ϵ-CONDITION SPECTRUM OF OPERATORS

D. Sukumar

Department of Mathematics
IIT Madras

January 6, 2007

T. J. Ransford.
Generalised spectra and analytical multivalued function
J.London Math. Soc (1984).

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1. $1 \in \Omega$,
2. $0 \notin \Omega$,
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Ransford's Spectrum

$$\sigma_\Omega(x) = \{\lambda \in \mathbb{C} : x - \lambda \notin \Omega\}.$$

DEFINITION (PSEUDOSPECTRUM $(0 < \epsilon < 1)$)

$$\Lambda_\epsilon(a) := \left\{ \lambda \in \mathbb{C} : \left\|(\lambda - a)^{-1}\right\| \geq \frac{1}{\epsilon} \right\},$$

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DEFINITION (ϵ- CONDITION SPECTRUM)

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2. $\sigma(a) \subseteq \sigma_\epsilon(a)$ for every $a \in A$, $0 < \epsilon < 1$
3. If $0 < \epsilon_1 < \epsilon_2 < 1$ then $\sigma_{\epsilon_1}(a) \subseteq \sigma_{\epsilon_2}(a)$ for every $a \in A$

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$$\sigma(a) = \bigcap_{0<\epsilon<1} \sigma_\epsilon(a)$$
5. $\sigma_\epsilon(a)$ is non empty compact subset of $\mathbb{C}$ for every $a \in A$
6. The map $a \to \sigma_\epsilon(a)$ is upper semi continuous function from A to compact subsets of $\mathbb{C}$.

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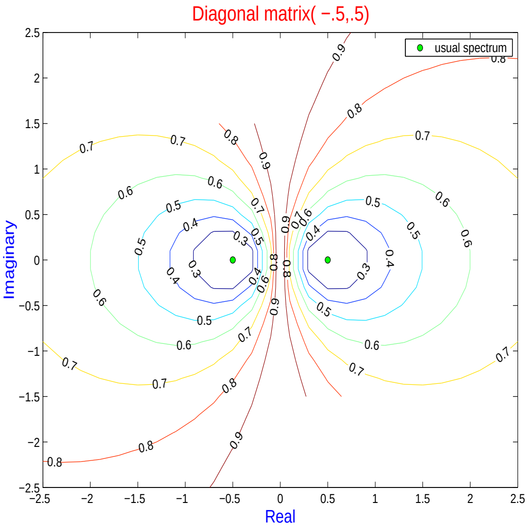
7. $\sigma_\epsilon(\alpha + \beta a) = \alpha + \beta\sigma_\epsilon(a)$ for all $\alpha, \beta \in \mathbb{C}$



Diagonal matrix(−.5,.5)
usual spectrum
Imaginary
Real

Theorem (No isolated points)

Let A be a complex unital Banach algebra and $a \in A$ such that $a \neq \lambda$ for every $\lambda \in \mathbb{C}$. Then $\sigma_\epsilon(a)$ has no isolated points.

THEOREM (NO ISOLATED POINTS)

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THEOREM (FINITE COMPONENTS)

Let $a \in A$ and $0 < \epsilon < 1$. Then $\sigma_\epsilon(a)$ has a finite number of components and every component of $\sigma_\epsilon(a)$ contains an element from $\sigma(a)$.

Condition spectral radius $r_\epsilon(a)$ is defined as

$$r_\epsilon(a) := \sup\{|z| : z \in \sigma_\epsilon(a)\}.$$

THEOREM

Let A be a Banach algebra and $a \in A$. Then

$$r(a) \leq r_\epsilon(a) \leq \frac{1+\epsilon}{1-\epsilon} \|a\|$$

Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then, the numerical range of T, denoted by $W(T)$, is defined by,

$$W(T) := \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

Theorem (Numerical range)

$$\sigma_\epsilon(a) \subseteq W(T) + B\left(0, \frac{2\epsilon}{1-\epsilon}\|T\|\right)$$

THEOREM

Let $T : \mathcal{H} \to \mathcal{H}$ *be a bounded linear operator on a Hilbert space* $\mathcal{H}$ *and* $\lambda \in \sigma_\epsilon(T)$.

1. *If* T *is a normal then*

$$d(\lambda, \sigma(T)) \leq \frac{2\epsilon}{1-\epsilon} \|T\|.$$

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3. *If T is a self-adjoint then*

$$|Im(\lambda)| \leq \frac{2\epsilon}{1-\epsilon} \|T\|.$$

Let $\mathcal{H}$ be finite dimensional Hilbert space.

THEOREM

Let M be a matrix of order $n \times n$. Then the following sets are equal.

1. $\sigma_\epsilon(M) = \{z \in \mathbb{C} : \|z - M\| \left\|(z - M)^{-1}\right\| \geq \epsilon^{-1}\}$
2. $B = \{z \in \mathbb{C} : \exists\, u \in \mathbb{C}^n$ *with* $\|u\| = 1$ *s.t* $\|(z - M)u\| \leq \epsilon \|(z - M)\|\}$
3. $C = \{z \in \mathbb{C} : z \in \sigma(M + E)$ *for some* E *with* $\|E\| \leq \epsilon \|(z - M)\|\}$

Mark Embree and Lloyd N. Trefethen, *Generalizing eigenvalue theorems to pseudospectra theorems*, SIAM J. Sci. Comput. **23** (2001), no. 2, 583–590 (electronic). MR MR1861266 (2002k:15019)

T. J. Ransford, *Generalised spectra and analytic multivalued functions*, J. London Math. Soc. (2) **29** (1984), no. 2, 306–322. MR MR744102 (85f:46091)

Thank you