Eigenvalues, eigenvectors and applications

Dr. D. Sukumar

Department of Mathematics
Indian Institute of Technology Hyderabad

Recent Trends in Applied Sciences with Engineering Applications
June 27-29, 2013
Department of Applied Science
Government Engineering College,Kozhikode, Kerala

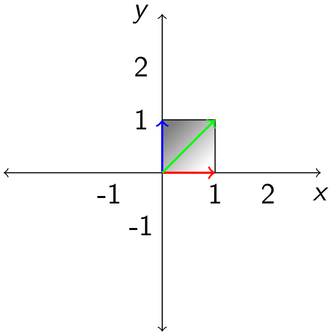
Maps which preserve

- Origin
- lines passing through origin
- parallelograms with one corner as origin

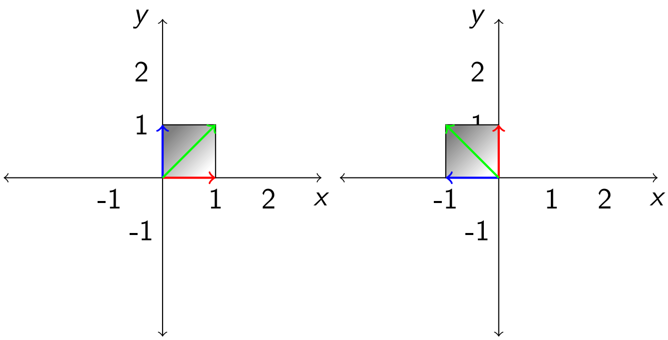
Outline

1. Linear transformations on plane
   - Typical Examples
   - Properties
2. Eigen values
   - Eigen value and eigen vector
3. Markov Matrices
   - Formation
   - Interpretation
   - Properties

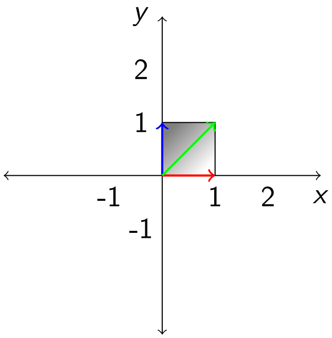
Rotation $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



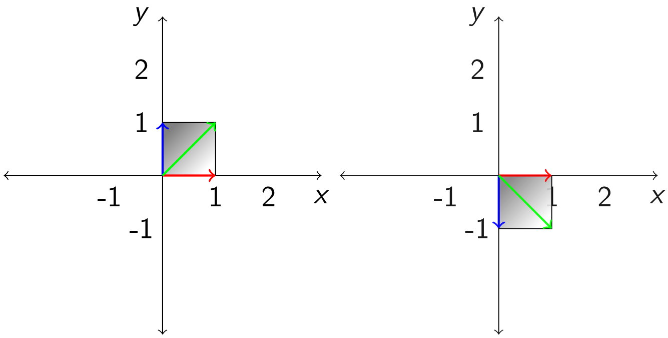
Rotation $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



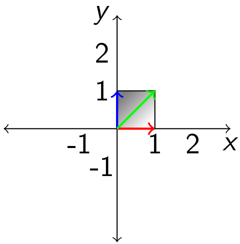
Reflection $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



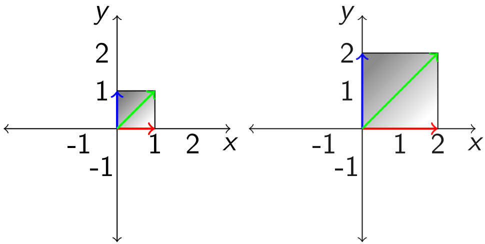
Reflection $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



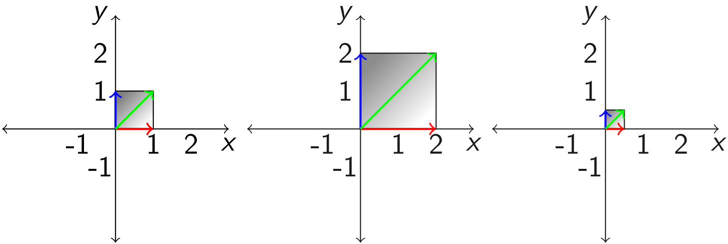
Expansion $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ Compression $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$



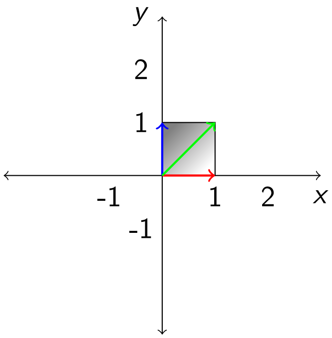
Expansion $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ Compression $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$



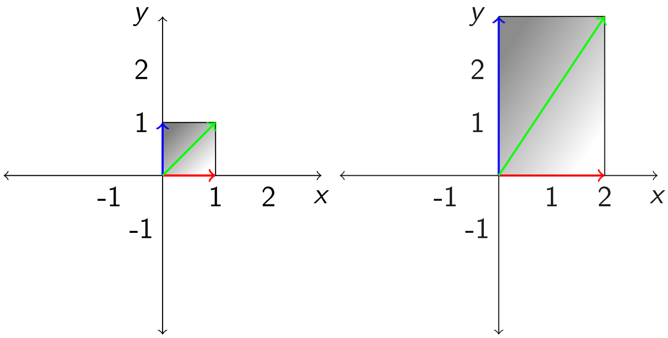
Expansion $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ Compression $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$



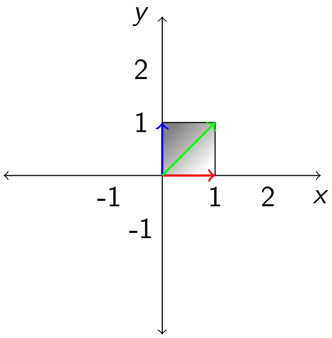
Multi-scaling or Stretching $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$



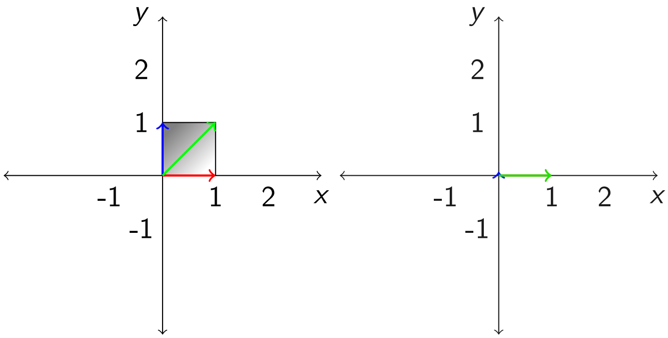
Multi-scaling or Stretching $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$



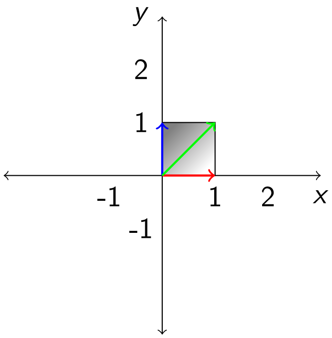
Projection $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$



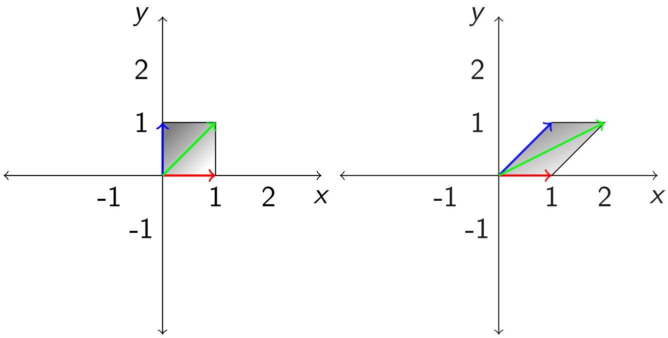
Projection $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$



Shear transformation $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$



Shear transformation $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$



Outline

Properties

- Area

Properties

- Area
- Eigen vectors

Properties

- Area
- Eigen vectors
- Eigen values

Properties

- Area
- Eigen vectors
- Eigen values
- Determinant

Properties

- Area
- Eigen vectors
- Eigen values
- Determinant
- Diagonalizable

Table of properties

Map	Area	Fixed Dir Eigenvector	Scale in FD Eigenvalue	Det	Diagonable
Rotation	1	NO	NO	1	NO
Reflection	1	x-axis, y-axis	1,-1	-1	Yes
Expansion	4	x-axis, y-axis	2, 2	4	Yes
Compression	1/4	x-axis, y-axis	1/2,1/2	1/4	Yes
Multi-scaling	6	x-axis, y-axis	2,3	6	Yes
Projection	0	x-axis, y-axis	1,0	0	Yes
Shear	1	x-axis	1	1	NO

Table: Properties

Problem

Big Problem

- Getting a common opinion from individual opinion

Purpose

Problem

Big Problem

- Getting a common opinion from individual opinion
- From individual preference to common preference

Purpose

Problem

Big Problem

- Getting a common opinion from individual opinion
- From individual preference to common preference

Purpose

- Showing all steps of this process using linear algebra

Problem

Big Problem

- Getting a common opinion from individual opinion
- From individual preference to common preference

Purpose

- Showing all steps of this process using linear algebra
- Mainly using eigenvalues and eigenvectors

Outline

Eigen value

Let A a be square matrix.

- Eigen values of A are solutions or roots of

$$\det(A - \lambda I) = 0.$$

Eigen value

Let A a be square matrix.

- Eigen values of A are solutions or roots of

$$\det(A - \lambda I) = 0.$$

- If

$$Ax = \lambda x \quad \text{or} \quad (A - \lambda I)x = 0,$$

for a non-zero vector x then

Eigen value

Let A a be square matrix.

- Eigen values of A are solutions or roots of

$$\det(A - \lambda I) = 0.$$

- If

$$Ax = \lambda x \quad \text{or} \quad (A - \lambda I)x = 0,$$

for a non-zero vector x then
  - λ is an eigenvalue of A and

Eigen value

Let A a be square matrix.

- Eigen values of A are solutions or roots of

$$\det(A - \lambda I) = 0.$$

- If

$$Ax = \lambda x \quad \text{or} \quad (A - \lambda I)x = 0,$$

  for a non-zero vector x then
  - λ is an eigenvalue of A and
  - x is an eigenvector corresponding to the eigenvalue λ.

Example

Consider the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$.

Example

Consider the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$.

Example (Eigen value)

$$A - \lambda I = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 5 \\ 3 & -\lambda \end{bmatrix}$$

Example

Consider the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$.

Example (Eigen value)

$$A - \lambda I = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 5 \\ 3 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(-\lambda) - (3 \times 5) = \lambda^2 - 2\lambda - 15$$

Example

Consider the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$.

Example (Eigen value)

$$A - \lambda I = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 5 \\ 3 & -\lambda \end{bmatrix}$$

$\det(A - \lambda I) = (2-\lambda)(-\lambda) - (3 \times 5) = \lambda^2 - 2\lambda - 15$

The roots of the polynomial are the eigen values: -3 and 5.

Example

Consider the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$.

Example (Eigen value)

$$A - \lambda I = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 5 \\ 3 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(-\lambda) - (3 \times 5) = \lambda^2 - 2\lambda - 15$$

The roots of the polynomial are the eigen values: -3 and 5.

Example (When $\lambda = -3$)

$$\begin{bmatrix} 2-(-3) & 5 \\ 3 & -(-3) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigen vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Example

Consider the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$.

Example (Eigen value)

$$A - \lambda I = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 5 \\ 3 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(-\lambda) - (3 \times 5) = \lambda^2 - 2\lambda - 15$$

The roots of the polynomial are the eigen values: -3 and 5.

Example (When $\lambda = -3$)

$$\begin{bmatrix} 2-(-3) & 5 \\ 3 & -(-3) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigen vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Example (When $\lambda = 5$)

$$\begin{bmatrix} 2-5 & 5 \\ 3 & -5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigen vector $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$

For the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$

- the eigenvalues are -3 and 5

For the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$

- the eigenvalues are -3 and 5
- the eigenvector corresponding to -3 is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$

- the eigenvalues are -3 and 5
- the eigenvector corresponding to -3 is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- the eigenvector corresponding to 5 is $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} =$$

For the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$

- the eigenvalues are -3 and 5
- the eigenvector corresponding to -3 is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- the eigenvector corresponding to 5 is $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} =$$

For the matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$

- the eigenvalues are -3 and 5
- the eigenvector corresponding to -3 is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- the eigenvector corresponding to 5 is $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 5 \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Outline

1. Linear transformations on plane
   - Typical Examples
   - Properties
2. Eigen values
   - Eigen value and eigen vector
3. Markov Matrices
   - Formation
   - Interpretation
   - Properties

5 Volunteers

1. Data: Each one should give your marking for each one of you.

5 Volunteers

1. Data: Each one should give your marking for each one of you. For example: Isha: (I)100, (J)20, (K)50, (L)30, (M)0

5 Volunteers

1. Data: Each one should give your marking for each one of you. For example: Isha: (I)100, (J)20, (K)50, (L)30, (M)0
2. Normalizing or averaging:

5 Volunteers

1. Data: Each one should give your marking for each one of you. For example: Isha: (I)100, (J)20, (K)50, (L)30, (M)0
2. Normalizing or averaging: How much you have given from the total marking

   Total mark: 100+20+50+30+0=200

5 Volunteers

1. Data: Each one should give your marking for each one of you. For example: Isha: (I)100, (J)20, (K)50, (L)30, (M)0
2. Normalizing or averaging: How much you have given from the total marking

   Total mark: 100+20+50+30+0=200

   Normal mark: 100/200, 20/200,50/200 , 30/200, 0/200

5 Volunteers

1. Data: Each one should give your marking for each one of you. For example: Isha: (I)100, (J)20, (K)50, (L)30, (M)0
2. Normalizing or averaging: How much you have given from the total marking

   Total mark: 100+20+50+30+0=200

   Normal mark: 0.50, 0.1, 0.25, 0.15, 0.0

5 Volunteers

1. Data: Each one should give your marking for each one of you. For example: Isha: (I)100, (J)20, (K)50, (L)30, (M)0
2. Normalizing or averaging: How much you have given from the total marking

   Total mark: 100+20+50+30+0=200

   Normal mark: 0.50, 0.1, 0.25, 0.15, 0.0
3. Arranging: Writing them in a matrix form

$$\begin{pmatrix} 0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\ 0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\ 0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\ 0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\ 0.0 & 0.2 & 0.0 & 0.2 & 0.2 \end{pmatrix}$$

	Isha	*Jose*	*Kumar*	*Latha*	*Mani*
Isha	0.5	0.8	0.0	0.4	0.2
Jose	0.1	0.32	0.0	0.0	0.2
Kumar	0.25	0.24	1.0	0.0	0.2
Latha	0.15	0.16	0.0	0.4	0.2
Mani	0.0	0.2	0.0	0.2	0.2
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
	sum is 1	1	1	1	1

Markov Matrix

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & & m_{nn} \end{bmatrix}_{n\times n}$$

Definition (Markov Matrix)

A real $n \times n$ matrix is called a Markov matrix if

1. Each entry is non-negative: $m_{ij} \geq 0$ for all $1 \leq i \leq n, 1 \leq j \leq n$.

Markov Matrix

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & & m_{nn} \end{bmatrix}_{n\times n}$$

Definition (Markov Matrix)

A real $n \times n$ matrix is called a Markov matrix if

1. Each entry is non-negative: $m_{ij} \geq 0$ for all $1 \leq i \leq n, 1 \leq j \leq n$.
2. Each column sum is 1: $\sum_{i=1}^{n} m_{ij} = 1$ for each $1 \leq j \leq n$.

Outline

Interpretation

- We have formed a Markov matrix from individual opinions.

$$\begin{pmatrix} 0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\ 0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\ 0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\ 0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\ 0.0 & 0.2 & 0.0 & 0.2 & 0.2 \end{pmatrix}$$

Interpretation

- We have formed a Markov matrix from individual opinions.
- If Isha's opinion is full value then Isha is better than others.

$$\begin{pmatrix} 0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\ 0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\ 0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\ 0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\ 0.0 & 0.2 & 0.0 & 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.1 \\ 0.25 \\ 0.15 \\ 0 \end{pmatrix}$$

Interpretation

- We have formed a Markov matrix from individual opinions.
- If Isha's opinion is full value then Isha is better than others.
- If Jose's opinion is full value then Isha is better than others.

$$\begin{pmatrix} 0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\ 0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\ 0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\ 0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\ 0.0 & 0.2 & 0.0 & 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.32 \\ 0.24 \\ 0.16 \\ 0.2 \end{pmatrix}$$

Interpretation

- We have formed a Markov matrix from individual opinions.
- If Isha's opinion is full value then Isha is better than others.
- If Jose's opinion is full value then Isha is better than others.
- If Mani's opinion is full value then all are equal.

$$\begin{pmatrix} 0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\ 0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\ 0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\ 0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\ 0.0 & 0.2 & 0.0 & 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{pmatrix}$$

Interpretation

- We have formed a Markov matrix from individual opinions.
- If Isha's opinion is full value then Isha is better than others.
- If Jose's opinion is full value then Isha is better than others.
- If Mani's opinion is full value then all are equal.
- If all opinions are equal value then Isha is better than others.

$$\begin{pmatrix} 0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\ 0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\ 0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\ 0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\ 0.0 & 0.2 & 0.0 & 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.380 \\ 0.124 \\ 0.338 \\ 0.182 \\ 0.120 \end{pmatrix}$$

Interpretation

- We have formed a Markov matrix from individual opinions.
- If Isha's opinion is full value then Isha is better than others.
- If Jose's opinion is full value then Isha is better than others.
- If Mani's opinion is full value then all are equal.
- If all opinions are equal value then Isha is better than others.
- If opinion has different value then Kumar is better than others.

$$\begin{pmatrix} 0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\ 0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\ 0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\ 0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\ 0.0 & 0.2 & 0.0 & 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.4 \\ 0.1 \\ 0.2 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.320 \\ 0.072 \\ 0.394 \\ 0.186 \\ 0.100 \end{pmatrix}$$

Interpretation

- We have formed a Markov matrix from individual opinions.
- If Isha's opinion is full value then Isha is better than others.
- If Jose's opinion is full value then Isha is better than others.
- If Mani's opinion is full value then all are equal.
- If all opinions are equal value then Isha is better than others.
- If opinion has different value then Kumar is better than others.

$$\begin{pmatrix} 0.5 & 0.8 & 0.0 & 0.4 & 0.2 \\ 0.1 & 0.32 & 0.0 & 0.0 & 0.2 \\ 0.25 & 0.24 & 1.0 & 0.0 & 0.2 \\ 0.15 & 0.16 & 0.0 & 0.4 & 0.2 \\ 0.0 & 0.2 & 0.0 & 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

- When the opinion values matches with the conclusion value?. For which opinion value we will get the same conclusion value.

Big Problem

- Getting a common opinion from individual opinion

Big Problem

- Getting a common opinion from individual opinion
- From individual preference to common preference

Big Problem

- Getting a common opinion from individual opinion
- From individual preference to common preference
- For a Markov matrix M, we need a stable opinion value x.

Big Problem

- Getting a common opinion from individual opinion
- From individual preference to common preference
- For a Markov matrix M, we need a stable opinion value x.
- Find a vector x such that

$$Mx = x$$

Big Problem

- Getting a common opinion from individual opinion
- From individual preference to common preference
- For a Markov matrix M, we need a stable opinion value x.
- Find a vector x such that

$$Mx = x$$

- For M, find an eigenvector x corresponding to eigenvalue 1.

Outline

1. Linear transformations on plane
   - Typical Examples
   - Properties
2. Eigen values
   - Eigen value and eigen vector
3. Markov Matrices
   - Formation
   - Interpretation
   - Properties

Properties

Theorem

For every Markov matrix 1 is surely an eigenvalue.

Properties

Theorem

For every Markov matrix 1 is surely an eigenvalue.

Proof.

For a Markov matrix M every column sum is 1. Consider the transpose matrix M^T. Now its row sum is 1. Let $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

- $M^T e = e$

$\square$

Properties

Theorem

For every Markov matrix 1 is surely an eigenvalue.

Proof.

For a Markov matrix M every column sum is 1. Consider the transpose matrix M^T. Now its row sum is 1. Let $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

- $M^T e = e$
- 1 is an eigenvalue of M^T with e as its eigenvector.

$\square$

Properties

Theorem

For every Markov matrix 1 is surely an eigenvalue.

Proof.

For a Markov matrix M every column sum is 1. Consider the transpose matrix M^T. Now its row sum is 1. Let $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

- $M^T e = e$
- 1 is an eigenvalue of M^T with e as its eigenvector.
- We know $\det(A) = \det(A^T)$, so $\det(M - \lambda I) = \det(M^T - \lambda I)$

□

Properties

Theorem

For every Markov matrix 1 is surely an eigenvalue.

Proof.

For a Markov matrix M every column sum is 1. Consider the transpose matrix M^T. Now its row sum is 1. Let $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

- $M^T e = e$
- 1 is an eigenvalue of M^T with e as its eigenvector.
- We know $\det(A) = \det(A^T)$, so $\det(M - \lambda I) = \det(M^T - \lambda I)$
- Hence 1 is an eigenvalue of M

□

Properties

Theorem

For every Markov matrix 1 is surely an eigenvalue.

Proof.

For a Markov matrix M every column sum is 1. Consider the transpose matrix M^T. Now its row sum is 1. Let $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

- $M^T e = e$
- 1 is an eigenvalue of M^T with e as its eigenvector.
- We know $\det(A) = \det(A^T)$, so $\det(M - \lambda I) = \det(M^T - \lambda I)$
- Hence 1 is an eigenvalue of M
- But eigen vectors of A and A^T need not be same.

Further Properties

The previous result guaranteed the eigenvalue 1, but there is a problem if it has more eigenvectors corresponding to eigenvalue 1.

Further Properties

The previous result guaranteed the eigenvalue 1, but there is a problem if it has more eigenvectors corresponding to eigenvalue 1.

Theorem

For a Markov matrix M with all entries $m_{ij} > 0$,

1. *1 is always an eigenvalue.*

Further Properties

The previous result guaranteed the eigenvalue 1, but there is a problem if it has more eigenvectors corresponding to eigenvalue 1.

Theorem

For a Markov matrix M with all entries $m_{ij} > 0$,

1. *1 is always an eigenvalue.*
2. *Other eigenvalues have magnitude less than 1 ($|\lambda| < 1$).*

Further Properties

The previous result guaranteed the eigenvalue 1, but there is a problem if it has more eigenvectors corresponding to eigenvalue 1.

Theorem

For a Markov matrix M with all entries $m_{ij} > 0$,

1. *1 is always an eigenvalue.*
2. *Other eigenvalues have magnitude less than 1 (* $|\lambda| < 1$ *).*
3. *There is an eigenvector corresponding to the eigenvalue 1.*

Further Properties

The previous result guaranteed the eigenvalue 1, but there is a problem if it has more eigenvectors corresponding to eigenvalue 1.

Theorem

For a Markov matrix M with all entries $m_{ij} > 0$,

1. *1 is always an eigenvalue.*
2. *Other eigenvalues have magnitude less than 1 ($|\lambda| < 1$).*
3. *There is an eigenvector corresponding to the eigenvalue 1.*
4. *This eigenvector has all positive entries with sum 1.*

Further Properties

The previous result guaranteed the eigenvalue 1, but there is a problem if it has more eigenvectors corresponding to eigenvalue 1.

Theorem

For a Markov matrix M with all entries $m_{ij} > 0$,

1. *1 is always an eigenvalue.*
2. *Other eigenvalues have magnitude less than 1 ($|\lambda| < 1$).*
3. *There is an eigenvector corresponding to the eigenvalue 1.*
4. *This eigenvector has all positive entries with sum 1.*
5. *Such an eigenvector is unique.*

Finding the stable opinion

Now we know that every Markov matrix with positive entries has a unique eigenvector corresponding to the eigenvalue $\lambda = 1$.

Finding the stable opinion

Now we know that every Markov matrix with positive entries has a unique eigenvector corresponding to the eigenvalue $\lambda = 1$.

$$M - \lambda I = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & & m_{nn} \end{bmatrix}_{n\times n} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}_{n\times n}$$

Finding the stable opinion

Now we know that every Markov matrix with positive entries has a unique eigenvector corresponding to the eigenvalue $\lambda = 1$.

$$M - \lambda I = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & & m_{nn} \end{bmatrix}_{n\times n} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}_{n\times n}$$

$$(M - \lambda I)x = \begin{bmatrix} m_{11} - \lambda & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} - \lambda & & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & & m_{nn} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Finding the stable opinion

Now we know that every Markov matrix with positive entries has a unique eigenvector corresponding to the eigenvalue $\lambda = 1$.

$$M - \lambda I = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & & m_{nn} \end{bmatrix}_{n\times n} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}_{n\times n}$$

$$(M - \lambda I)x = \begin{bmatrix} m_{11}-\lambda & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22}-\lambda & & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & & m_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Solve $(M - \lambda I)x = 0$ by Gauss Elimination Process

Other Views

- It is a simpler version of page-rank.

Other Views

- It is a simpler version of page-rank.
- It represents transition matrix and steady state.

Other Views

- It is a simpler version of page-rank.
- It represents transition matrix and steady state.
- Markov process in input-output business models.

Other Views

- It is a simpler version of page-rank.
- It represents transition matrix and steady state.
- Markov process in input-output business models.
- ...

Questions

Questions

Thank you

You know better

In aeronautical engineering eigenvalues may determine whether the flow over a wing is laminar or turbulent.

In electrical engineering they may determine the frequency response of an amplifier or the reliability of a national power system.

In structural mechanics eigenvalues may determine whether an automobile is too noisy or whether a building will collapse in an earth-quake.

In probability they may determine the rate of convergence of a Markov process.

In ecology they may determine whether a food web will settle into a steady equilibrium.

In numerical analysis they may determine whether a discretization of a differential equation will get the right answer or how fast a conjugate gradient iteration will converge.