Vector Calculus

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Plane curves, Space curves

Vector fields

Parametric curves

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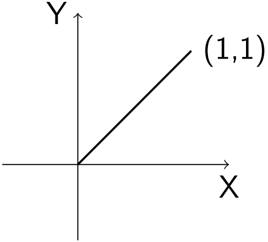

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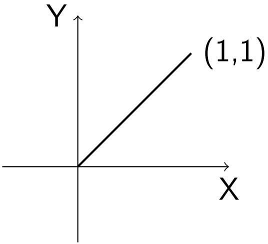

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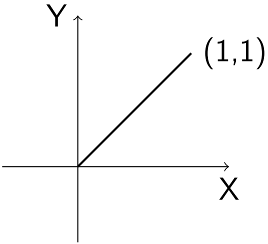

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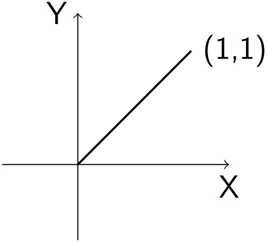

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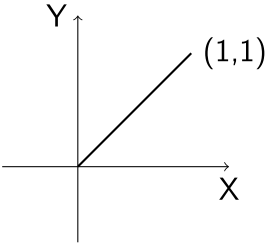

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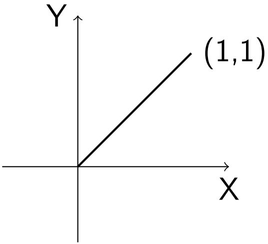

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- $r(t) = (t/2)\mathbf{i} + (t/2)\mathbf{j}; \quad 0 \leq t \leq 2$

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Let C be a space curve and $f : \mathbb{R}^3 \to \mathbb{R}$ is a function defined on this curve. Then the path integral/line integral is

$$\int_C f(x, y, z)ds = \int_a^b f(g(t), h(t), k(t))|v(t)|dt$$

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where the line element is
$ds = |v(t)|dt = |dr/dt|dt = \sqrt{(\frac{dg}{dt})^2 + (\frac{dh}{dt})^2 + (\frac{dk}{dt})^2}$.

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3. Evaluate $\int_a^b f(g(t), h(t), k(t))|v(t)|dt$

Exercise

1. Integrate $f(x,y) = \frac{x^3}{y}$ over the curve
$$C : y = x^2/2; \quad 0 \le x \le 2$$
$\frac{10\sqrt{5}-2}{3}$

2. Integrate $f(x,y) = x + y$ over the curve
$$C : x^2 + y^2 = 4$$
in the first quadrant from (2,0) to (0,2) 8.

3. Integrate $f(x,y,z) = x - 3y^2 + z$ over the line segment C joining the origin and the point $(1,1,1)$ 0

4. Integrate the same function on another path with (0,0,0) to (1,1,0) and (1,1,1). $-\frac{(\sqrt{2}+3)}{2}$

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  - Vector fields

Draw the vector fields of

1. $F(x, y) = (x, y)$
2. $F(x, y) = -y\mathbf{i} + x\mathbf{j}$
3. $M(x, y) = y, N(x, y) = x$
4. $F(x, y) = \frac{-y\mathbf{i}+x\mathbf{j}}{\sqrt{x^2+y^2}}$

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Fluid Flow when F is velocity field.

Circulation and Flux

Let C be closed curve.

Circulation Integration of scalar component in the tangent direction of F.

$$\oint_C F\cdot Tds = \oint_C F\cdot dr = \oint_C F\cdot\frac{dr}{dt}dt = \oint_C Mdx+Ndy$$

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Flux integration of scalar component in the normal direction of F.

$$\oint_C F \cdot n ds = \oint_C F \cdot (T \times k) ds = \oint_C M dy - N dx$$