Vector Calculus

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Multiple Integrals

Rectangular Co-ordinates

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  What is integration of constant function?

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- Initially we restrict to rectangular regions,
- General regions later.
- First decode what is the domain of the function and co-domain of the function. $f : R \to \mathbb{R}$ where R is the region.

$$R := \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$$

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- Let the small area be ΔA_0 and height $f(x_0, y_0)$ so the expression $f(x_0, y_0)A_0$ represents the volume.
- Do this process in the whole region and we will end up with integration of the function $f(x, y)$.

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_K$$

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- When the refinement really converges? - integration theory.
- But we assume something more 'continuity' of the function. Though the property is not necessary. This integrals have lot of properties like the single integrals.

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$$V = \iint_R f(x, y) dA = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_K$$

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Example

Calculate the volume under the plane $z = f(x, y) = 4 + x - y$ over the rectangular region

$$R = 0 \leq x \leq 1, \quad 0 \leq y \leq 2$$

Let us first try to see the surface.

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Hence the volume under the plane is given by

$$\begin{aligned}
\int_0^1 \left(\int_0^2 (4 + x - y)dy \right) dx &= \int_0^1 \left[4y + xy - \frac{y^2}{2} \right]_0^2 dx \\
&= \int_0^1 (6 + 2x)dx \\
&= \left[6x + x^2 \right]_0^1 \\
&= 7 Units
\end{aligned}$$

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Doing it in the other direction, that is in y- direction. For a fixed y we will have the picture. To get the volume, we should integrate over such all possible areas. $\int_0^2 A(y)dy$ Look out only $A(y)$ and see that

$$A(y) = \int_0^1 (4 + x - y)dx$$

Hence the volume under the plane is given by

$$\begin{aligned}
\int_0^2 \left(\int_0^1 (4 + x - y)dx\right) dy &= \int_0^2 \left[4x + \frac{x^2}{2} - xy\right]_0^1 dx \\
&= \int_0^2 \left(\frac{9}{2} - y\right) dy \\
&= \left[\frac{9}{2}y - \frac{y^2}{2}\right]_0^2 \\
&= 7 Units
\end{aligned}$$

Rectangle Co-ordinates

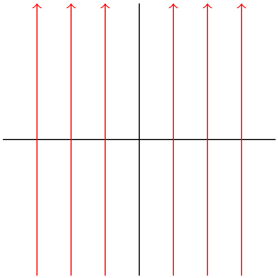

Figure: $x =$constant, $y =$constant

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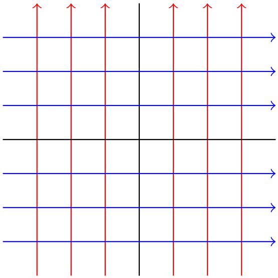

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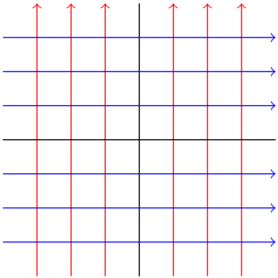

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Fubini's Theorem(first form)(1907)

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Sketch the region of integration and evaluate

1. $\int_0^{\pi} \int_0^{\sin(x)} y dy dx$ $\pi/4$

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3. $\int_1^2 \int_y^{y^2} dx dy$ 5/6

Sketch the region of integration and write an equivalent double integral with the order of integration reversed

1. $\int_0^2 \int_{y-2}^0 dxdy$
2. $\int_0^2 \int_0^{4-y^2} ydxdy$
3. $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dxdy$

1. Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x$, $x = 0$ and $x + y = 2$ in the XY-plane. $4/3$
2. What region R in the XY-plane minimizes the value of

$$\int\int (x^2 + y^2 - 9) dA$$

Sketch the regions bounded by the given lines and curves. Then express the regions area as an iterated double integrals and evaluate the integral

1. Co-ordinate axes and the line $x + y = 2$ 2
2. The curve $y = e^x$ and the lines $y = 0$, $x = 0$ and $x = \ln 2$ 1
3. The parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$ $4/3$
4. $\int_0^3 \int_{-x}^{x(2-x)} dydx$