Vector Calculus

Dr. D. Sukumar

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Green's theorem

Divergence of a vector field

Flux density at a point

Let $F = M(x, y)i + N(x, y)j$

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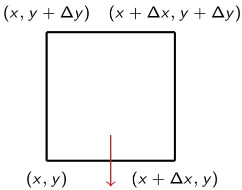
Flux across the edges

Bottom $F(x,y)\cdot(-j)\,\Delta x = -N(x,y)\,\Delta x$



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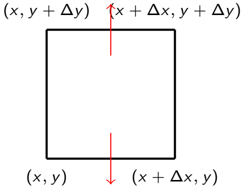

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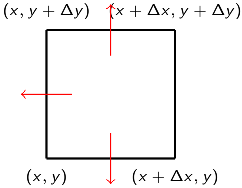

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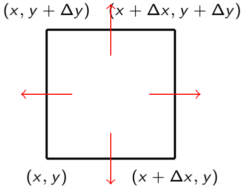

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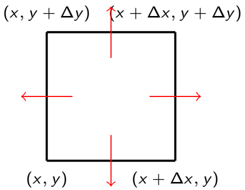

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Combining opposite forces we get

$[N(x,y+\Delta y) - N(x,y)]\,\Delta x \approx \frac{\partial N}{\partial y}\Delta y\Delta x$

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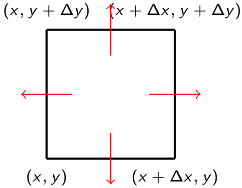

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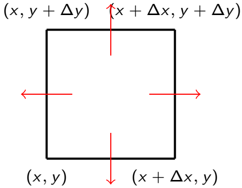

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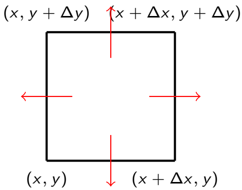

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$[N(x,y+\Delta y) - N(x,y)]\,\Delta x \approx \frac{\partial N}{\partial y}\Delta y\Delta x$

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On adding we get $\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x\Delta y$

Total flux per unit area $= \frac{\text{Flux across the boundary}}{\text{Rectangle area}}$

Flux density/the divergence of $F = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$

Curl of a vector field

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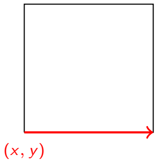

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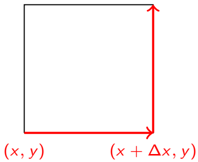

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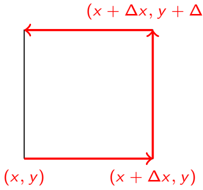

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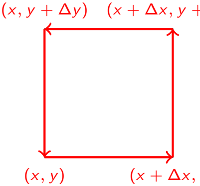

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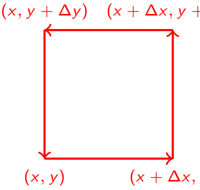
$$-\left[M\,(x,y+\Delta y) - M(x,y)\right]\Delta x = -\frac{\partial M}{\partial y}\Delta y\Delta x$$



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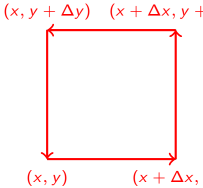

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Paring the opposite directions we get

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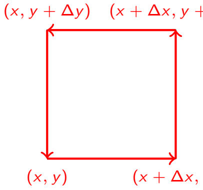
Adding them we get $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y$



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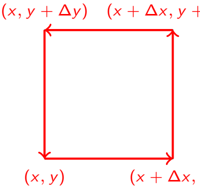

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Circulation density $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

Green's theorem

Normal form

Now by knowing that divergent represents the flux density (net flow) at a point

Theorem (Flux divergence or Normal form)

$$\oint_C F \cdot n dr = \oint_C M dy - N dx = \int\int_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

- The sum of the Flux density at all points is equal to sum of F in the normal direction at the boundary curve

Green's theorem

Tangent form

Now by knowing that curl represents the circulation density at a point

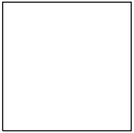
Theorem (Circulation-Curl or Tangent form)

$$\oint F \cdot T dr = \oint_C M dx + N dx = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

- Sum of the Circulation density at all point is equal to sum of F in the tangent direction at the boundary curve.

Intuitive Proof

Green's Theorem Normal Form



In every line common to two adjacent boxes, the normal components across the line have opposite directions and hence net result is zero. What left out is boundary lines, hence the net result is just concentrated on the boundary.

Intuitive Proof

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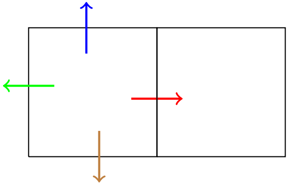

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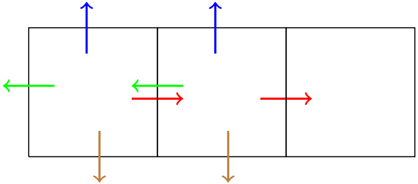

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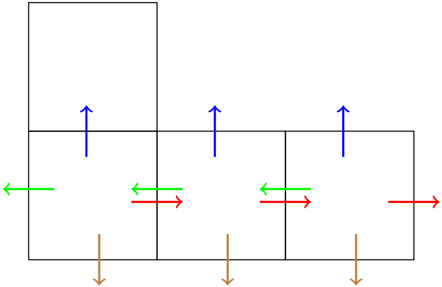

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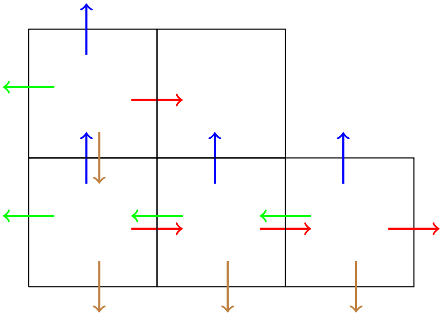

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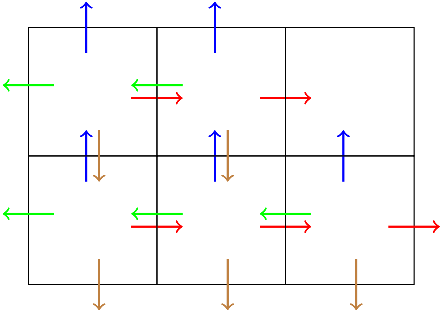

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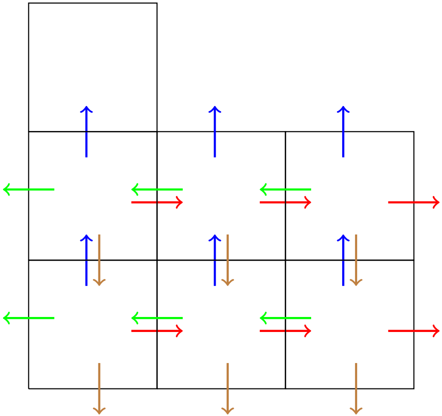

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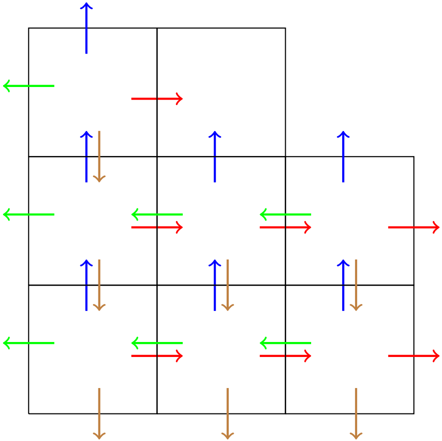

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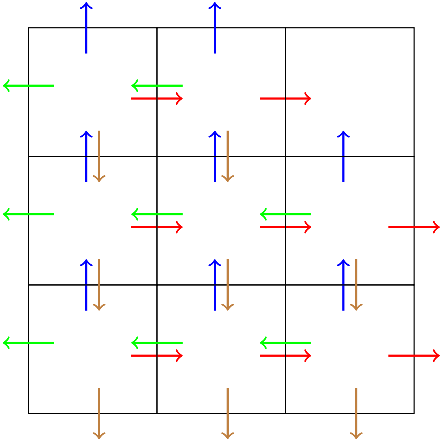

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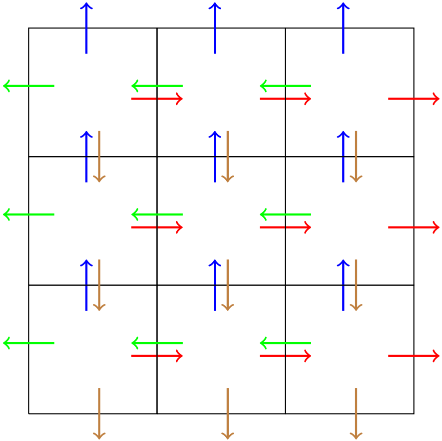

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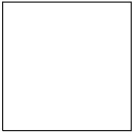
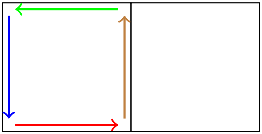
Green's Theorem Tangent Form



Every internal line adjacent to two points contributes twice but in opposite direction and hence the net becomes zero on the internal lines. What left out is boundary lines, hence the net result is just concentrated on the boundary.

Intuitive Proof

Green's Theorem Tangent Form



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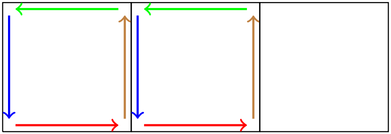

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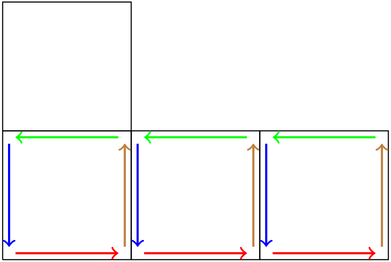

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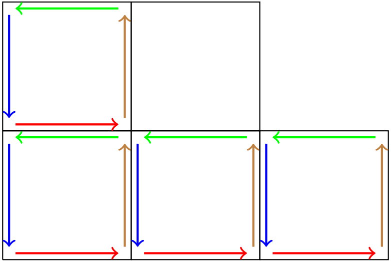

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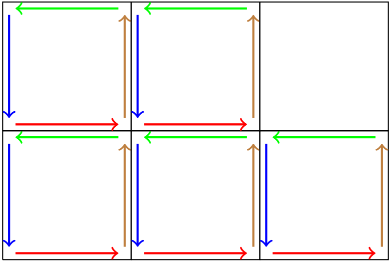

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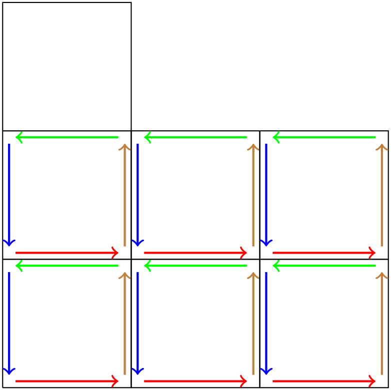

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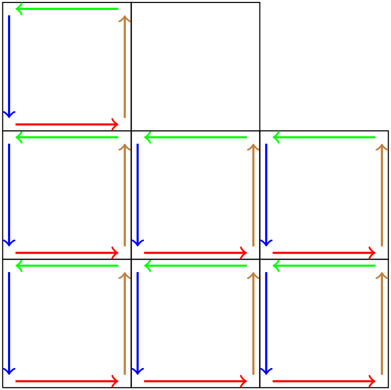

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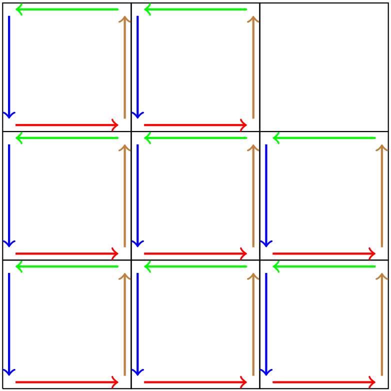

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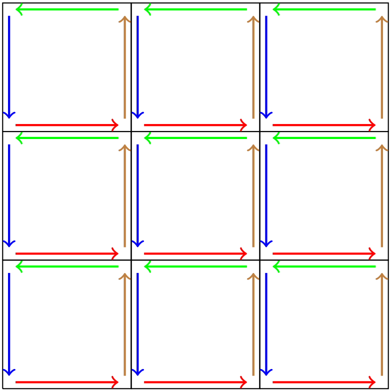
Intuitive Proof

Green's Theorem Tangent Form



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Conditions

$$\oint_C F \cdot n dr = \oint_C M dy - N dx = \int\int_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$\oint F \cdot T dr = \oint_C M dx + N dy = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

To make this expression a valid one we need some conditions on the domain and functions

Condition on $F = Mi + Nj$

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To make this expression a valid one we need some conditions on the domain and functions

Condition on $F = Mi + Nj$

1. M, N partial derivatives are continuous at each point an open set containing the region R.

Conditioins on curve.

1. Simple curve.
2. Closed curve.
3. Finitely many non differential points (picewise smooth)

Connecting conditions:

1. On each piece of the curve C, the functions M, N are integrable.

Under these conditions. Green's Theorem holds.

Proof

Tangent or Circulation-curl form

Let C be the smooth simple closed curve such that any line parallel to the axis does not intersect it more than two points. Integrate $\frac{\partial M}{\partial y}$ with respect to y from $f_1(x)$ to $f_2(x)$

Proof

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$$\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y} \, \mathrm{d}y = M(x, f_2(x)) - M(x, f_1(x))$$

Proof

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$$\begin{aligned}
\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,\mathrm{d}y\mathrm{d}x &= \int_a^b M(x, f_2(x)) - M(x, f_1(x))\,\mathrm{d}x \\
&= -\int_0^a M(x, f_2(x))\,\mathrm{d}x - \int_a^b M(x, f_1(x))\,\mathrm{d}x \\
&= -\int_{C_1} M\,\mathrm{d}x - \int_{C_2} M\,\mathrm{d}x \\
&= -\oint_c M dx
\end{aligned}$$

Proof.

Similarly, we get

$$\int_c^d \int_{g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x}\,\mathrm{d}y \quad = \quad \oint_c N dy$$

Adding them we get,

$$\oint (M dx + N dy) = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

□

Exercise

1. Verify both forms of the Green's theorem on disk $x^2 + y^2 \leq a$ and its boundary circle $r = a\cos t\ \mathbf{i} + a\sin t\ \mathbf{j}, 0 \leq t \leq 2\pi$
   1.1 $F = -y\mathbf{i} + x\mathbf{j}$
   1.2 $F = y\mathbf{i}$
   1.3 $F = 2x\mathbf{i} - 3y\mathbf{j}$
   1.4 $F = -x^2y\mathbf{i} + xy^2\mathbf{j}$
2. Apply Green's theorem to evaluate $\oint y^2dx + x^2dy$ over the triangle bounded by $x = 0,\ x + y = 1,\ y = 0$
3. Show that the value of $\oint xy^2dx + (x^2y + 2x)dy$ around any square depends only on the area of the square not on its location.