# Vector Calculus

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$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \quad a \leq u \leq b, c \leq v \leq d$$

Range of $\mathbf{r}$ is the surface $\mathbf{S}$, Domain of $\mathbf{r}$ is in the $u - v$ plane.

# Surface Area

## Explicit form

Area of the surface $f(x, y, z) = c$ over a closed and bounded plane region $R$ is

$$\text{Surface area } = \iint_S d\sigma = \int\int_R \frac{|\nabla f|}{|\nabla f . p|} dA$$

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The integral of $g$ over $S$

$$\text{Surface Integral } = \iint_S g d\sigma = \int\int_R g(x, y, z) \frac{|\nabla f|}{|\nabla f . p|} dA$$

# Surface Area

## Parametric form

$\mathbf{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k},\ a \le u \le b \quad c \le v \le d$

Area of surface is

$$\iint_S d\sigma = \iint_{uv-region} |r_u \times r_v| dudv$$

Surface integral of $k$ over $S$

$$\iint_S k d\sigma = \iint_{uv-region} k(f,g,h)|r_u \times r_v| dudv$$

# Green's Theorem

## Tangent form or Ciculation-Curl form

$$\oint_c M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA$$

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$$\oint_c Mdx + Ndy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$\oint_C F \cdot dr = \iint_R (\nabla \times F) \cdot k \, dA$$

- $C$ is a simple, closed, smooth curve in counterclockwise direction
- $R$ is the region enclosed by $C$
- $dA$ is area element
- $dr$ is tangential length

# Stoke's Theorem

The circulation of $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ around the boundary $C$ of an oriented surface $S$ in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of $\nabla \times F \cdot n$ over $S$

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Use Stoke's Theorem to calculate the circulation of the Field $F = x^2 i + 2xj + z^2 k$ around the curve $C$: The ellipse $4x^2 + y^2 = 4$ in the $xy$ plane counter clockwise when viewed from above.

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$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix}$$

$$(\nabla \times F) \cdot k = 2$$

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$$= 4\left(\frac{\pi}{2} - 0\right) = 2\pi$$

# Exercise

## Stoke's Theorem

Use Stoke's theorem to calculate the flux of the curl of the field $F$ across the surface $S$ in the direction of the outward unit normal $n$.

1. $F = 2zi + 3xj + 5yk$
   $S : z + x^2 + y^2 = 4$ $12\pi$
2. $F = 2zi + 3xj + 5yk$
   $S : \mathbf{r}(r, \theta) = (r\cos\theta)i + (r\sin\theta)j + (4 - r^2)k$
   $0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi$
3. $F = x^2yi + 2y^3zj + 3zk$
   $S : \mathbf{r}(r, \theta) = (r\cos\theta)i + (r\sin\theta)j + rk$
   $0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi$ $\frac{-\pi}{4}$

Write your registration number and name on the top.

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Example cs15btech1104**3** means a=4 b=3

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1. by rectangle co-ordinates $(x, y)$

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3. by polar co-ordinates. $(r, \theta)$