frac{d\Delta}{dt} = \frac{1}{8\pi}\left[(\Gamma_1 + \Gamma_2)\frac{s_1^2 - s_2^2}{s_3^2} + (\Gamma_2 + \Gamma_3)\frac{s_2^2 - s_3^2}{s_1^2} + (\Gamma_3 + \Gamma_1)\frac{s_3^2 - s_1^2}{s_2^2}\right]. \tag{9c}$$

We do not include derivations of either (9a) or (9c) in this paper, but the solution procedure is as follows. Given equations for the velocities of the three corners of a triangle, it is clear, in principle, that one can write equations for the time rate of change of the triangle sides and the triangle area. The geometrical considerations required for such a "direct" derivation are, however, somewhat involved. A straightforward algebraic development can be achieved by appealing to the Hamiltonian formulation [14] of (2) introduced already by Kirchhoff in 1876 and well covered in the texts cited [8–11]. If we set

$$H = -\frac{1}{4\pi}\left(\Gamma_1\Gamma_2 \log s_3^2 + \Gamma_2\Gamma_3 \log s_1^2 + \Gamma_3\Gamma_1 \log s_2^2\right), \tag{10}$$

it is not difficult to verify that (2) may be written

$$\Gamma_\alpha \frac{dx_\alpha}{dt} = \frac{\partial H}{\partial y_\alpha}, \quad \Gamma_\alpha \frac{dy_\alpha}{dt} = -\frac{\partial H}{\partial x_\alpha} \quad (\alpha = 1,2,3). \tag{11}$$

Thus, x_α and $\Gamma_\alpha y_\alpha$ are canonically conjugate variables, and one can introduce a Poisson bracket [15,16]

$$[f,g] = \sum_{\alpha=1}^{3} \frac{1}{\Gamma_\alpha}\left(\frac{\partial f}{\partial x_\alpha}\frac{\partial g}{\partial y_\alpha} - \frac{\partial f}{\partial y_\alpha}\frac{\partial g}{\partial x_\alpha}\right). \tag{12}$$

In the context of point vortex dynamics this development dates back at least to a 1905 paper by Laura [17]. From the fundamental Poisson brackets,

$$[x_1,\Gamma_1 y_1] = [x_2,\Gamma_2 y_2] = [x_3,\Gamma_3 y_3] = 1, \tag{13}$$

with all other brackets of two coordinates equal to zero, one builds up the algebra to produce results such as [13]

$$\left[s_1^2,s_2^2\right] = -8\frac{\Delta}{\Gamma_3}, \quad \left[s_2^2,s_3^2\right] = -8\frac{\Delta}{\Gamma_1}, \quad \left[s_3^2,s_1^2\right] = -8\frac{\Delta}{\Gamma_2}. \tag{14}$$

Then, since the evolution of any function of the coordinates is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f,H], \tag{15}$$

and H is given by (10), one finds (9a) and (9c) after straightforward calculation of Poisson brackets.

III. EVOLUTION OF VORTEX TRIANGLE GEOMETRY

One can show [7] that the vortex velocities may be written quite simply in terms of the interior angles of the vortex triangle and the radius of the circumscribed circle, R. These results allow us to derive equations of motion for R and for the velocity of the center of the circumcircle, Z. If the angles of the triangle are denoted A,B,C as shown in Fig. 1, then we have the geometrical relations

$$s_1 = 2R\,\sin A, \quad s_2 = 2R\,\sin B, \quad s_3 = 2R\,\sin C. \tag{16}$$

For later reference, we also note the relation

$$R = \frac{s_1 s_2 s_3}{4|\Delta|} \tag{17a}$$

which gives, upon substitution of (16),

$$|\Delta| = 2R^2 \sin A\,\sin B\,\sin C. \tag{17b}$$

The interior angles have the fundamental relation

$$A + B + C = \pi. \tag{18}$$

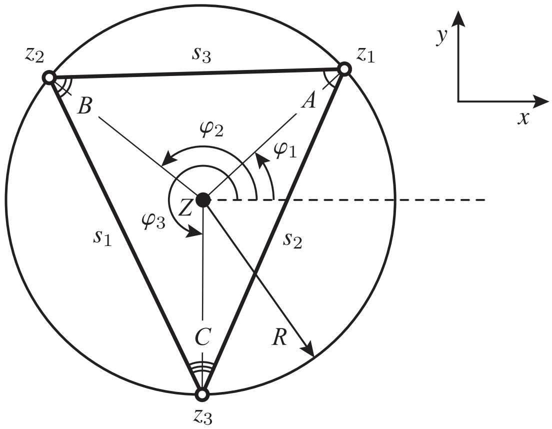

FIG. 1. Definition of the geometrical variables. Open circles mark the vortex locations z_1, z_2, z_3. Filled circle marks the center of the circumcircle, Z. The dashed line is parallel to the x axis.

These relations allow us to derive equations for the evolution of the triangle shape, location, and orientation.

When the vortices become collinear, both the radius and the center of the circumcircle go to infinity, and the positions of the vortices in terms of the circumcircle become ill defined. Since this geometrical approach breaks down as the vortices pass through a collinear state, we can choose, without any loss of generality, to always label the vortices such that they are oriented counterclockwise, which gives $\Delta > 0$ in all of the subsequent analyses.

A. Evolution of the circumcircle radius and center

As shown in Fig. 1, the vortex coordinates can be written as

$$z_1 = Z + \mathrm{Re}^{i\varphi_1}, \quad z_2 = Z + \mathrm{Re}^{i\varphi_2}, \quad z_3 = Z + \mathrm{Re}^{i\varphi_3}, \tag{19}$$

where φ_α measures the angle made by the position vector of vortex α with respect to the x (horizontal) axis. If the three vortices appear counterclockwise, as we have assumed, it follows from elementary geometry that the interior angles of the vortex triangle are given by

$$\varphi_2 - \varphi_1 = 2C, \quad \varphi_3 - \varphi_2 = 2A, \quad \varphi_1 - \varphi_3 = 2B - 2\pi. \tag{20}$$

With this notation, we can write the equation of motion for, e.g., vortex 1 as

$$\dot{\bar{z}}_1 = \frac{1}{2\pi i R}\left(\frac{\Gamma_2}{e^{i\varphi_1} - e^{i\varphi_2}} + \frac{\Gamma_3}{e^{i\varphi_1} - e^{i\varphi_3}}\right) = \frac{e^{-i\varphi_1}}{2\pi i R}\left(\frac{\Gamma_2}{1 - e^{i2C}} + \frac{\Gamma_3}{1 - e^{-i2B}}\right),$$

where the overdot denotes the time derivative. This expression may be written as

$$\dot{z}_1 = \frac{e^{i\varphi_1}}{4\pi R}[\Gamma_2 \cot C - \Gamma_3 \cot B + i(\Gamma_2 + \Gamma_3)]. \tag{21}$$

The real multiple of $e^{i\varphi_1}$ in (21) gives the radial velocity of vortex 1 relative to the circumcircle, namely,

$$(\dot{z}_1)_{\mathrm{rad}} = \frac{\Gamma_2 \cot C - \Gamma_3 \cot B}{4\pi R}. \tag{22a}$$

The real multiple of $ie^{i\varphi_1}$ in (21) gives the tangential velocity in the positive (counterclockwise) direction, namely,

$$(\dot{z}_1)_{\tan} = \frac{\Gamma_2 + \Gamma_3}{4\pi R}. \tag{22b}$$

Let the center of the circumcircle be $Z = X + iY$. We may then equate $(\dot{z}_1)_{\rm rad}$ in (22a) to the sum of the rate of change of R plus the projection of $\dot{Z}$ onto the radial direction from Z to z_1, giving

$$\dot{R} + \dot{X}\cos\varphi_1 + \dot{Y}\sin\varphi_1 = \frac{\Gamma_2\cot C - \Gamma_3\cot B}{4\pi R}. \tag{23a}$$

Similarly, we may equate $(\dot{z}_1)_{\tan}$ in (22b) to the sum of $R\dot{\varphi}_1$ and the tangential component of $\dot{Z}$ at the position of vortex 1, giving

$$R\dot{\varphi}_1 - \dot{X}\sin\varphi_1 + \dot{Y}\cos\varphi_1 = \frac{\Gamma_2 + \Gamma_3}{4\pi R}. \tag{23b}$$

Alternatively, we may simply differentiate (19) to find that

$$e^{-i\varphi_1}\dot{z}_1 = e^{-i\varphi_1}\dot{Z} + \dot{R} + iR\dot{\varphi}_1. \tag{24}$$

The real and imaginary parts of this equation give (23a) and (23b), respectively. Similar relations hold for the velocity components of vortices 2 and 3.

From (23a) and the corresponding equations for vortices 2 and 3, we have a system of equations that may be written in matrix form using the Cartesian coordinates of the vortices as

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{R} \end{bmatrix} = \frac{1}{8\pi\Delta}\begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \\ R\sin(2A) & R\sin(2B) & R\sin(2C) \end{bmatrix}\begin{bmatrix} \Gamma_2\cot C - \Gamma_3\cot B \\ \Gamma_3\cot A - \Gamma_1\cot C \\ \Gamma_1\cot B - \Gamma_2\cot A \end{bmatrix}, \tag{25}$$

where we have used the assumption that $\Delta > 0$. To write the equation for $\dot{R}$ in its most transparent form, we collect terms proportional to each of the circulations Γ_1, Γ_2, and Γ_3, giving

$$\begin{aligned}\dot{R} = \frac{1}{16\pi R\sin A\sin B\sin C}&\{\Gamma_1[\cot B\sin(2C) - \cot C\sin(2B)] \\ &+ \Gamma_2[\cot C\sin(2A) - \cot A\sin(2C)] + \Gamma_3[\cot A\sin(2B) - \cot B\sin(2A)]\},\end{aligned} \tag{26a}$$

or, again using (18),

$$\begin{aligned}\frac{dR^2}{dt} = \frac{1}{4\pi}&[\Gamma_1\cot B\cot C(\cot B - \cot C) \\ &+ \Gamma_2\cot C\cot A(\cot C - \cot A) + \Gamma_3\cot A\cot B(\cot A - \cot B)].\end{aligned} \tag{26b}$$

For the motion of the center of the circumcircle we find from (25) that

$$\begin{aligned}\dot{Z} = \dot{X} + i\dot{Y} = \frac{1}{8\pi i\Delta}&[(z_1 - z_2)(\Gamma_1\cot B - \Gamma_2\cot A) \\ &+ (z_2 - z_3)(\Gamma_2\cot C - \Gamma_3\cot B) + (z_3 - z_1)(\Gamma_3\cot A - \Gamma_1\cot C)].\end{aligned} \tag{27a}$$

By regrouping terms we find

$$\dot{Z} = \frac{1}{8\pi i\Delta}[(Q + iP)(\cot A + \cot B + \cot C) - \gamma_1(z_1\cot A + z_2\cot B + z_3\cot C)], \tag{27b}$$

where γ_1 is given in (1a). The advantage of using this form of the equation is that, if $\gamma_1 = 0$, the second term in square brackets vanishes; if instead $\gamma_1 \neq 0$, we can arrange for the center of vorticity (4) to be at the origin, in which case $Q + iP = 0$. Note that since $A + B + C = \pi$ (18),

$$\cot A + \cot B + \cot C = \tfrac{1}{2}(\sin^2 A + \sin^2 B + \sin^2 C) \geqslant 0,$$

with the equality occurring only when the vortices are collinear. By using (19) and collecting the coefficients of Γ_1, Γ_2, and Γ_3, we can rewrite (27b) as

$$\begin{aligned}\dot{Z} = \frac{\mathrm{R}e^{i\varphi_1}}{4\pi\Delta}&[\Gamma_1(e^{-iB}\cot C\ \sin B - e^{iC}\cot B\ \sin C)\\ &+\Gamma_2(e^{iC}\cot A\ \sin C - e^{iA}\cot C\ \sin A + 2i\,e^{-iB}\sin A\ \cos C)\\ &+\Gamma_3(e^{-iA}\cot B\ \sin A - e^{-iB}\cot A\ \sin B + 2i\,e^{iC}\sin A\ \cos B)].\end{aligned} \tag{27c}$$

B. Evolution of the triangle shape

Next, we seek equations of motion for the interior angles A, B, C. From (16), we have

$$\cot A\frac{dA}{dt} = \frac{1}{2R\ \sin A}\frac{ds_1}{dt} - \frac{1}{R}\frac{dR}{dt}. \tag{28}$$

Here, from (9a),

$$\frac{ds_1}{dt} = \frac{\Gamma_1\Delta}{\pi s_1}\left[\frac{s_3^2 - s_2^2}{s_3^2 s_2^2}\right] = \frac{\Gamma_1}{8\pi R}\left[\frac{\cos(2B) - \cos(2C)}{\sin B\ \sin C}\right] = \frac{\Gamma_1}{4\pi R}\sin A(\cot B - \cot C). \tag{29}$$

Combining (26b), (28), and (29) gives

$$\begin{aligned}\cot A\frac{dA}{dt} = \frac{1}{8\pi R^2}&[\Gamma_1(1 - \cot B\cot C)(\cot B - \cot C)\\ &- \Gamma_2\cot C\ \cot A(\cot C - \cot A)\ - \Gamma_3\cot A\ \cot B(\cot A - \cot B)].\end{aligned} \tag{30a}$$

For completeness we write out the corresponding equations for dB/dt and dC/dt:

$$\begin{aligned}\cot B\frac{dB}{dt} = \frac{1}{8\pi R^2}&\,[\Gamma_2(1 - \cot C\ \cot A)(\cot C - \cot A)\\ &- \Gamma_3\cot A\ \cot B(\cot A - \cot B)\ - \Gamma_1\cot B\ \cot C(\cot B - \cot C)],\end{aligned} \tag{30b}$$

$$\begin{aligned}\cot C\frac{dC}{dt} = \frac{1}{8\pi R^2}&\,[\Gamma_3(1 - \cot A\ \cot B)(\cot A - \cot B)\\ &- \Gamma_1\cot B\ \cot C(\cot B - \cot C)\ - \Gamma_2\cot C\ \cot A(\cot C - \cot A)].\end{aligned} \tag{30c}$$

Equations (26b) and (30) form an autonomous four-dimensional dynamical system embedded in the six-dimensional system (2).

There is, in addition, the obvious constraint on the three angles (18), so the system consisting of (26b) and (30) may be thought of as three dimensional. We saw earlier that the system comprised of (9a) and (9c) was also three dimensional because of (9b), except for those instants when the three vortices become collinear. The role of collinear configurations shows up in a different way in the system consisting of (26b) with (30): Collinear configurations are singularities of these equations wherein $R \to \infty$. It is thus clear that one has to stop and consider how to continue the solution beyond such a singularity in this formulation.

C. Evolution of the triangle orientation

We have used the transformations (19) and (20) to change variables from the vortex positions z_1, z_2, z_3 to the geometric variables Z, R, A, B, and C. This change of variables involves two steps: first a change from z_1, z_2, z_3 to Z, R, and φ_1, φ_2, φ_3 given by (19); and then a further change of variables from φ_1, φ_2, φ_3 to A, B, C given by (20). We see from (20) that given Z, R, A, B, and C, we cannot recover z_1, z_2, and z_3 unless one of the angles φ_1,φ_2,φ_3 is also known.

We previously used the real part of (24), i.e., (23a), to determine the evolution of R and Z in Sec. III A; we now use the imaginary part of (24) to determine the evolution of φ_1. Substituting (21) and (27c) into (24) and taking the imaginary part gives, after some algebraic manipulation,

$$\dot{\varphi}_1 = \frac{1}{4\pi\Delta}\{\Gamma_1[\cot B\,\sin^2 C + \cot C\,\sin^2 B] + \Gamma_2[\cot C\,\sin^2 A - \cot A\,\sin^2 C + \sin 2A] + \Gamma_3[\cot B\,\sin^2 A - \cot A\,\sin^2 B + \sin 2A]\}. \tag{31}$$

Equations (26), (27c), (30), and (31) are seven equations for the seven variables Z, R, A, B, C, and φ_1; one of the angles A, B, or C may be eliminated by using the constraint (18). These equations determine the evolution of the vortex triangle and provide an alternate description of three-vortex motion—equivalent to the classic representation in (2)—except at instants of time when the vortices become collinear.

D. Integrals of motion

We return to a consideration of the integrals of motion. In the reduced description provided by (9a), conservation of L (7) follows immediately by dividing the left-hand sides in (9a) by Γ_1, Γ_2, and Γ_3, respectively, and adding. The Hamiltonian (10) is also an integral of the motion. This again follows easily from (9a) by dividing the left-hand sides by $\Gamma_1 s_1^2$, $\Gamma_2 s_2^2$, and $\Gamma_3 s_3^2$, respectively, and adding.

In terms of the variables R, A, B, C we have from (16) that

$$L = 4\gamma_3 R^2\left(\frac{\sin^2 A}{\Gamma_1} + \frac{\sin^2 B}{\Gamma_2} + \frac{\sin^2 C}{\Gamma_3}\right) \tag{32a}$$

and

$$H = -\frac{1}{2\pi}\left[\gamma_2 \log\frac{R}{R_0} + \gamma_3\left(\frac{\log\sin A}{\Gamma_1} + \frac{\log\sin B}{\Gamma_2} + \frac{\log\sin C}{\Gamma_3}\right)\right], \tag{32b}$$

where R_0 is a constant length scale. Here we have used the fact that the equations of motion are unchanged up to additive constants in the Hamiltonian, as (10) and (32b) differ by the additive constant $-\frac{1}{2\pi}\gamma_2\log(2R_0)$. Equations (32) must be integrals of (26b) and (30), as may be verified directly. The verification, however, takes a few steps of not entirely transparent algebra, and one may wonder if these integrals would have been discovered working within the dynamical system (26b) and (30) without the general background we have given. We leave the details to the reader.

In addition we have a purely geometrical integral, viz., $A + B + C = \pi$ (18). That the sum $A + B + C$ has vanishing time derivative also follows by adding the equations of motion for A, B, and C, (30), and noting that the net coefficient of each of Γ_1, Γ_2, and Γ_3 vanishes. Here one needs to use relations such as

$$\frac{1}{\cot A} = \frac{\cot B + \cot C}{1 - \cot B\,\cot C},$$

which holds when $A + B + C = \pi$.

E. EOM for the center of the circumcircle revisited

We have already derived the equation of motion (EOM) for the center of the circumcircle Z. In this section we show how that result can be obtained using the canonical formalism. In the next section we then show how the equation of motion for R follows from the equation of motion for Z.

We return to the coordinates of the vortices and note that (9b) is equivalent to

$$4i\Delta = \bar{z}_1(z_2 - z_3) + \bar{z}_2(z_3 - z_1) + \bar{z}_3(z_1 - z_2). \tag{33}$$

From the fundamental Poisson brackets (13), which in terms of the complex vortex coordinates read

$$[z_\alpha, z_\beta] = 0, \quad [z_\alpha, \bar{z}_\beta] = -\frac{2i}{\Gamma_\alpha}\delta_{\alpha\beta}, \tag{34}$$

it follows that

$$[z_1,\Delta] = -\frac{z_2 - z_3}{2\Gamma_1}, \tag{35a}$$

$$[z_2,\Delta] = -\frac{z_3 - z_1}{2\Gamma_2}, \tag{35b}$$

$$[z_3,\Delta] = -\frac{z_1 - z_2}{2\Gamma_3}. \tag{35c}$$

Next, we pause to derive a well-known expression for the center of the circumcircle in terms of the positions of the vertices of the triangle. By definition of the circumcircle and its radius we have $|z_1 - Z| = |z_2 - Z| = |z_3 - Z| = R$. Thus,

$$\begin{aligned} z_1\bar{z}_1 + |Z|^2 - z_1\overline{Z} - \bar{z}_1 Z &= R^2, \\ z_2\bar{z}_2 + |Z|^2 - z_2\overline{Z} - \bar{z}_2 Z &= R^2, \\ z_3\bar{z}_3 + |Z|^2 - z_3\overline{Z} - \bar{z}_3 Z &= R^2. \end{aligned} \tag{36}$$

By eliminating $|Z|^2 - R^2$ from these relations we obtain

$$(\bar{z}_1 - \bar{z}_2)Z + (z_1 - z_2)\overline{Z} = z_1\bar{z}_1 - z_2\bar{z}_2, \tag{37a}$$

$$(\bar{z}_2 - \bar{z}_3)Z + (z_2 - z_3)\overline{Z} = z_2\bar{z}_2 - z_3\bar{z}_3, \tag{37b}$$

and solving these two linear equations for Z and $\overline{Z}$ we find

$$Z = \frac{|z_1|^2(z_2 - z_3) + |z_2|^2(z_3 - z_1) + |z_3|^2(z_1 - z_2)}{\bar{z}_1(z_2 - z_3) + \bar{z}_2(z_3 - z_1) + \bar{z}_3(z_1 - z_2)}. \tag{38}$$

From (33) the denominator is seen to be $4i\Delta$.

Since we have equations of motion for the vortex positions and for Δ (9c), we can, in principle, derive an equation of motion for Z from (38). In order to do so we calculate the Poisson bracket $[Z,H]$. Using (35a) we get

$$[z_1, 4i\Delta Z] = 4i([z_1,\Delta]Z + [z_1,Z]\Delta) = 4i\left(\frac{z_3 - z_2}{2\Gamma_1}Z + [z_1,Z]\Delta\right).$$

However, from the fundamental Poisson brackets, and in view of (38), the left-hand side is clearly

$$[z_1, 4i\Delta Z] = [z_1, |z_1|^2(z_2 - z_3)] = z_1(z_3 - z_2)\frac{2i}{\Gamma_1}.$$

Thus,

$$[z_1, Z] = -\frac{(z_2 - z_3)(z_1 - Z)}{2\Delta\Gamma_1}, \tag{39a}$$

and by permutation of indices,

$$[z_2, Z] = -\frac{(z_3 - z_1)(z_2 - Z)}{2\Delta\Gamma_2}, \tag{39b}$$

$$[z_3, Z] = -\frac{(z_1 - z_2)(z_3 - Z)}{2\Delta\Gamma_3}. \tag{39c}$$

As a corollary,

$$[\Gamma_1 z_1 + \Gamma_2 z_2 + \Gamma_3 z_3, Z] = 0. \tag{40}$$

Taking the Poisson bracket of (37a) with $\bar{z}_3$, we get

$$(\bar{z}_1 - \bar{z}_2)[\bar{z}_3, Z] + (z_1 - z_2)[\bar{z}_3, \overline{Z}] = 0.$$

Here $[\bar{z}_3, \bar{Z}] = \overline{[z_3, Z]}$, since the Poisson bracket operations are all in terms of real-valued quantities. Thus,

$$(\bar{z}_1 - \bar{z}_2)[\bar{z}_3, Z] - (z_1 - z_2)\frac{(\bar{z}_1 - \bar{z}_2)(\bar{z}_3 - \bar{Z})}{2\Delta\Gamma_3} = 0,$$

or

$$[\bar{z}_3, Z] = \frac{(z_1 - z_2)(\bar{z}_3 - \bar{Z})}{2\Delta\Gamma_3}. \tag{41a}$$

By permutation of indices,

$$[\bar{z}_2, Z] = \frac{(z_3 - z_1)(\bar{z}_2 - \bar{Z})}{2\Delta\Gamma_2}, \tag{41b}$$

$$[\bar{z}_1, Z] = \frac{(z_2 - z_3)(\bar{z}_1 - \bar{Z})}{2\Delta\Gamma_1}. \tag{41c}$$

We are now in a position to calculate

$$\begin{aligned}\Gamma_1\Gamma_2\big[Z, \log s_3^2\big] &= \frac{\Gamma_1\Gamma_2}{s_3^2}[Z,(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)]\\ &= \frac{\Gamma_1\Gamma_2}{s_3^2}\{[Z,z_1 - z_2](\bar{z}_1 - \bar{z}_2) + [Z,\bar{z}_1 - \bar{z}_2](z_1 - z_2)\}.\end{aligned}$$

Here, by (39a) and (41c),

$$\begin{aligned}[Z,z_1](\bar{z}_1 - \bar{z}_2) + [Z,\bar{z}_1](z_1 - z_2) &= \frac{(z_2 - z_3)}{2\Delta\Gamma_1}[(z_1 - Z)(\bar{z}_1 - \bar{z}_2) - (\bar{z}_1 - \bar{Z})(z_1 - z_2)]\\ &= -\frac{(z_2 - z_3)(\bar{z}_1 - \bar{z}_2)}{\Delta\Gamma_1}\left[Z - \frac{1}{2}(z_1 + z_2)\right],\end{aligned}$$

where in the last step (37a) has been used in the form

$$(z_1 - z_2)\bar{Z} = -(\bar{z}_1 - \bar{z}_2)Z + |z_1|^2 - |z_2|^2.$$

Thus,

$$\begin{aligned}[Z,z_1](\bar{z}_1 - \bar{z}_2) + [Z,\bar{z}_1](z_1 - z_2) &= \frac{(z_2 - z_3)}{2\Delta\Gamma_1}[2Z(\bar{z}_1 - \bar{z}_2) - z_1\bar{z}_2 + \bar{z}_1 z_2 + |z_1|^2 - |z_2|^2]\\ &= -\frac{(z_2 - z_3)(\bar{z}_1 - \bar{z}_2)}{\Delta\Gamma_1}\left[Z - \frac{1}{2}(z_1 + z_2)\right].\end{aligned}$$

Similarly,

$$\begin{aligned}[Z,z_2](\bar{z}_1 - \bar{z}_2) + [Z,\bar{z}_2](z_1 - z_2) &= \frac{(z_3 - z_1)}{2\Delta\Gamma_2}[(z_2 - Z)(\bar{z}_1 - \bar{z}_2) - (\bar{z}_2 - \bar{Z})(z_1 - z_2)]\\ &= -\frac{(z_3 - z_1)(\bar{z}_1 - \bar{z}_2)}{\Delta\Gamma_2}\left[Z - \frac{1}{2}(z_1 + z_2)\right].\end{aligned}$$

Subtracting we have

$$\begin{aligned}\Gamma_1\Gamma_2\big[Z,\log s_3^2\big] &= \frac{\Gamma_1\Gamma_2(\bar z_1-\bar z_2)}{\Delta s_3^2}\left[-\frac{(z_2-z_3)}{\Gamma_1}+\frac{(z_3-z_1)}{\Gamma_2}\right]\left[Z-\frac{1}{2}(z_1+z_2)\right]\\ &= \frac{\bar z_1-\bar z_2}{\Delta s_3^2}[\gamma_1 z_3-(Q+iP)]\left[Z-\frac{1}{2}(z_1+z_2)\right]\\ &= \frac{\gamma_1 z_3-(Q+iP)}{\Delta}\,\frac{Z-\frac{1}{2}(z_1+z_2)}{z_1-z_2},\end{aligned}$$

with similar terms from the other two terms in the Hamiltonian.

Now, in terms of the angles in the vortex triangle

$$\frac{Z-\frac{1}{2}(z_1+z_2)}{z_1-z_2}=-\frac{1}{2}\frac{e^{i\varphi_1}+e^{i\varphi_2}}{e^{i\varphi_1}-e^{i\varphi_2}}=-\frac{1}{2}\frac{1+e^{2iC}}{1-e^{2iC}}=\frac{1}{2i}\frac{\cos C}{\sin C}=-\frac{1}{2}i\,\cot C.$$

Thus,

$$-\frac{\Gamma_1\Gamma_2}{4\pi}\big[Z,\log s_3^2\big]=\frac{(Q+iP)-\gamma_1 z_3}{8\pi i\Delta}\cot C.$$

The contributions to $[Z,H]$ from the other two terms in the Hamiltonian follows by permutation of indices, and we obtain again (27b), namely,

$$\begin{aligned}\dot Z=[Z,H]=\frac{1}{8\pi i\Delta}&[(Q+iP)(\cot A+\cot B+\cot C)\\ &-\gamma_1(z_1\cot A+z_2\cot B+z_3\cot C)\,].\end{aligned}$$

However, it is clear that the geometrical derivation is much simpler than the direct algebraic approach.

F. EOM for the radius of the circumcircle revisited

In Sec. III A we derived the equation of motion for R. A different derivation is obtained by starting from the geometrical result (17a) and using the equations of motion for the sides, (9a), and for the area, (9c). These may then be combined to produce (26b).

Here we pursue a somewhat different avenue. We note the following relation between Z and R: Multiply the first of (36) by Γ_1, the second by Γ_2, and the third by Γ_3, and add the results. This gives

$$I_0+\gamma_1 Z\bar Z-(Q+iP)\bar Z-(Q-iP)Z=\gamma_1 R^2, \tag{42a}$$

where Q and P are as in (3) and I_0 as in (5a).

If the sum of the vortex circulations, γ_1, vanishes, the projection of the vector from the origin to the circumcenter, Z, onto the (constant) linear impulse is constant. Thus, the circumcenter travels along a line perpendicular to $Q+iP$. Analysis [18,19] shows that the vortices periodically become collinear for all initial conditions. As the vortices become collinear, Z recedes to infinity along a line perpendicular to $Q+iP$. In other words, the collinear vortices are situated along $Q+iP$. If $Q=P=0$, the vortices must remain collinear and will, as the analysis shows [18,19], rotate like a rigid body.

The same conclusions are reached from (27b). For $\gamma_1=0$ in that equation, the second term in square brackets is absent and $\dot Z$ is an imaginary number times $Q+iP$.

In the general case, $\gamma_1\neq 0$, we may view (42a) as an example of the parallel axis theorem (6). Using (5b) to write $I_Z=I_{cv}+\gamma_1|Z-z_{cv}|^2$, and combining this equation with (5a) allows us to write (42a) as (for $\gamma_1\neq 0$)

$$|Z-z_{cv}|^2=R^2-\frac{I_{cv}}{\gamma_1}. \tag{42b}$$

Since z_{cv} (4) and I_{cv} are dynamical invariants, this equation shows that if we have determined the evolution of Z, we also have the evolution of R. If we place z_{cv} at the origin, which requires $\gamma_1 \neq 0$, we have that $|Z|^2 - R^2$ is a constant of the motion.

Equation (42b) has an interesting geometrical interpretation. By (7), $L = \gamma_1 I_{\text{cv}}$, and the sign of I_{cv}/γ_1 is the same as the sign of L. If $L = 0$, (42b) reduces to

$$|Z - z_{\text{cv}}|^2 = R^2. \tag{43}$$

Thus, while the circumcircle is, in general, time dependent, it evolves in such a way that one point on the circumcircle is always at the center of vorticity.

If $L > 0$, then (42b) can be written as

$$|Z - z_{\text{cv}}|^2 + \left|\frac{I_{\text{cv}}}{\gamma_1}\right| = R^2, \tag{44}$$

and we see that the center of vorticity must lie inside the circumcircle for all times. Furthermore, the circumradius R has a lower bound, namely, $R_{\min} = \sqrt{I_{\text{cv}}/\gamma_1}$, which corresponds to the circumcenter coinciding with the center of vorticity. On the other hand, if $L < 0$, (42b) may be written as

$$|Z - z_{\text{cv}}|^2 = R^2 + \left|\frac{I_{\text{cv}}}{\gamma_1}\right|, \tag{45}$$

and we see that the center of vorticity must lie outside the circumcircle for all times. In this case, there is a disk of radius $\sqrt{I_{\text{cv}}/\gamma_1}$ centered at z_{cv} that is a forbidden region for the circumcenter.

IV. SPECIAL SOLUTIONS

The equations of motion for R (26), A, B, C (30), Z (27), and φ_1 (31) provide a different perspective on three-vortex motion than that given by (2) or (9). We utilize this feature to extract some results on three-vortex motion that are more difficult to derive from other forms.

A. Motions with constant *R*

Motions for which R is constant are known to exist, e.g., the relative equilibria (and for the collinear relative equilibria R is infinite). One might ask if there are others. When R is a constant, from (28) and (29) we have simplified expressions for $\dot{A}$, $\dot{B}$, and $\dot{C}$, viz.,

$$\frac{dA}{dt} = \frac{\Gamma_1}{8\pi R^2}\frac{\cot B - \cot C}{\cot A}, \quad \frac{dB}{dt} = \frac{\Gamma_2}{8\pi R^2}\frac{\cot C - \cot A}{\cot B}, \quad \frac{dC}{dt} = \frac{\Gamma_3}{8\pi R^2}\frac{\cot A - \cot B}{\cot C}. \tag{46}$$

The two dynamical integrals of these equations, (32), can be rearranged to give

$$\frac{\sin^2 A}{\Gamma_1} + \frac{\sin^2 B}{\Gamma_2} + \frac{\sin^2 C}{\Gamma_3} = \frac{L}{4\gamma_3 R^2} = \lambda_1 \tag{47a}$$

and

$$\frac{\log \sin A}{\Gamma_1} + \frac{\log \sin B}{\Gamma_2} + \frac{\log \sin C}{\Gamma_3} = -\frac{2\pi H + \gamma_2 \log(R/R_0)}{\gamma_3} = \lambda_2, \tag{47b}$$

where the λ_i are constants. These integrals define surfaces in a $(\sin A, \sin B, \sin C)$ space, and these two surfaces intersect (at most) in a curve. Adding to these equations the condition $A + B + C = \pi$ (18), we can at most expect isolated triples (A, B, C) to give solutions, so that $\dot{A} = \dot{B} = \dot{C} = 0$. It then follows from (46) that we must either have $\cot A = \cot B = \cot C$ or $\tan A = \tan B = \tan C$; in the latter case we also have $R \to \infty$. The first restriction yields the equilateral triangle, $A = B = C = \frac{\pi}{3}$. The second restriction gives the collinear relative equilibria as a singular limit. Thus, *there are no three-vortex motions with constant, finite R other than the equilateral triangle configurations.*

Expressions for the constants of motion simplify for the equilateral triangle equilibria. Substitution of $A = B = C = \pi/3$ into (17b) gives the area of the triangle as

$$\Delta = \frac{3\sqrt{3}}{4} R^2. \tag{48}$$

Substitution into (47) reduces the equations for the impulse and the Hamiltonian to

$$L = 3\gamma_2 R^2, \tag{49a}$$

$$H = -\frac{\gamma_2}{2\pi} \log\left[\frac{\sqrt{3}}{2}\left(\frac{R}{R_0}\right)\right]. \tag{49b}$$

Since R is a constant, we may choose to define $R_0 = R$ without any loss of generality; by (49b) this choice establishes $H = 0$ as the value of the Hamiltonian for all equilibrium motions with constant R. If $\gamma_2 = 0$, we also have $L = 0$ regardless of the value of R; the equilateral triangle configurations with $\gamma_2 = 0$ and $R =$ constant (corresponding to $L = 0$ and $H = 0$, respectively) are the limiting cases of the self-similar motion discussed in Sec. IV B. For $\gamma_2 \neq 0$, by (49a) we see that equilateral triangle equilibria with a finite and nonzero R can exist only if $\gamma_2 L > 0$.

Now consider the motion of the center Z of the circumcircle, given generally by (27). Substituting $A = B = C = \pi/3$, we obtain

$$\dot{Z} = \frac{e^{i\varphi_1}}{12\pi R}[-2i\Gamma_1 + (\sqrt{3}+i)\Gamma_2 - (\sqrt{3}-i)\Gamma_3], \tag{50}$$

which shows that the time dependence of $\dot{Z}$ comes only through $\varphi_1(t)$. To find an expression for $\varphi_1(t)$, we substitute $A = B = C = \pi/3$ into (31) and obtain $\dot{\varphi}_1 = \Omega =$ constant, where

$$\Omega = \frac{\gamma_1}{6\pi R^2}. \tag{51}$$

Integrating $\dot{\varphi}_1$ gives $\varphi_1(t) = \Omega t$, where we have chosen the orientation of the coordinate axis to be such that $\varphi_1(0) = 0$.

To solve for the motion of Z, consider first the case when the sum of vortex strengths vanishes, i.e., $\gamma_1 = 0$. We will then have $\dot{\varphi}_1 = 0$ and $\dot{Z} =$ constant, corresponding to translating equilibria. The motion of the circumcenter can be written in terms of the constants of the motion by integrating (27b) to give

$$Z(t) = \frac{Q + iP}{6\pi i R^2} t = \frac{(Q+iP)\gamma_2}{2\pi i L} t, \tag{52}$$

where we have chosen $Z(0) = 0$. This expression for $Z(t)$ matches the previously known result [20, Eq. (2.15)].

Next, consider the motion of Z when $\gamma_1 \neq 0$, in which case we can choose the origin of coordinates such that the center of vorticity is at the origin, i.e., $z_{cv} = (Q + iP)/\gamma_1 = 0$. With this choice, (27b) reduces to

$$\dot{Z} = \frac{-\gamma_1}{8\sqrt{3}\pi\Delta i}(z_1 + z_2 + z_3).$$

By (19) we have

$$z_1 + z_2 + z_3 = 3Z + \mathrm{R}e^{i\varphi_2}(e^{-i2C} + 1 + e^{i2A}) = 3Z,$$

since $A = B = C = \pi/3$, so that $\dot{Z} = i\,\Omega\, Z$, where Ω is as defined in (51). Integration of this equation for $\dot{Z}$ gives

$$Z = \rho e^{i\Omega t}, \tag{53a}$$

where we have again chosen $Z(0) = 0$. By combining (7), (42b), and (49a) we have

$$\rho = \left(R^2 - \frac{L}{\gamma_1^2}\right)^{1/2} = \sqrt{L}\left(\frac{1}{3\gamma_2} - \frac{1}{\gamma_1^2}\right)^{1/2}. \tag{53b}$$

With the circumcenter and the vortex triangle moving in circular motion with the same angular velocity Ω (51), the vortex system is moving in rigid body rotation. This motion and the rotation rate are consistent with the known result for the rotating equilibrium configuration [20, Eq. (2.16)].

The radius of the circumcenter's path, ρ, in (53b) is real valued for all allowed values of γ_1, γ_2, and L. If $\gamma_2 < 0$, then from the condition that $\gamma_2 L > 0$ we have $L < 0$, and ρ will be real according to the first equality in (53b). If $\gamma_2 = 0$, then $L = 0$, giving $\rho = R$. If $\gamma_2 > 0$, then $L > 0$, and according to the second equality in (53b) we must have

$$\gamma_1^2 > 3\gamma_2 \Rightarrow \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 > \gamma_2 \tag{54}$$

for ρ to be real; it can be shown that this inequality holds for any choice of vortex strengths. Finally, if we consider $\rho = 0$, we are led to the equality $\gamma_1^2 = 3\gamma_2$. The only possible solution for this equality is $\Gamma_1 = \Gamma_2 = \Gamma_3$; this can be seen by solving $\gamma_1^2 = 3\gamma_2$ as a quadratic for Γ_3. In this case we have both "geometrical symmetry," i.e., the equilateral triangle configuration, and "physical symmetry," i.e., symmetry in the vortex strengths. As expected, this symmetry leads to $Z = 0$, and the circumcenter coincides with the center of vorticity.

The translating equilibria given by (52) do not contain stationary equilibria as a special case since $\gamma_1 = 0 \Rightarrow Q + iP \neq 0$ and $\gamma_2 \neq 0$. On the other hand, the rotating equilibria given by (53) also do not contain stationary equilibria as a special case; by (51), taking $\Omega = 0$ and $\gamma_1 \neq 0$ requires $R \to \infty$, corresponding to collinear equilibria. We conclude that there are no stationary equilateral triangle equilibria, which is consistent with the known theory [20].

B. Motions with invariant triangle shape

By (26b), motions for which A, B, C are constant imply either that R^2 is constant or that it grows linearly with time. The former case yields the equilateral triangle relative equilibria discussed in Sec. IV A. In the latter case we have

$$R(t) = R_0\sqrt{1 - \frac{t}{\tau}}, \tag{55}$$

where R_0 is the initial value of R and the time scale τ is given by

$$\begin{aligned}\frac{4\pi R_0^2}{\tau} = &-[\Gamma_1 \cot B \cot C(\cot B - \cot C) + \Gamma_2 \cot C \cot A(\cot C - \cot A)\\ &+ \Gamma_3 \cot A \cot B(\cot A - \cot B)].\end{aligned}$$

But when A, B, C are constants we have from (30) that this expression can be written in any of the forms

$$-\frac{4\pi R_0^2}{\tau} = \Gamma_1(\cot B - \cot C) = \Gamma_2(\cot C - \cot A) = \Gamma_3(\cot A - \cot B). \tag{56}$$

These may be the simplest expressions known for τ which, when positive, gives the time of collapse [21,22]. When $\tau < 0$ we instead have self-similar expansion.

The constants of motion (32) enable us to establish the necessary and sufficient conditions for self-similar motion. The condition that the angles A, B, C be constant (but R is not constant) means

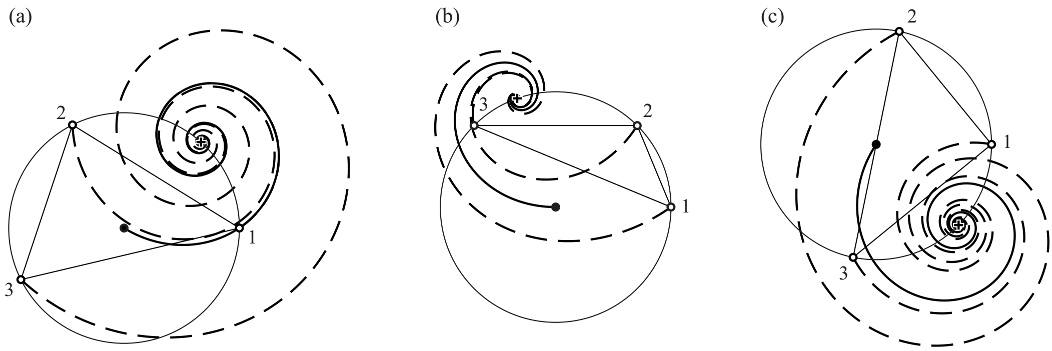

FIG. 2. Vortex triangle trajectories in self-similar collapse. Triangles and circumcircles are shown with light solid lines. Open circles mark the initial vortex positions, labeled by vortex number; closed disks mark the initial position of the circumcenter; and the $+$ symbols mark the point of collapse. Vortex trajectories are shown with dashed lines, and circumcenter trajectories are shown with heavy solid lines. For each triangle, the circumcircle has initial radius $R_0 = 1$; the remaining parameters for each example are (a) $(\Gamma_1,\Gamma_2,\Gamma_3) = (1,1,-1/2)$, $(A,B,C) = (45°,77.4°,57.6°)$, $H = -0.003\,99$, $\tau = 32.9$; (b) $(\Gamma_1,\Gamma_2,\Gamma_3) = (1,-2,-2)$, $(A,B,C) = (45°,112.5°,22.5°)$, $H = -0.110$, $\tau = 4.44$; and (c) $(\Gamma_1,\Gamma_2,\Gamma_3) = (1,-1/3,1/2)$, $(A,B,C) = (90°,50.4°,39.6°)$, $H = -0.003\,98$, $\tau = 30.8$.

that $L = 0$ and $\gamma_2 = 0$ in order for (32) to be constants of motion. On the other hand, if $L = 0$ and $\gamma_2 = 0$, we get the following set of equations:

$$\frac{\sin^2 A}{\Gamma_1} + \frac{\sin^2 B}{\Gamma_2} + \frac{\sin^2 C}{\Gamma_3} = 0, \tag{57a}$$

$$\frac{\log\,\sin A}{\Gamma_1} + \frac{\log\,\sin B}{\Gamma_2} + \frac{\log\,\sin C}{\Gamma_3} = -\frac{2\pi H}{\gamma_3}. \tag{57b}$$

These two equations intersect in a curve, as we saw in the previous section. Together with the condition $A + B + C = \pi$ (18), this means that there can at most be isolated triples as solutions. Thus, the angles A, B, C must be constant. We conclude that $L = 0$ and $\gamma_2 = 0$ are necessary and sufficient conditions for self-similar motion.

The geometric formulation facilitates a detailed description of self-similar motion. By (43), the circumcircle passes through the center of vorticity, z_{cv}, when $L = 0$. Also by (43), as $R \to 0$, we have $Z \to z_{\text{cv}}$, and the point of collapse coincides with the center of vorticity. Thus, for self-similar collapse, the point of collapse always lies on the circumcircle. It appears that a number of other results can be derived for self-similar motion using this geometric formulation; we intend to present these results in a future paper.

Examples of self-similar vortex motion are shown in Fig. 2. Given a set of vortex strengths and one of the angles in the triangle, (57a) and the condition $A + B + C = \pi$ are used to determine the other two angles in the triangle. The value of the Hamiltonian is then determined from (57b), and the value of τ is determined by (56). The evolution of the triangle orientation is given by solving (31), and then the motion of the circumcenter can be determined using (27c). Finally the vortex trajectories themselves are obtained using (19) and (20).

V. SUMMARY AND OUTLOOK

The three-vortex problem has a long, rich history. Previous approaches to describing and solving the three-vortex problem have focused primarily on the dynamics of the intervortex distances. In a complementary approach, we have obtained the equations of motion using a different set of variables. Starting from geometry and using the known equations of motion, we have shown that the motion of three vortices can be regarded as the motion of the center of the circle that circumscribes the three vortices, plus a motion of the three vortices about this circumcenter. The motion about the circumcenter consists of two parts: the first part is determined by the four autonomous equations for R (26) and A,B,C (30), and the second part is determined by the equation for φ_1 (31), which gives the orientation of the triangle in the plane. With the constants of motion L and H (32) and the geometrical constraint on the angles (18), the embedded dynamical system consisting of R,A,B,C is integrable. The transformation from the variables z_α to Z, R, and φ_α is not a canonical transformation. What this implies, however, is not clear.

We have examined some of the special solutions to these equations and have shown that these solutions exist under conditions that can be determined from the constants of motion. The geometric approach used here reveals some aspects of these well-known special solutions that had not been considered previously. These aspects will be described in detail in a future paper. One might expect to find other solutions to these equations that throw light on the three-vortex problem from a different perspective.

The geometrical method has been developed here in the context of the motion of three point vortices on the unbounded plane. However, this approach is not necessarily limited to this problem, since the main idea rests on the fact that the constants of motion only depend on the intervortex distances. There are several similar problems where the method developed here may be applicable, such as the motion of three point vortices on a sphere, which is of geophysical interest [11]. The viscous evolution of three point vortices, i.e., the Navier-Stokes evolution of initial conditions given by inviscid point vortex configurations, are also of considerable interest [23,24], and a similar geometric approach may provide new insight. We also mention recent work on the Euler-α system [25,26] whose singular solutions, the α-point vortices, also exhibit self-similar collapse. Such solutions have applications to the discussion of two-dimensional turbulence, and the α-point vortex system has a Hamiltonian structure that depends only on the inter-α-point vortex distances [26]. Finally, one may speculate if the extension of the geometrical method to $N > 3$ point vortices might provide a new perspective on N vortex motion.

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