Vector Calculus

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Surfaces

Curves in plane has three forms

- Explicit form: $y = f(x)$

Surfaces in space has three forms

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- Explicit form:$z = f(x, y)$
- Implicit form:$f(x, y, z) = 0$
- Parametric (vector) form:

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \quad a \leq u \leq b, c \leq v \leq d$$

Range of **r** is the surface **S**, Domain of **r** is in the u, v plane.

Surface Area

Explicit form

Area of the surface $f(x, y, z) = c$ over a closed and bounded plane region R is

$$\text{Surface area } = \iint_S d\sigma = \int\int_R \frac{|\nabla f|}{|\nabla f . p|} dA$$

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The integral of g over S

$$\text{Surface Integral } = \iint_S g d\sigma = \int\int_R g(x, y, z) \frac{|\nabla f|}{|\nabla f.p|} dA$$

Find the area of the surface cut from the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$

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$\nabla f = 2xi + 2yj - k, |\nabla f| = \sqrt{4x^2 + 4y^2 + 1},\ p = k$

$\nabla f \cdot p = -1$ and $|\nabla f \cdot p| = 1$

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$$\text{SA} = \int\int_R \frac{|\nabla f|}{|\nabla f \cdot p|} dA = \iint_{x^2+y^2 \le 2} \left(\sqrt{4x^2 + 4y^2 + 1}\right) dxdy$$

$$= \int\int_{r \le \sqrt{2}} \sqrt{4r^2 + 1}\, rdrd\theta = \int\int_{r \le \sqrt{2}} \sqrt{4r^2 + 1} \frac{dr^2}{2} d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1} \frac{dr^2}{2} d\theta = \frac{1}{2} \int_0^{2\pi} \frac{1}{4} \left[\frac{\left(4r^2 + 1\right)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^{\sqrt{2}} d\theta$$

$$= \frac{1}{2}\frac{1}{4} \int_0^{2\pi} \left[\frac{2}{3}(27 - 1)\right] d\theta = \frac{1}{3}\frac{1}{4}(26 \times 2\pi) = \frac{1}{4}\frac{52\pi}{3} = \frac{13\pi}{3}$$

Find the area of the region cut from the plane $x + 2y + 2z - 5$ by the cylinder whose walls are $x = y^2$ and $x = 2 - y^2$.

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$$
\begin{aligned}
&= \iint_R \frac{|\nabla f|}{|\nabla f \cdot p|} dA = \iint \frac{3}{2} dA \\
&= \frac{3}{2}\left\{\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \mathrm{d}x\mathrm{d}y + \int_1^2 \int_{-\sqrt{2-x}}^{\sqrt{2-x}} \mathrm{d}x\mathrm{d}y\right\} \\
&= \frac{3}{2}\left\{2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 - 2\left[\frac{(2-x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^2\right\} \\
&= \frac{3}{2}\left\{\frac{4}{3} + \frac{4}{3}\right\} = 4
\end{aligned}
$$

Exercise

1. Integrate $g(x, y, z) = x + y + z$ over the portion of the plane $2x + 2y + z = 2$ that lies in the first octant. 2

Parametrized surfaces

$\mathbf{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k},\ a \le u \le b \quad c \le v \le d$

Area of surface is

$$\iint_S d\sigma = \iint_{uv-region} |r_u \times r_v| dudv$$

Surface integral of k over S

$$\iint_S k d\sigma = \iint_{uv-region} k(f,g,h)|r_u \times r_v| dudv$$

Find the surface area of the portion of the cone $z = 2\sqrt{x^2 + y2}$ between the planes $z = 2$ and $z = 6$

$$
\begin{aligned}
S(r,\theta) &= rcos\theta i + rsin\theta j + 2rk \\
r_r &= cos\theta i + rsin\theta j + 2k \\
r_\theta &= -rsin\theta i + rcos\theta j + \theta k \\
|r_r \times r_\theta| &= \begin{vmatrix} i & j & k \\ cos\theta & \sin\theta & 2 \\ -rsin\theta & r\cos\theta & 0 \end{vmatrix} \\
&= i(-2rcos\theta) - j(2rsin\theta) + k(rcos^2\theta + rsin^2\theta) \\
&= |-2rcos\theta i - 2rsin\theta j + rk| \\
&= \sqrt{4r^2cos^2\theta + 4r^2sin^2\theta + r^2} \\
&= r\sqrt{5}
\end{aligned}
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$Area = \int_0^{2\pi}\int_1^3 r\sqrt{5}\mathrm{d}r\mathrm{d}\theta = 8\sqrt{5}\pi$

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Exercise

1. The first octant of the portion of the cone $z = \frac{\sqrt{x^2+y^2}}{2}$ between the planes $z = 0$ and $z = 3$
2. The portion of the plane $x + y + z = 1$
   2.1 inside cylinder $x^2 + y^2 = 9$
   2.2 inside cylinder $y^2 + z^2 = 9$
3. Integrate $k(x, y, z) = x^2$ over the unit sphere $x^2 + y^2 + z^2 = 1$