Progeny of spectrum

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Gujarat.

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1. Part I
   - Spectrum
   - Broad themes

2. Part II
   - Condition spectrum
   - Continuity

1. Part I
   - Spectrum
   - Broad themes
2. Part II

Spectrum

Eigenvalue - Finite dimensional spaces

Let $A \in M_{n\times n}(\mathbb{C})$.

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- not invertible $\sim$ not one-one $\sim$ not onto.

$$A : \mathbb{C}^n \to \mathbb{C}^n$$

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Rich results are available for the case when $\mathcal{X}$ is a Banach space or a Hilbert space $\mathcal{H}$. That is when $T \in B(\mathcal{X})$ or $T \in B(\mathcal{H})$

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Let $a \in \mathcal{A}$.

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As we need scalar multiplication, addition, product/multiplication, we consider an unital algebra.
Further, in the case of Unital Banach algebra $\mathcal{A}$ we get $\sigma(a)$ be a non-empty compact subset of $\mathbb{C}$.

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Broad themes

Variations and notions

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For $a \in \mathcal{A}$ and $Inv(\mathcal{A})$ denote the invertible elements of $\mathcal{A}$.

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- Application - Approximation -*Pseudospectrum*

Exponential spectrum

Understanding - Math curiosity

Let $\mathcal{A}$ be a unital Banach algebra and $Exp(\mathcal{A})$ denote the set of exponential elements of the Banach algebra.

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- Called as exponential spectrum of a

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non commutativity of exponential spectrum

For $a, b \in \mathcal{A}$, does the spectrum commute

$$\sigma(ab) = \sigma(ba)?$$

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There exists $a, b \in C(\mathbb{S}^4, M_2(\mathbb{C}))$

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where $\mathbb{S}^4 := \{(z_0, z_1, z_2) \in \mathbb{C}^3 : \sum_{i=0}^{2} |z_i|^2 = 1, Imz_2 = 0\}$

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Does there exist a Banach space E and operators S, T on E such that

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Ransford spectrum

Axiomatic (unifying) Abstraction

$$\sigma(a) := \{\lambda \in \mathbb{C} : a - \lambda 1 \notin Inv(\mathcal{A})\}$$

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$$\sigma_{\Omega}(a) := \{\lambda \in \mathbb{C} : a - \lambda 1 \notin \Omega\}$$

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- Pseudoconvexity of Ω gives
  1. non-empty, compactness
  2. spectral radius formula

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Psuedospectrum

Application - Approximation

Let $\mathcal{A}$ be a complex unital Banach algebra with unit.

Definition (ϵ- pseudo spectrum ($\epsilon > 0$))

$$\Lambda_\epsilon(a) := \left\{ \lambda \in \mathbb{C} : \lambda - a \notin Inv(\mathcal{A}) \text{ or } \|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon} \right\}$$

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Axiomatic, Approximation, Application

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3. $\sigma_\epsilon(a)$ has finite components and each component has a spectral value.
4. If $\lambda \in \sigma_\epsilon(a)$ then $|\lambda| \leq \dfrac{1+\epsilon}{1-\epsilon}\|a\|$.

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[3] S. H. Kulkarni and D. Sukumar. The condition spectrum. *Acta Sci. Math. (Szeged)*, 74(3-4):625–641, 2008

1 Part I

2 Part II
- Condition spectrum
- Continuity

Properties as a set valued map (correspondance)

$$\sigma_\epsilon(a) := \left\{ \lambda \in \mathbb{C} : \|a - \lambda\| \|(a-\lambda)^{-1}\| \geq \frac{1}{\epsilon} \right\}$$

[4] Arundhathi Krishnan and S. H. Kulkarni. Pseudospectrum of an element of a Banach algebra. *Oper. Matrices*, 11(1):263–287, 2017

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- $C_\epsilon : a \to \sigma_\epsilon(a)$

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Hemi-continuity of pseudospectrum[4].

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Set valued map

Definition (Upper and lower hemicontinuous)

A correspondence $\phi : X \twoheadrightarrow Y$ between topological space is upper hemicontinuous at the point $x \in X$ if every neighbourhood U of $\phi(x)$ there is a neighbourhood V of x such that

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continuous at $x \in X$ if it is both upper and lower hemicontinuous at x.

Continuity of condition spectrum

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- $C_a : \epsilon \to \sigma_\epsilon(a)$ is upper hemicontinuous[5].

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- Similarly results for L_ϵ using the sub-correspondence.

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Unavoidable assumption

Interior point of the boundary

Study of Shargarodsky problem [6]

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When is the interior of $L\Lambda_\epsilon(a)$ empty?.

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Questions

Thank you.