

Vector Calculus

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Vector fields

Gradient

Divergence

Curl

▶ Now

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- ▶ **Vector fields**

Draw the vector fields of

1. $F(x, y) = (x, y)$
2. $F(x, y) = -y\mathbf{i} + x\mathbf{j}$
3. $M(x, y) = y, N(x, y) = x$
4. $F(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}}$

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Fluid Flow when F is velocity field.

Circulation and Flux

Let C be closed curve.

Circulation Integration of scalar component in the tangent direction of F .

$$\oint_C F \cdot T ds = \oint_C F \cdot dr = \oint_C F \cdot \frac{dr}{dt} dt = \oint_C M dx + N dy$$

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Flux integration of scalar component in the normal direction of F .

$$\oint_C F \cdot n ds = \oint_C F \cdot (T \times k) ds = \oint_C M dy - N dx$$

Gradient field, Potential function

Let $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$

1. gradient of ϕ ($\text{grad}\phi$).

$$\nabla\phi = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k \right) \phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

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2. Every ϕ (with necessary conditions) creates a vector field $\nabla\phi$. These fields are called **gradient field**. If the vector field F is coming from such a function ϕ then ϕ is called a **potential function of the vector field**.

Excercise

1. Find the gradient field of $g(x, y, z) = e^z - \ln(x^2 + y^2)$
2. Give a formula $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ for the vector field in the plane that has the property that F points towards the origin with magnitude inversely proportional to the square of the distance from the point (x, y, z) to origin.

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2. Condition for the exactness or conservativeness of $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}; \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}; \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$$

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Partial derivative of first two equations tells

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By Euler's theorem on mixed derivatives, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Exercise

1. Which of the fields are conservative?

1.1 $yzi + xzj + xyk$

1.2 $yi + (x + z)j - yk$

2. Find the potential function ϕ for the field F

2.1 $2xi + 3yj + 4zk$

2.2 $(\ln x + \sec^2(x + y))i + (\sec^2(x + y) + \frac{y}{y^2 + z^2})j + \frac{z}{y^2 + z^2}k$

Gradient f

∇f , grad f

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- ▶ **Scalar to vector:** $f \rightarrow \nabla f$

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- ▶ f has maximum change in the direction of ∇f
- ▶ f has minimum change in the direction of $-\nabla f$

Divergent F

$\text{div} \cdot F, \nabla \cdot F$

$$\begin{aligned}\nabla \cdot F &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \cdot (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) \\ &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\end{aligned}$$

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- ▶ **vector to scalar:** $F \rightarrow \nabla \cdot F$
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- ▶ $\text{div } F = \text{outflow} - \text{inflow}$ (source or sink)
- ▶ $\nabla \cdot F = 0$ incompressible, solenoidal

Curl F

Curl F , $\nabla \times F$

$$\begin{aligned}\nabla \times F &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \times (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}\end{aligned}$$

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- ▶ Circulation density

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- ▶ $\text{Curl } F \approx \text{Rotation about an axis,}$
- ▶ $\nabla \times F = 0$ implies Irrotational.