Banach algebras with natural optimal radius of open ball at each invertible element

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October 13, 2017

A	be a complex unital Banach algebra with unit e.
$G(A)$	Invertible elements of A
$Sing(A)$	Singular elements of A respectively.
$\sigma(a)$	The spectrum of $a \in A$
$\rho(a)$	The resolvent of $a \in A$.
$r(a)$	Spectral radius of $a \in A$

Let $\phi : A \to \mathbb{C}$ is linear functional

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For $0 < \epsilon < 1$

$$\sigma_\epsilon(a) = \left\{ \lambda \in \mathbb{C} : \|a - \lambda\| \left\|(a - \lambda)^{-1}\right\| \geq \frac{1}{\epsilon} \right\}$$

and ϕ is δ-multiplicative if

$$\forall a, b \in A, \quad |\phi(ab) - \phi(a)\phi(b)| \leq \delta \|a\| \|b\|$$

Theorem

$G(A)$ is an open set in A.

$$a \in G(A) \Rightarrow B\left(a, \frac{1}{\|a^{-1}\|}\right) \subseteq G(A)$$

Is this the biggest ball?

Does there exists a $s \in Sing(A)$ such that $\|s - a\| = \frac{1}{\|a^{-1}\|}$

Definition (B)

An element $a \in G(A)$ is said to satisfy condition (B) if the biggest open ball centered at a, contained in $G(A)$, is of radius $\frac{1}{\|a^{-1}\|}$ i.e

$$\overline{B\left(a, \frac{1}{\|a^{-1}\|}\right)} \cap Sing(A) \neq \phi.$$

We say a Banach algebra A satisfies condition (B) if every $a \in G(A)$ satisfies condition (B).

- A Banach algebra A satisfying condition (B), every member of the $\sigma_\epsilon(a)$ is a spectral value of a perturbed a.
- Further if A is Banach algebra satisfying condition (B), and $a \in A$, then for every open set Ω containing $\sigma(a)$, there exists $0 < \epsilon < 1$ such that $\sigma_\epsilon(a) \subset \Omega$.

For any $a \in A$, $r(a) = \|a\|$ if and only if $\|a^2\| = \|a\|^2$

Theorem (Sufficient condition)

Let $a \in G(A)$ such that $\|(a^{-1})^2\| = \|a^{-1}\|^2$, then a satisfies condition (B).

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Theorem (Sufficient condition)

Let $a \in G(A)$ such that $\|(a^{-1})^2\| = \|a^{-1}\|^2$, then a satisfies condition (B).

Proof.

Since $\|(a^{-1})^2\| = \|a^{-1}\|^2$, by the compactness of spectrum there exists $\lambda_0 \in \sigma(a)$ such that

$$\frac{1}{\|a^{-1}\|} = \frac{1}{r(a^{-1})} = \inf\{|\lambda| : \lambda \in \sigma(a)\} = |\lambda_0|.$$

The element $s = a - \lambda_0 \in A$ can be taken as a singular element in the boundary of $B\left(a, \frac{1}{\|a^{-1}\|}\right)$ with the required property. □

Theorem

Let A be a commutative Banach algebra. Then $a \in G(A)$ satisfies condition (B) if and only if $\|(a^{-1})^2\| = \|a^{-1}\|^2$.

Proof.

If a satisfies (B), there exists $s \in Sing(A)$ such that

$$\begin{aligned}\|a^{-1}\|^2 &= \frac{1}{\|a-s\|^2} \\ &\leq \frac{1}{\|(a-s)^2\|} = \frac{1}{\|a^2-(sa+as-s^2)\|} \leq \|(a^{-1})^2\|,\end{aligned}$$

where $sa + as - s^2 \in Sing(A)$ as A is commutative. Thus we have $\|a^{-1}\|^2 = \|(a^{-1})^2\|$. □

Corollary

Let A be a finite dimensional Banach algebra that satisfies condition (B). Then A is commutative if and only if $\|a^2\| = \|a\|^2$ for every $a \in A$.

Proof.

Invertible elements are dense. □

Example (The converse not true if A is non-commutative)

For this, we will see later that any invertible operator on a Hilbert space satisfies condition (B),
If J is invertible matrix such that J^{-1} is a Jordan matrix with $r(J) < 1$, then

$$r(J^{-1}) \neq \|J^{-1}\|.$$

Example (Do not satisfy (B))

Let $C^1[0,1]$ be the space of all complex valued functions on [0,1] with continuous derivative equipped with the norm

$$\|f\| = \|f\|_\infty + \|f'\|_\infty \quad \text{for all } f \in C^1[0,1].$$

Then $(C^1[0,1], \|.\|)$ is a commutative semi-simple Banach function algebra. Consider the function $f(x) = e^x$ for all $x \in [0,1]$ and notice that

$$\|(f^{-1})^2\| \neq \|f^{-1}\|^2.$$

Theorem

Let A be a commutative Banach algebra that satisfies condition (B), then A is isomorphic to a uniform algebra.

Example (Converse not true)

Let $A = \mathbb{C}^2$. Then $(A, \|.\|_1)$ is isomorphic to $(A, \|.\|_\infty)$, a uniform algebra. But $(A, \|.\|_1)$ does not satisfy condition (B), as $r(a, b) < \|(a, b)\|_1$ if and only if (a, b) is invertible.

Let $\phi : A \to B$ be an isometric Banach algebra isomorphism. Then ϕ preserves condition (B).

Example (Isometry cannot be dropped)

Let X be a locally compact Hausdorff space and X^∞ denote the one point compactification of X.

- $C(X^\infty)$, (being a uniform algebra) satisfies condition (B).
- Let $C_0(X)$. $C_0(X)$ is unital if and only if X compact.
- Let $C_0(X)^e$ denote the *unitization* of $C_0(X)$.
- In particular take $X = (1, \infty)$.
- $\left(\frac{1}{x^2}, 1\right)$ has the inverse $\left(\frac{-1}{1+x^2}, 1\right)$ in $C_0((1,\infty))^e$.
- $\left(\frac{1}{x^2}, 1\right)$ does not satisfy condition (B).
- Define the map $\psi : C_0((1,\infty))^e \to C((1,\infty)^\infty)$ by $\psi(f,\lambda) = f + \lambda e$, where $e(x) = 1$ for every $x \in (1,\infty)^\infty$ and each $f \in C_0((1,\infty))$ is extended by assigning zero to the point ∞.
- ψ is a Banach algebra isomorphism, but not an isometry.

From the next example we see that finite dimensional Banach algebras may fail to satisfy condition (B).

Example

Consider $\ell^1(\mathbb{Z}_2) = \{f | f : \mathbb{Z}_2 \longrightarrow \mathbb{C}\}$ with the norm $\|f\| = |f(0)| + |f(1)|$ and multiplication defined by convolution as

$$(f * g)(0) = f(0)g(0) + f(1)g(1)$$

$$(f * g)(1) = f(0)g(1) + f(1)g(0).$$

Here the identity element being $(e(0), e(1)) = (1, 0)$. It is easy to verify that $f = (1, 0)$ and $g = (0, i)$ satisfies condition (B) but $f + g$ does not.

Now we use polar decomposition of invertible elements in a C*-algebra to prove condition (B) in the same.

Theorem

Let A be any C-algebra, then A satisfies condition (B).*

Now we use polar decomposition of invertible elements in a C^*-algebra to prove condition (B) in the same.

Theorem

Let A be any C^*-algebra, then A satisfies condition (B).

Corollary

If H is a Hilbert space then $B(H)$ satisfies condition (B).

If we consider a Banach space instead of a Hilbert space, we have a sufficient condition. $T \in B(X)$ is called norm attaining if there exists an element $x \in X$ with $\|x\| = 1$, such that $\|Tx\| = \|T\|$.

Theorem

Let $T \in G(B(X))$ such that T^{-1} is norm attaining, then T satisfies condition (B).

Corollary

If X is finite dimensional, then any $T \in B(X)$ attains its norm, and hence, $B(X)$ satisfies condition (B).

Example (Norm attaining is not necessary)

Let the Hilbert space $(\ell^2, \|.\|_2)$ and $\{e_n\}_{n\in\mathbb{N}}$ be the standard complete orthonormal basis. Consider $T \in B(H)$ defined by

$$T(e_n) = \left(1 + \frac{1}{(n+1)}\right) e_n \quad n \geq 1.$$

Then T is invertible and satisfies condition (B) as H is a Hilbert space, but T^{-1} is not norm attaining.

(B)

- $C(X)$, X compact T_2
- M_n
- C^* algebra
- $B(H)$, H a Hilbert space

Does not have (B)

- $C^1[0,1]$
- $\ell^1(\mathbb{Z}_2)$
- $B(X)$, X a Banach space

[1, 2, 3, 4]

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Thank you