Vector Calculus

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Green's Theorem

Tangent form or Ciculation-Curl form

$$\oint_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Green's Theorem

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$$\oint_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$\oint_C F \cdot dr = \iint_R (\nabla \times F) \cdot k \, dA$$

Stoke's Theorem

$$\oint_C F \cdot dr = \iint_S \nabla \times F \cdot n \, d\sigma$$

Green's Theorem

(Normal form or Flux-Divergence form)

$$\oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$$

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$$\oint_C F \cdot n \, ds = \iint_R \nabla \cdot F \, dA$$

- C is a simple, closed, smooth curve
- R is the region enclosed by C
- dA is area element
- ds is length element

$$\iint_S F \cdot n \, d\sigma = \iiint_D \nabla \cdot F \, dV.$$

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- S is a simple, closed, oriented surface.
- D is solid regin bounded by S
- $d\sigma$ surface area element
- dV is volume element

The Divergence Theorem

Gauss

The flux of a vector field $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ across a closed oriented surface S in the direction of the surface's outward unit normal field n equals the integral of $\nabla \cdot F$ (divergence of F) over the region D enclosed by the surface:

$$\iint_S F \cdot n \, d\sigma = \iiint_D \nabla \cdot F \, dV.$$

$F = yi + xyi - zk$
D : The region inside the solid cylinder $x^2 + y^2 \leq 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$

$$\nabla \cdot F = 0 + x - 1 = x - 1$$

$$\begin{aligned}
\iiint_D \nabla \cdot F \, dV &= \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{x^2+y^2} (x-1) dzdydx \\
&= \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x-1)(x^2+y^2) dydx \\
&= \int_0^2 (x-1)[x^2 y + \frac{y^3}{3}]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \\
&= \int_0^2 (x-1)(2x^2\sqrt{4-x^2} + \frac{2}{3}(4-x)^2\sqrt{4-x^2}) dx \\
&= \frac{1}{3}\int_0^2 (x-1)\sqrt{4-x^2}[6x^2 + 2(16 - 8x + 8x^2)] dx \\
&= \frac{1}{3}\int_0^2 (x-1)\sqrt{4-x^2}[8x^2 - 8x + 16] dx \\
&= -16\pi
\end{aligned}$$

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Exercise

Divergence theorem

Use divergence theorem to calculate outward flux

1. $F = (y - x)\mathbf{i} + (z - y)\mathbf{j} + (y - x)\mathbf{k}$
   D :The cube bounded by the planes $x \pm 1$, $y \pm 1$ and $z \pm 1$.
   -16
2. $F = x^2\mathbf{i} - 2xy\mathbf{j} + 3xz\mathbf{k}$
   D :The region cut from the first octant by the sphere
   $x^2 + y^2 + z^2 = 4$ 3π

- F is conservative, F is irrotational$\Longrightarrow$ Ciruculation$= 0$
- F is incompressible, $\nabla.F$ is $0 \Longrightarrow$ Flux$= 0$

Fundamental Theorem of Calculus

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&= \text{total outward flux of } F \text{ across the boundary} \\
&= \int_{[a,b]} \nabla \cdot F dx
\end{aligned}$$

Integral of the differential operator acting on a field over a region equal the sum of (or integral of) field components appropriate to the operator on the boundary of the region

Scalar integration

1. Integration

Scalar integration

1. Integration
   - Area Between curves

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   - Area Between curves
   - Volume by cross section

Scalar integration

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   - Area Between curves
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   - Surface area of revolution by

Scalar integration

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   - Surface area of revolution by
     - Disk

Scalar integration

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   - Surface area of revolution by
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Scalar integration

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     - Shell

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1. Integration
   - Area Between curves
   - Volume by cross section
   - Surface area of revolution by
     - Disk
     - Washer
     - Shell
2. Double integral

Scalar integration

1. Integration
    - Area Between curves
    - Volume by cross section
    - Surface area of revolution by
        - Disk
        - Washer
        - Shell
2. Double integral
    - Cartesian co-ordinates

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3. Triple integrals

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   - Polar co-ordinates
3. Triple integrals
   - Rectangular

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3. Triple integrals
   - Rectangular
   - Cylindrical

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   - Rectangular
   - Cylindrical
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4. Change of variable

Scalar integration

1. Integration
   - Area Between curves
   - Volume by cross section
   - Surface area of revolution by
     - Disk
     - Washer
     - Shell
2. Double integral
   - Cartesian co-ordinates
   - Polar co-ordinates
3. Triple integrals
   - Rectangular
   - Cylindrical
   - Spherical
4. Change of variable
   - Jacobian

Vector integration

5. Line integral

Vector integration

5. Line integral
6. Vector fields

Vector integration

5. Line integral
6. Vector fields
   - Gradient

Vector integration

5. Line integral
6. Vector fields
    - Gradient
    - Divergent – Flux density

Vector integration

5. Line integral
6. Vector fields
   - Gradient
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   - Curl– Circulation density

Vector integration

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7. Green's theorem

Vector integration

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   - Normal form

Vector integration

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   - Normal form
   - Tangent form

Vector integration

5. Line integral
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   - Gradient
   - Divergent – Flux density
   - Curl– Circulation density
7. Green's theorem
   - Normal form
   - Tangent form
8. Surface integral

Vector integration

5. Line integral
6. Vector fields
   - Gradient
   - Divergent – Flux density
   - Curl– Circulation density
7. Green's theorem
   - Normal form
   - Tangent form
8. Surface integral
   - Equation form

Vector integration

5. Line integral
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   - Gradient
   - Divergent – Flux density
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   - Normal form
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8. Surface integral
   - Equation form
   - Parametric form

Vector integration

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   - Gradient
   - Divergent – Flux density
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7. Green's theorem
   - Normal form
   - Tangent form
8. Surface integral
   - Equation form
   - Parametric form
9. Stoke's theorem

Vector integration

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   - Normal form
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8. Surface integral
   - Equation form
   - Parametric form
9. Stoke's theorem
10. Gauss divergence theorem

Test

- No particular Model.
- Only exact answer will carry full marks.

Best wishes