Condition spectrum results that generalizes usual spectrum results

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1. Introduction
2. Results

Definition (Spectrum)

The spectrum of A is defined as

$$\sigma(A) := \{z \in \mathbb{C} : z - A \notin inv(A)\}$$

Definition (Pseudospectrum)

Let $\varepsilon > 0$. The ε-pseudospectrum of a matrix A is defined as

$$\Lambda_\varepsilon(A) := \left\{z \in \mathbb{C} : z - A \notin inv(A) \text{ or } \left\|(z-A)^{-1}\right\| \geq \varepsilon\right\}.$$

Definition (Condition spectrum)

Let $0 < \varepsilon < 1$. The ε-condition spectrum of a A is defined as

$$\sigma_\varepsilon(A) := \left\{z \in \mathbb{C} : z - A \notin inv(A) \text{ or } \left\|(z-A)^{-1}\right\| \|z - A\| \geq \frac{1}{\varepsilon}\right\}$$

Consider solving the system of equations

$$Ax - \lambda x = b.$$

- Spectrum $\sigma(A) \leftrightarrow$ uniqueness of the solution.
- Pseudospectrum of $\Lambda_\varepsilon(A) \leftrightarrow$ computational aspects of the solution.
- Condition spectrum $\sigma_\varepsilon(A) \leftrightarrow$ computational stability aspect of deriving the solution.

Lemma

1. *For every* $0 < \varepsilon < 1$, $\sigma_\varepsilon(A)$ *is non empty.*
2. *For every* $0 < \varepsilon < 1$, $\sigma(A) \subseteq \sigma_\varepsilon(A)$.
3. *For every* $0 < \varepsilon < 1$, $\sigma_\varepsilon(A)$ *is compact.*
4. *For every* $0 < \varepsilon < 1$, $\sigma_\varepsilon(A)$ *has no isolated points.*

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Theorem

The following sets are equivalent.

1. $\sigma_\varepsilon(A) = \{z \in \mathbb{C} : \left\|(z - A)^{-1}\right\| \|z - A\| \geq \varepsilon^{-1}\}$
2. $\{z \in \mathbb{C} : \exists\, u \in \mathbb{C}^n, \|u\| = 1 \text{ with } \|(z - A)u\| \leq \varepsilon \|(z - A)\|\}$
3. $\{z \in \mathbb{C} : z \in \sigma(A + E) \text{ for some } E \text{ with } \|E\| \leq \varepsilon \|(z - A)\|\}$

Singularity

Theorem

$$A \textit{ is singular} \iff 0 \in \sigma(A)$$

$$\left\|A^{-1}\right\| \geq \varepsilon^{-1} \iff 0 \in \Lambda_\varepsilon(A)$$

$$\|A\| \left\|A^{-1}\right\| \geq \varepsilon^{-1} \iff 0 \in \sigma_\varepsilon(A)$$

Bounds for the spectral radius

Theorem

$$\lambda \in \sigma(A) \Rightarrow |\lambda| \leq \|A\|$$

$$\lambda \in \Lambda_\varepsilon(A) \Rightarrow |\lambda| \leq \|A\| + \varepsilon$$

$$\lambda \in \sigma_\varepsilon(A) \Rightarrow |\lambda| \leq \frac{1+\varepsilon}{1-\varepsilon} \|A\|$$

Digonalization

Theorem

A has N distinct eigenvalues $\Rightarrow$ *A is diagonalizable.*

$\Lambda_{\varepsilon}(A)$ *has N distinct components* $\Rightarrow$ *A is diagonalizable.*

$\sigma_{\varepsilon}(A)$ *has N distinct components* $\Rightarrow$ *A is diagonalizable.*

Lower bound of the inverse

Theorem

$$\left\|(z-A)^{-1}\right\| \geq \frac{1}{d(z,\sigma(A))}$$

$$\left\|(z-A)^{-1}\right\| \geq \frac{1}{d(z,\Lambda_\varepsilon(A))+\varepsilon}$$

$$\left\|(z-A)^{-1}\right\| \geq \frac{1}{d(z,\sigma_\varepsilon(A))+\frac{2\varepsilon}{1-\varepsilon}\|A\|}$$

Preserving similarity

Theorem

$$A = SBS^{-1} \Rightarrow \sigma(A) = \sigma(B)$$

$$A = SBS^{-1} \Rightarrow \Lambda_{\varepsilon}(A) \subseteq \Lambda_{\kappa(S)\varepsilon}(B)$$

$$A = SBS^{-1} \Rightarrow \sigma_{\varepsilon}(A) \subseteq \sigma_{\kappa(S)^2\varepsilon}(B)$$

Similarity transformation through orthogonal and unitary matrices preserves the condition spectrum.

Transient behavior

Theorem

$$\max_{\lambda \in \sigma(A)} |\lambda| > 1 \Rightarrow \sup_{k \geq 0} \left\| A^k \right\| = \infty$$

$$\max_{\lambda \in \Lambda_\varepsilon(A)} |\lambda| > 1 + M\varepsilon \Rightarrow \sup_{k > 0} \left\| A^k \right\| > M$$

$$\max_{\lambda \in \sigma_\varepsilon(A)} |\lambda| > \frac{1 + M^2\varepsilon}{1 - M\varepsilon} \Rightarrow \sup_{k \geq 0} \left\| A^k \right\| > M \quad \textit{whenever} \quad M \leq \frac{1}{\varepsilon}$$

Transient behavior

Theorem

$\lambda \in \sigma(A) \Rightarrow \left\|A^k\right\| \geq |\lambda|^k$ *for all k*

$\lambda \in \Lambda_\varepsilon(A) \Rightarrow \left\|A^k\right\| \geq |\lambda|^k - \dfrac{k\varepsilon \|A\|^{k-1}}{1 - k\varepsilon / \|A\|}$ *for all k such that* $k\varepsilon < \|A\|$

$\lambda \in \sigma_\varepsilon(A) \Rightarrow \left\|A^k\right\| \geq |\lambda|^k - \dfrac{ks \|A\|^{k-1}}{1 - ks / \|A\|}$ *for all k such that* $(2k+1)\varepsilon < 1$ *where* $s = \dfrac{2\varepsilon}{1-\varepsilon}\|A\|$

Gerschgorin's theorem

Theorem

$$\sigma(A) \subseteq \bigcup_{j=1}^{N} D(d_j, r_j)$$

$$\Lambda_\varepsilon(A) \subseteq \bigcup_{j=1}^{N} D(d_j, r_j + \sqrt{N}\varepsilon)$$

$$\sigma_\varepsilon(A) \subseteq \bigcup_{j=1}^{N} D\left(d_j, r_j + \sqrt{N}\frac{2\varepsilon}{1-\varepsilon}\|A\|\right)$$

Connection with numerical range

Theorem

$$W(A) \supseteq conv(\sigma(A))$$

$$W(A) \supseteq conv(\Lambda_\varepsilon(A)) \setminus \varepsilon \text{ - } border$$

$$W(A) \supseteq conv(\sigma_\varepsilon(A)) \setminus \varepsilon_1 \text{ - } border, \;\; here \; \varepsilon_1 = \frac{2\varepsilon}{1-\varepsilon} \|A\|$$

The notion $S \setminus \varepsilon$-border means the set of points $z \in \mathbb{C}$ such that $D(z,\varepsilon) \subseteq S$

Change under linear transformation

Theorem

For all $\alpha, \beta \in \mathbb{C}$

$$\sigma(\alpha + \beta A) = \alpha + \beta\sigma(A)$$

$$\Lambda_{\varepsilon|\beta|}(\alpha + \beta A) = \alpha + \beta\Lambda_{\varepsilon}(A)$$

$$\sigma_{\varepsilon}(\alpha + \beta A) = \alpha + \beta\sigma_{\varepsilon}(A)$$

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Thank you