WHEN IS A LINEAR FUNCTION ON A FUNCTION ALGEBRA ALMOST MULTIPLICATIVE?

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DEFINITION (δ-ALMOST MULTIPLICATIVE FUNCTION)

A linear map $\phi : A \to \mathbb{C}$ is said to be almost multiplicative if there exists a $\delta > 0$ such that

$$|\phi(ab) - \phi(a)\phi(b)| \leq \delta \, \|a\| \, \|b\| \quad \text{for all} \quad a, b \in A.$$

Definition (δ-almost multiplicative function)

A linear map $\phi : A \to \mathbb{C}$ is said to be almost multiplicative if there exists a $\delta > 0$ such that

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Definition (multiplicative function)

A linear map $\phi : A \to \mathbb{C}$ is said to be multiplicative if

$$\phi(ab) = \phi(a)\phi(b) \quad \text{for all} \quad a, b \in A.$$

Condition spectrum

Definition (ϵ- Condition spectrum ($0<\epsilon<1$))

$$\sigma_\epsilon(a) := \left\{ \lambda \in \mathbb{C} : \|\lambda - a\| \left\|(\lambda - a)^{-1}\right\| \geq \frac{1}{\epsilon} \right\},$$

Condition spectral radius $r_\epsilon(a)$ is defined as

$$r_\epsilon(a) := \sup\{|z| : z \in \sigma_\epsilon(a)\}.$$

Theorem

Let A be a Banach algebra and $a \in A$. Then

$$r_\epsilon(a) \leq \frac{1+\epsilon}{1-\epsilon} \|a\|$$

FUNCTION ALGEBRA

THEOREM

Let A be a function algebra and $\phi : A \to \mathbb{C}$ be linear such that $\phi(a) \in \sigma_\epsilon(a)$ for every a in A. Then ϕ is δ-almost multiplicative, where

$$\delta = \frac{2(2\epsilon + 1)}{\log\left(\epsilon^{-1}\right)}.$$

THEOREM

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THEOREM (GKŻ THEOREM)

Let A be complex Banach algebra and $\phi : A \to \mathbb{C}$ be a linear map with $\phi(1) = 1$. If, for every $a \in A$,

$$\phi(a) \in \sigma(a)$$

then ϕ is multiplicative.

Lemma

Let A be a complex function algebra and let $\lambda \in \sigma_\epsilon(a)$. Then,

1. $d\Big(\lambda, \sigma(a)\Big) \leq \dfrac{2\epsilon}{1-\epsilon}\,\|a\|$ *and*
2. *There exists an almost δ-multiplicative linear functional ψ such that $\lambda = \psi(a)$ and*

$$|\psi(1) - 1| < \frac{2\epsilon}{1-\epsilon} \qquad \delta = \frac{2\epsilon\left(3-\epsilon\right)^2}{(1-\epsilon)}$$

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