Vector Calculus

Dr. D. Sukumar

January 14, 2016

Change of variables

Cartesian, Cylindrical and Spherical

Change of variable

In evaluating the integration $\int_a^b f(x)dx$ we do a change of variable $t = 2x$

$$\int_a^b f(x)\, dx$$

Change of variable

In evaluating the integration $\int_a^b f(x)dx$ we do a change of variable $t = 2x$

$$\int_a^b f\left(\frac{t}{2}\right) dx$$

- $t = 2x \Rightarrow x = \frac{t}{2}$

Change of variable

In evaluating the integration $\int_a^b f(x)dx$ we do a change of variable $t = 2x$

$$\int_{2a}^{2b} f\left(\frac{t}{2}\right) dx$$

- $t = 2x \Rightarrow x = \frac{t}{2}$
- $x = a \Rightarrow t = 2a, x = b \Rightarrow t = 2b$

Change of variable

In evaluating the integration $\int_a^b f(x)dx$ we do a change of variable $t = 2x$

$$\int_{2a}^{2b} f\left(\frac{t}{2}\right) \frac{dt}{2}$$

- $t = 2x \Rightarrow x = \frac{t}{2}$
- $x = a \Rightarrow t = 2a, x = b \Rightarrow t = 2b$
- $x = \frac{t}{2} \Rightarrow dx = \frac{dt}{2}$

Change of variable

Let G be the region in u, v plane.

Change of variable

Let G be the region in u, v plane.
It is transformed into another region R in the xy-plane as a one-to-one map by the equation.

$$x = g(u, v), y = h(u, v).$$

Change of variable

Let G be the region in u, v plane.
It is transformed into another region R in the xy-plane as a one-to-one map by the equation.

$$x = g(u, v), y = h(u, v).$$

We call, R is the image of G, G is the pre-image of R.

Change of variable

Let G be the region in u, v plane.
It is transformed into another region R in the xy-plane as a one-to-one map by the equation.

$$x = g(u, v), y = h(u, v).$$

We call, R is the image of G, G is the pre-image of R.
Any function f defined on R can be thought of as a function $f(g(u, v), h(u, v))$ defined on G as well.

so

What is the relation between integral f over R and integral $f(g(u,v),h(u,v))$ over G. If

1. f,g,h have continuous partial derivatives and
2. $J(u,v)$ is zero only at isolated points then
   $\iint_R f(x,y)dxdy = \iint_G f(g(u,v),h(u,v))|J(u,v)|dudv$

Jacobian

Jacobian determinant or Jacobian of the co-ordinate transformation

$$x = g(u, v), y = h(u, v).$$

is

$$J(x, y; u, v) = \left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = \frac{\partial(x, y)}{\partial(u, v)} = J(u, v)$$

Example

Evaluate

$$\iint_R (2x^2 - xy - y^2)dxdy$$

enclosed by the triangular region with vertices (0,0), (1,1) and (1, -2) in the *xy*-plane by changing to the co-ordinates u, v given by the relation $u = x - y$; $v = 2x + y$.

Method

- Solve the system $u = x - y$; $v = 2x + y$ for x and y in-terms of u and v.

Method

- Solve the system $u = x - y$; $v = 2x + y$ for x and y in-terms of u and v.
- Find the Jacobian.

Method

- Solve the system $u = x - y$; $v = 2x + y$ for x and y in-terms of u and v.
- Find the Jacobian.
- Let R be the triangle region with vertices (0,0), (1,1) and (1, -2) in the xy-plane. Find the image of R and sketch the transformed region in uv-plane.

Method

- Solve the system $u = x - y$; $v = 2x + y$ for x and y in-terms of u and v.
- Find the Jacobian.
- Let R be the triangle region with vertices (0,0), (1,1) and (1, -2) in the xy-plane. Find the image of R and sketch the transformed region in uv-plane.
- Evaluate
$$\iint_R (2x^2 - xy - y^2) dxdy$$
enclosed by the triangular region.

Method

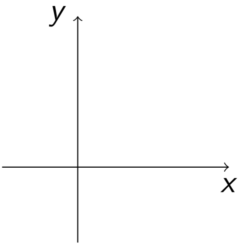
- Solve the system $u = x - y$; $v = 2x + y$ for x and y in-terms of u and v.
- Find the Jacobian.
- Let R be the triangle region with vertices (0,0), (1,1) and (1, -2) in the xy-plane. Find the image of R and sketch the transformed region in uv-plane.
- Evaluate
$$\iint_R (2x^2 - xy - y^2)dxdy$$
enclosed by the triangular region.
- Evaluate the same integral in the region bounded by the lines $R : y = -2x + 4, y = -2x + 7, y = x - 2, y = x + 1$

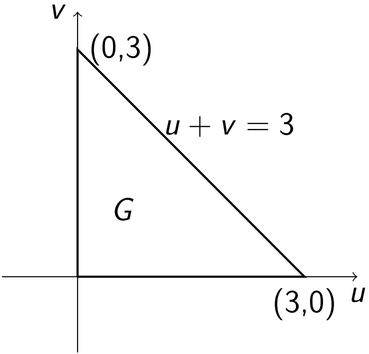
$u = x - y;\ v = 2x + y$

- $x = \frac{u+v}{3}; y = \frac{-2u+v}{3}$

$u = x - y; \; v = 2x + y$

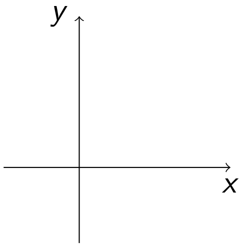
- $x = \frac{u+v}{3}; y = \frac{-2u+v}{3}$
- $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = \left| \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \right| = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$

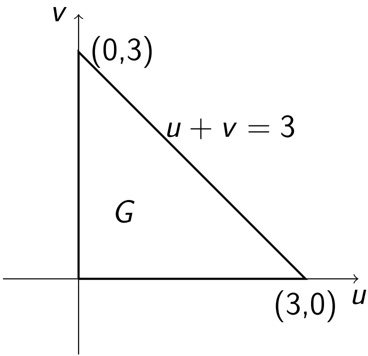




$u = x - y;\ v = 2x + y$

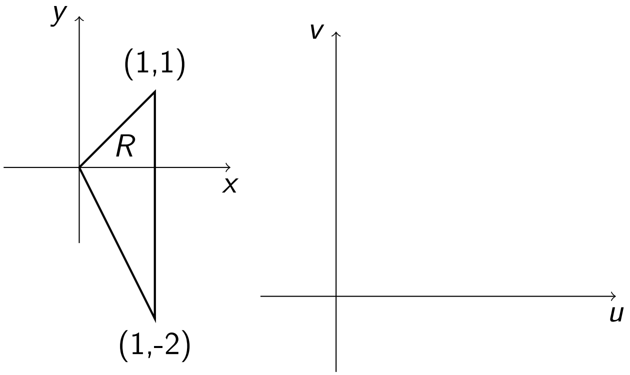
- $x = \frac{u+v}{3}; y = \frac{-2u+v}{3}$
- $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = \left| \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \right| = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$





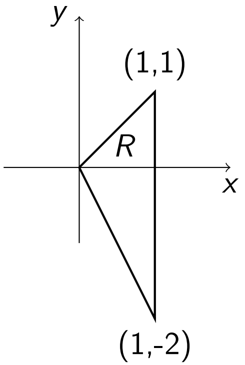
$u = x - y;\ v = 2x + y$

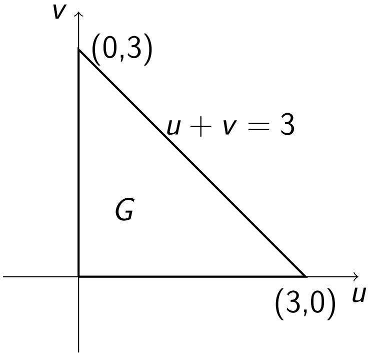
- $x = \frac{u+v}{3};\ y = \frac{-2u+v}{3}$
- $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = \left| \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \right| = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$



$u = x - y;\ v = 2x + y$

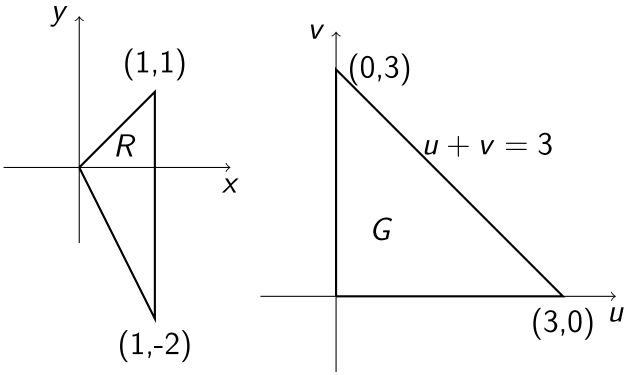
- $x = \frac{u+v}{3};\ y = \frac{-2u+v}{3}$
- $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = \left| \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \right| = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$





$u = x - y;\ v = 2x + y$

- $x = \frac{u+v}{3};\ y = \frac{-2u+v}{3}$
- $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \left|\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| = \left|\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix}\right| = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$



$$\int_0^3 \int_0^{v-3} (2x^2 - xy - y^2)\frac{1}{3}\,du\,dv$$

Example(cont.)

$$\iint_R (2x^2 - xy - y^2)\,dxdy$$

- $u = x - y; v = 2x + y \Rightarrow x = \frac{u+v}{3}; y = \frac{-2u+v}{3}$

Example(cont.)

$$\iint_R (2x^2 - xy - y^2)dxdy$$

- $u = x - y; v = 2x + y \Rightarrow x = \frac{u+v}{3}; y = \frac{-2u+v}{3}$
- $R : y = -2x + 4, y = -2x + 7, y = x + 1, y = x - 2 \Rightarrow$ $G : v = 4, v = 7, u = -1, u = 2$

Example(cont.)

$$\iint_R (2x^2 - xy - y^2)dxdy$$

- $u = x - y; v = 2x + y \Rightarrow x = \frac{u+v}{3}; y = \frac{-2u+v}{3}$
- $R : y = -2x + 4, y = -2x + 7, y = x + 1, y = x - 2 \Rightarrow$ $G : v = 4, v = 7, u = -1, u = 2$
- $dxdy = J(x, y; u, v)dudv = \frac{1}{3}dudv$

Example(cont.)

$$\iint_R (2x^2 - xy - y^2)dxdy$$

- $u = x - y; v = 2x + y \Rightarrow x = \frac{u+v}{3}; y = \frac{-2u+v}{3}$
- $R : y = -2x + 4, y = -2x + 7, y = x + 1, y = x - 2 \Rightarrow$ $G : v = 4, v = 7, u = -1, u = 2$
- $dxdy = J(x, y; u, v)dudv = \frac{1}{3}dudv$

$$\iint_G 2\left(\frac{u+v}{3}\right)^2 - \left(\frac{u+v}{3}\right)\left(\frac{-2u+v}{3}\right) - \left(\frac{-2u+v}{3}\right)^2 \frac{1}{3}dudv$$

Example(cont.)

$$\iint_R (2x^2 - xy - y^2)dxdy$$

- $u = x - y; v = 2x + y \Rightarrow x = \frac{u+v}{3}; y = \frac{-2u+v}{3}$
- $R: y = -2x + 4, y = -2x + 7, y = x + 1, y = x - 2 \Rightarrow$ $G: v = 4, v = 7, u = -1, u = 2$
- $dxdy = J(x, y; u, v)dudv = \frac{1}{3}dudv$

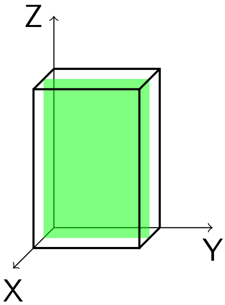
$$\int_{-1}^{2}\int_{4}^{7} 2\left(\frac{u+v}{3}\right)^2 - \left(\frac{u+v}{3}\right)\left(\frac{-2u+v}{3}\right) - \left(\frac{-2u+v}{3}\right)^2 \frac{1}{3}dud$$

$$\frac{33}{4}$$

Evaluate

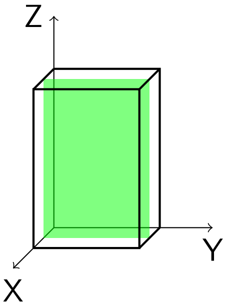
1. Find the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$
   1.1 $x = u\cos v, y = u\sin v, z = w$
   1.2 $x = 2u - 1,\ y = 3v - 4,\ z = \frac{1}{2}(w-4)$
2. Evaluate $\iint_R (3x^2 + 14xy + 8y^2)dxdy$ on the region $R : y = \frac{-3}{2}x + 1, y = \frac{-3}{2}x + 3, y = \frac{-1}{4}x, y = \frac{-1}{4}x + 1$ using the substitution $u = 3x + 2y,\ v = x + 4y$ 64/5

Cartesian co-ordinates

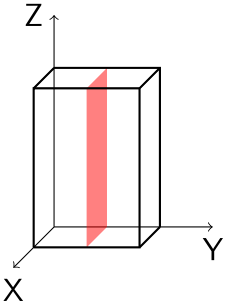


$$-\infty < x < \infty$$

Cartesian co-ordinates

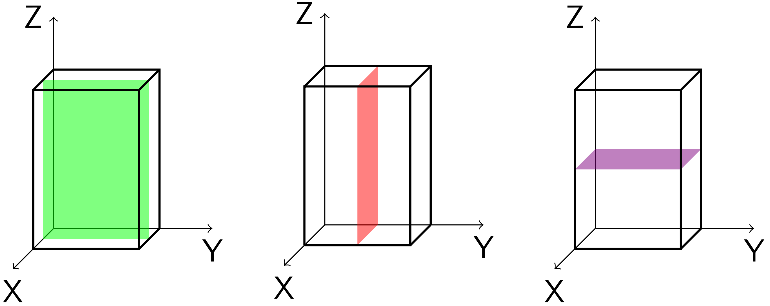


$$-\infty < x < \infty$$



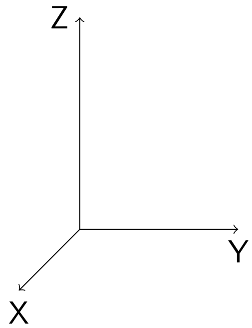
$$-\infty < y < \infty$$

Cartesian co-ordinates

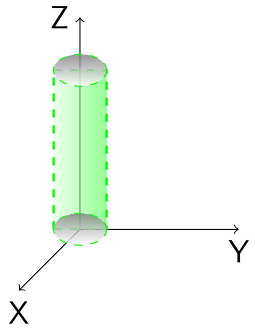


$-\infty < x < \infty$ $\quad$ $-\infty < y < \infty$ $\quad$ $-\infty < z < \infty$

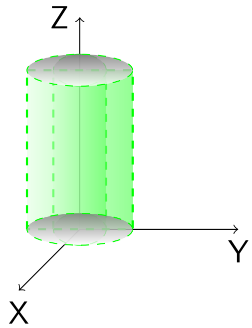
Cylindrical



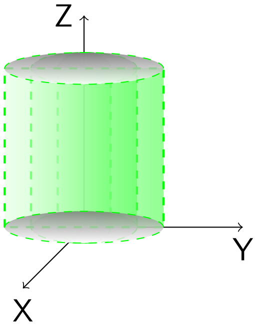
Cylindrical

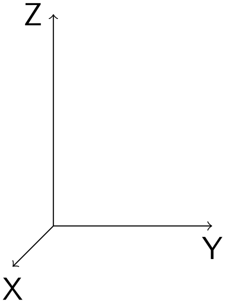


Cylindrical



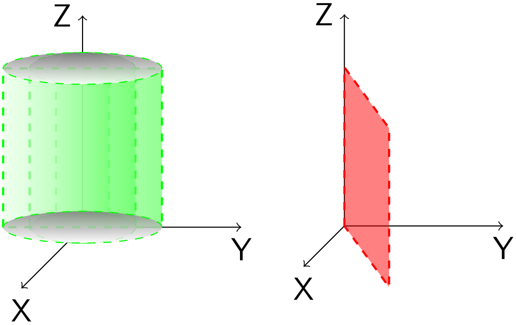
Cylindrical





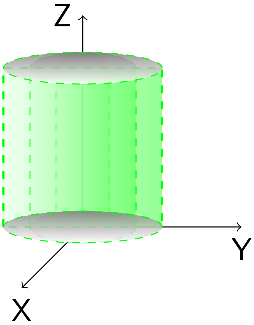
$0 \leq r < \infty$

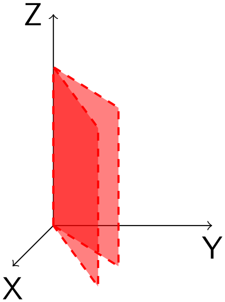
Cylindrical



$$0 \le r < \infty$$

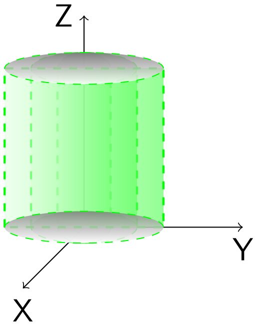
Cylindrical

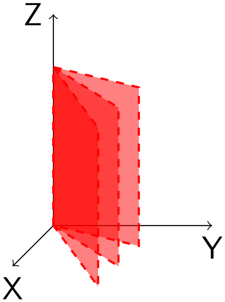




$$0 \leq r < \infty$$

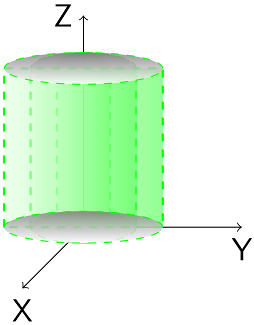
Cylindrical

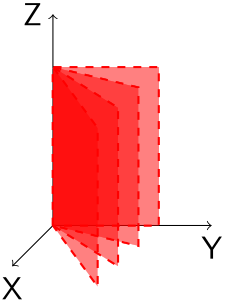


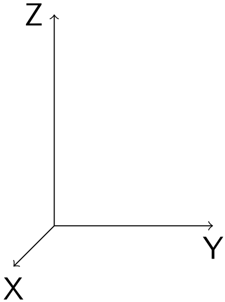


$$0 \leq r < \infty$$

Cylindrical

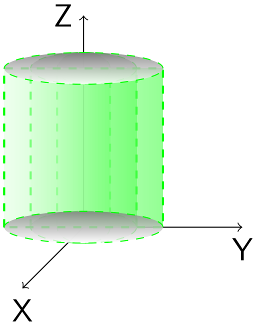


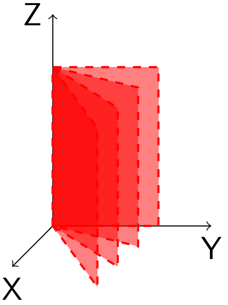


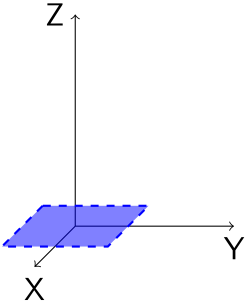


$0 \leq r < \infty$ $\qquad$ $0 \leq \theta \leq 2\pi$

Cylindrical



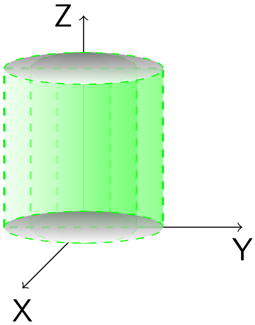


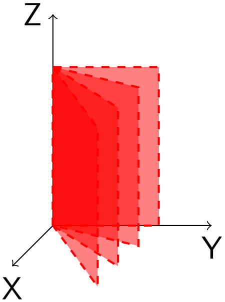


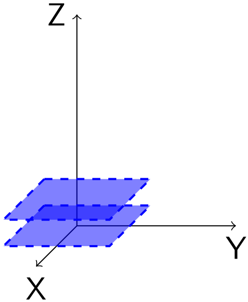
$0 \leq r < \infty$

$0 \leq \theta \leq 2\pi$

Cylindrical



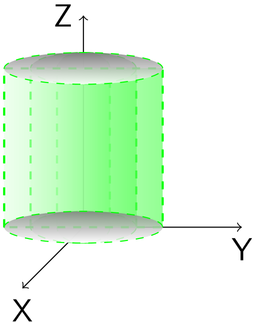


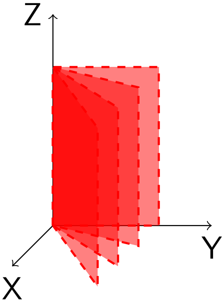


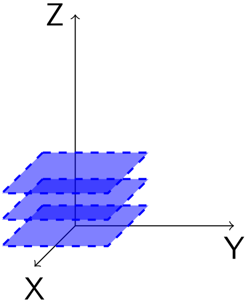
$0 \leq r < \infty$

$0 \leq \theta \leq 2\pi$

Cylindrical



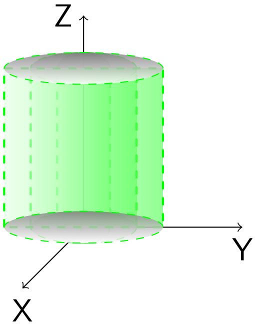




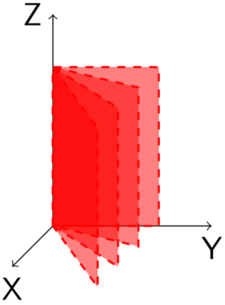
$0 \leq r < \infty$

$0 \leq \theta \leq 2\pi$

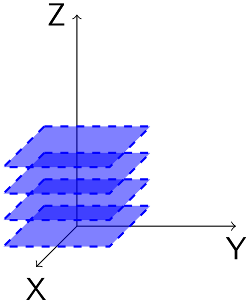
Cylindrical



$0 \leq r < \infty$

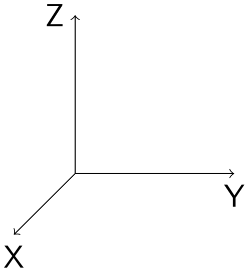


$0 \leq \theta \leq 2\pi$

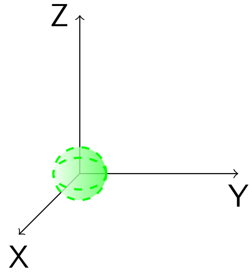


$-\infty < z < \infty$

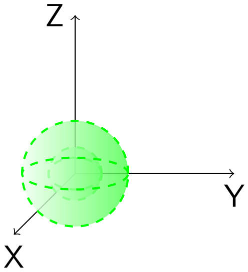
Spherical



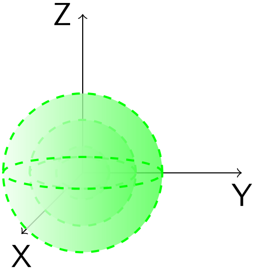
Spherical

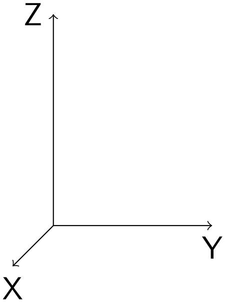


Spherical



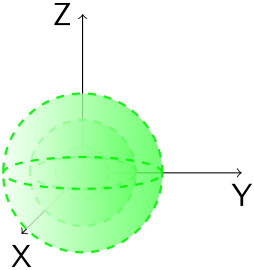
Spherical

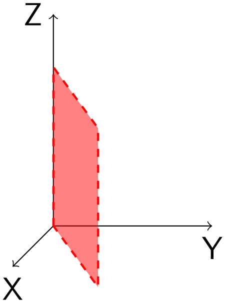




$$0 \leq \rho < \infty$$

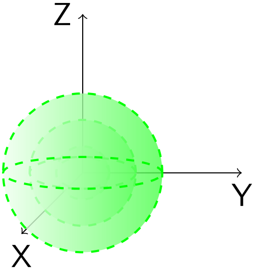
Spherical

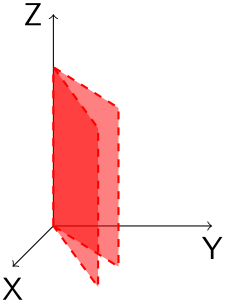




$$0 \leq \rho < \infty$$

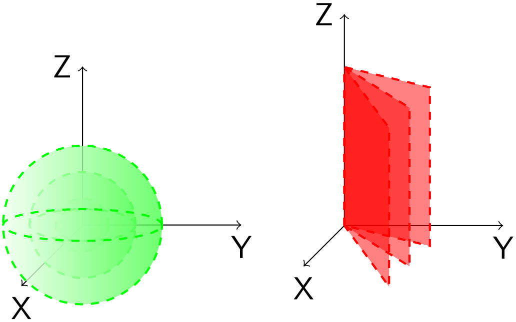
Spherical





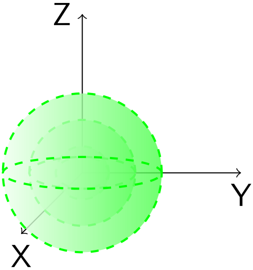
$$0 \leq \rho < \infty$$

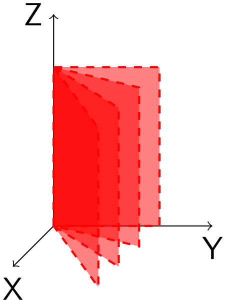
Spherical

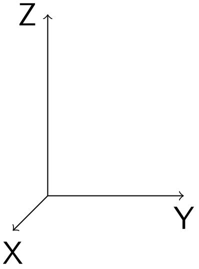


$$0 \leq \rho < \infty$$

Spherical



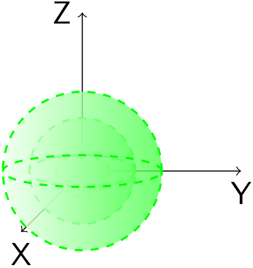


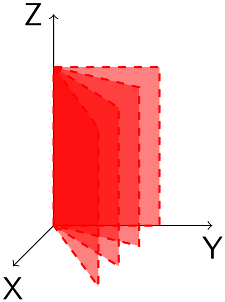


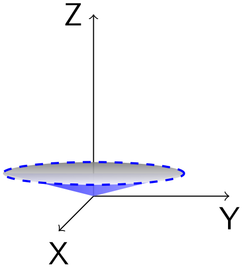
$$0 \leq \rho < \infty$$

$$0 \leq \theta \leq 2\pi$$

Spherical



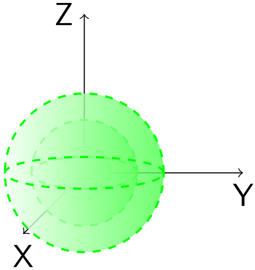


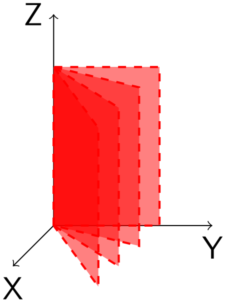


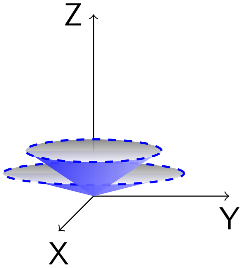
$0 \le \rho < \infty$

$0 \le \theta \le 2\pi$

Spherical



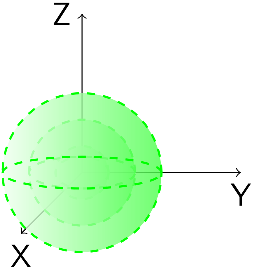




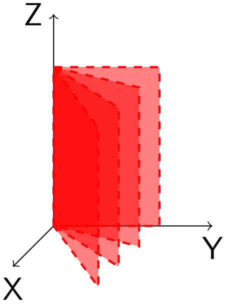
$0 \leq \rho < \infty$

$0 \leq \theta \leq 2\pi$

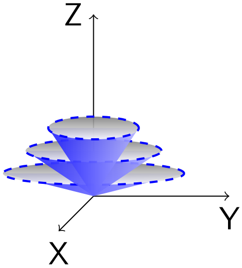
Spherical



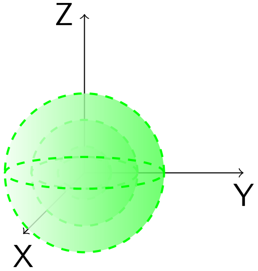
$0 \leq \rho < \infty$



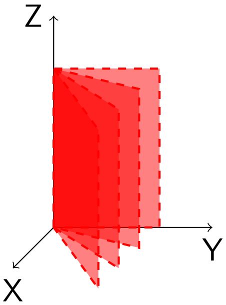
$0 \leq \theta \leq 2\pi$



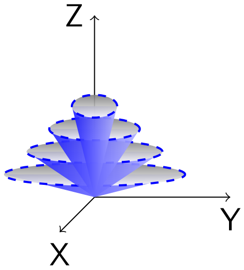
Spherical



$0 \leq \rho < \infty$



$0 \leq \theta \leq 2\pi$



$0 \leq \phi \leq \pi$

Multivariable function

A function f is given on a domain D

$$f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$$

Multivariable function

A function f is given on a domain D

$$f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$$

D:= Set form

Multivariable function

A function f is given on a domain D

$$f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$$

D:= Set form
$\{(x, y, z) | a \leq x \leq b, c \leq y \leq d, e \leq z \leq f\}$

Multivariable function

A function f is given on a domain D

$$f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$$

D:= Set form
$\{(x, y, z) | a \leq x \leq b, c \leq y \leq d, e \leq z \leq f\}$

D:= Description by words

Multivariable function

A function f is given on a domain D

$$f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$$

D:= Set form
$\{(x, y, z) | a \leq x \leq b, c \leq y \leq d, e \leq z \leq f\}$

D:= Description by words
Tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$

Multivariable function

A function f is given on a domain D

$$f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$$

D:= Set form
$\{(x, y, z) | a \leq x \leq b, c \leq y \leq d, e \leq z \leq f\}$

D:= Description by words
Tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$

D:= Picture

Triple Integral(Rectangular co-ordinates)

Describe the regions and setup the integral

1. Write six different iterated triple integrals for the volume of the rectangular solids in the first octant bounded by the co-ordinates planes and the planes $x = 1$,$y = 2$,$z = 3$
2. Region in the first octant enclosed by the cylinder $x^2 + z^2 = 4$ and the plane $y = 3$.
3. Region bounded by paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$

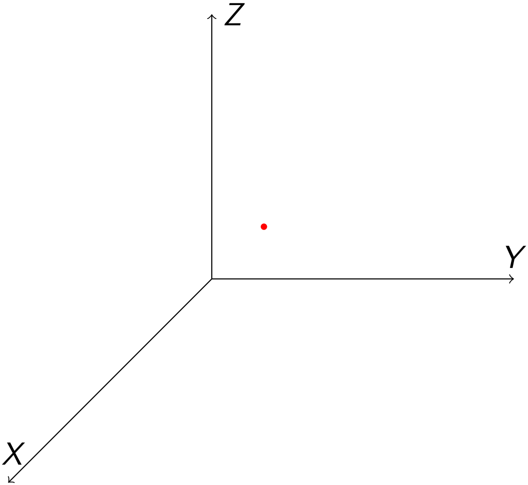
Triple integral (Cylindrical co-ordinates)

1. Solid enclosed by the cylinder $x^2 + y^2 = 4$ bounded above by the paraboloid $z = x^2 + y^2$
2. Cylinder whose base is the circle $r = 3cos\theta$ and whose top lies in the plane $z = 5 - x$
3. Find the average value of the function $f(r, \theta, z) = r$ over the region bounded by the cylinder $r = 1$ between the planes $z = -1$ and $z = 1$.

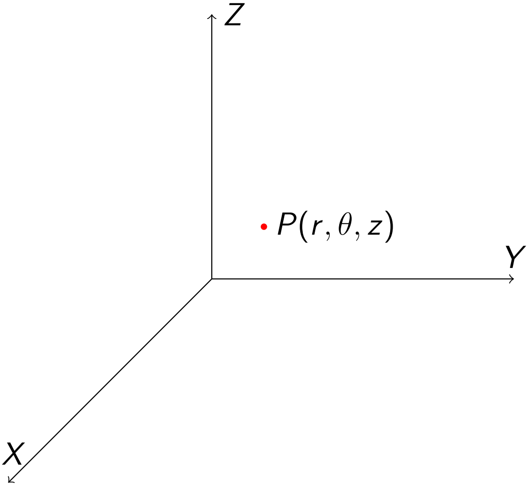
Triple integral (Spherical co-ordinates)

1. Region bounded below by $z = 0$, and above by the sphere $x^2 + y^2 + z^2 = 4$ side by $x^2 + y^2 = 1$.
2. Find the average value of the function $f(\rho, \phi, \theta)$ over the solid ball $\rho \leq 1$.

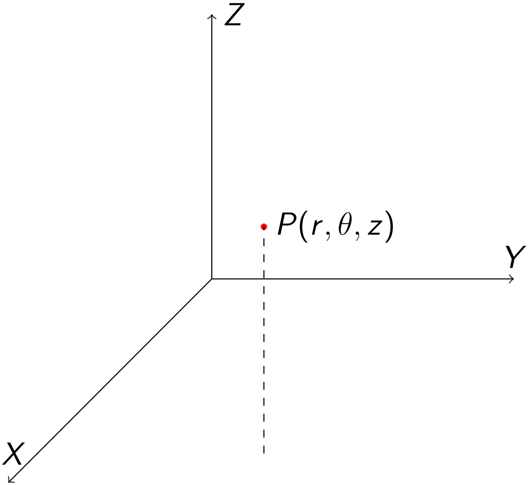
Cylindrical to Rectangular



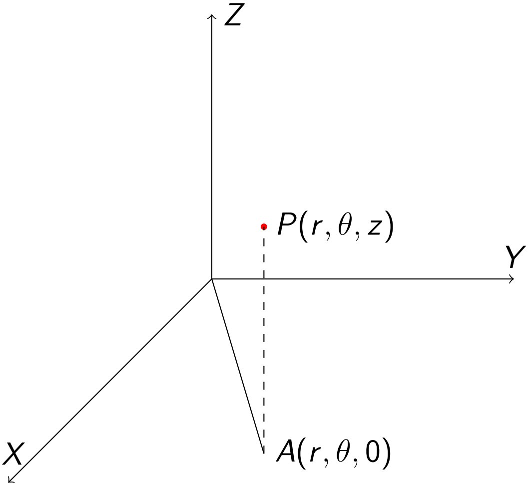
Cylindrical to Rectangular



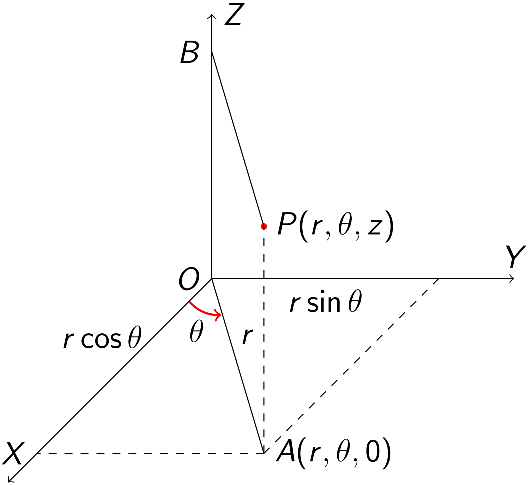
Cylindrical to Rectangular



Cylindrical to Rectangular



Cylindrical to Rectangular



Cylindrical

$$V = \iiint dz\, r\, dr d\theta$$

Cylindrical

$$V = \iiint dz\, r\, dr d\theta$$

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Cylindrical

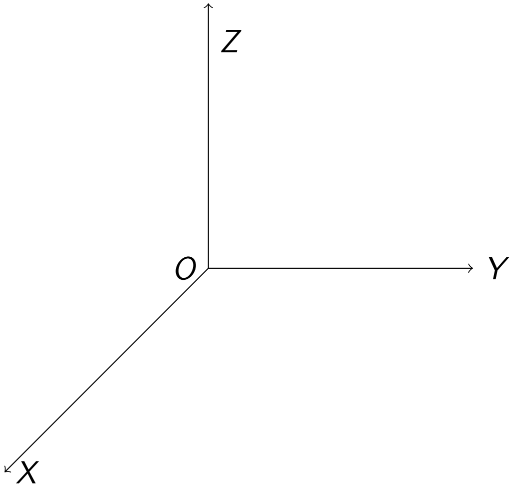
$$V = \iiint dz\, r\, dr d\theta$$

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

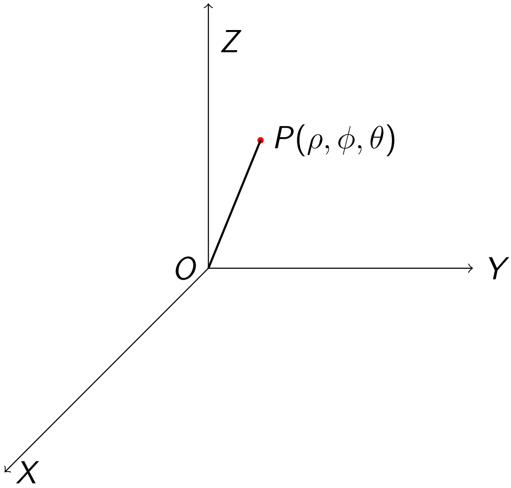
$$J(r,\theta,z) = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \left| \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = r$$

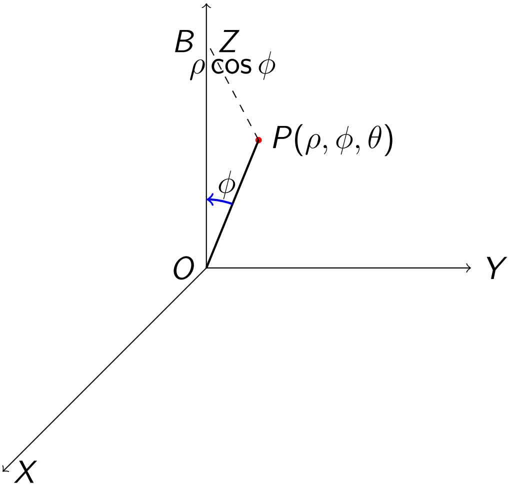
Spherical to Rectangular



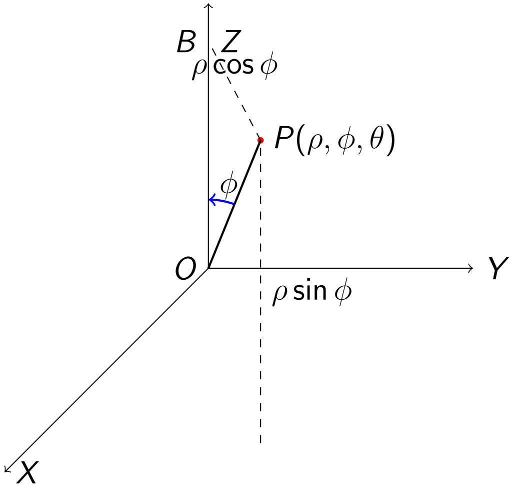
Spherical to Rectangular



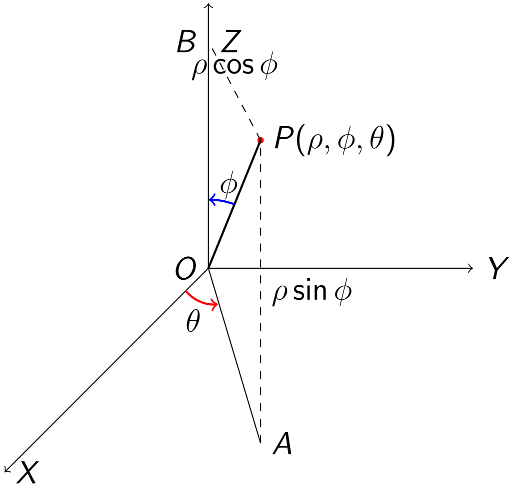
Spherical to Rectangular



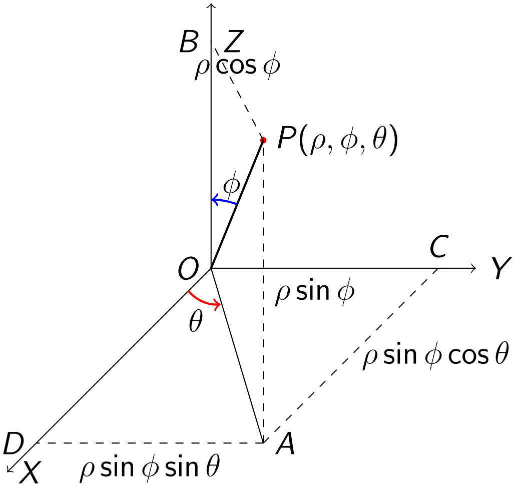
Spherical to Rectangular



Spherical to Rectangular



Spherical to Rectangular



Spherical

$$V = \iiint \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

Spherical

$$V = \iiint \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$