BANACH ALGEBRAS

1. Banach Algebras

The aim of this notes is to make familiarity with the basic concepts of Banach algebras.

Definition 1.1 (Algebra)**.** *Let $\mathcal{A}$ be a non-empty set. Then $\mathcal{A}$ is called an algebra if*

(1) $(\mathcal{A}, +, .)$ *is a vector space over a field* $\mathbb{F}$
(2) $(\mathcal{A}, +, \circ)$ *is a ring and*
(3) $(\alpha(a))b = \alpha(ab) = a\alpha b$ *for every* $\mathbb{F}$*, for every* $a, b \in \mathcal{A}$

Usually we write ab instead of $a \circ b$ for the product of a and b.

Definition 1.2. *An algebra $\mathcal{A}$ is said to be*

(1) ***real*** *or* ***complex*** *according to the field* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$.
(2) ***commutative*** *if* $(\mathcal{A}, +, \circ)$ *is commutative*
(3) ***unital*** *if* $(\mathcal{A}, +, \circ)$ *has a unit, usually denoted by* 1.
(4) *if $\mathcal{A}$ is unital and $a \in \mathcal{A}$. If there exists a $b \in \mathcal{A}$ such that*

$$ab = ba = 1,$$

then b is called an ***inverse*** *of a.*

Remark 1.3. The unit element in a Banach algebra is unique. Also if an element has an in inverse, then it is unique.

Definition 1.4 (Invertible Set)**.** *Let* $G(\mathcal{A}) := \{a \in \mathcal{A} : a \text{ is invertible in } \mathcal{A}\}$.

Note that $1 \in G(\mathcal{A})$ and $0 \neq G(\mathcal{A})$. The set $G(\mathcal{A})$ is a multiplicative group.

Definition 1.5. *Let $\mathcal{A}$ be an algebra and $\mathcal{B} \subseteq \mathcal{A}$. Then $\mathcal{B}$ is said to be a subalgebra if $\mathcal{B}$ it self is an algebra with respect to the operations of $\mathcal{A}$.*

Definition 1.6 (normed algebra)**.** *If $\mathcal{A}$ is an algebra and $\|\cdot\|$ is a norm on $\mathcal{A}$ satisfying*

$$\|ab\| \leq \|a\|\|b\|, \quad \text{for all } a, b \in \mathcal{A},$$

then $\|\cdot\|$ is called an algebra norm and $(\mathcal{A}, \|\cdot\|)$ is called a ***normed algebra****. A complete normed algebra is called a* ***Banach algebra****.*

Remark 1.7. We always denote the identity of a unital Banach algebra by 1 and assume that $\|1\| = 1$. In a normed algebra, the multiplication is both left and right continuous with respect to the algera norm.

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Example 1.8 (Finite dimensional)**.**
(1) Let $\mathcal{A} = \mathbb{C}$. Then with respect to the usual multiplication of complex numbers and the modulus, $\mathcal{A}$ is a Banach algebra.
(2) Let $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$, the set of $n \times n$ matrices with matrix addition, matrix multiplication and with Frobenius norm defined by
$$\|A\|_F = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}$$
is a non-commutative unital Banach algebra.

Exercise 1.9. *Show that* $\|A\| = \max_{1 \le i,j \le n} |a_{ij}|$ *defines a norm on* $\mathcal{M}_n(\mathbb{C})$ *but it is not an algebra norm.*

Example 1.10 (Continuous functions)**.** Observe for all the examples in this list, the addition and multiplication are pointwise and hence they are all commutative.

(1) Let K be a compact Hausdorff space and $\mathcal{A} = C(K)$, the set of al complex valued continuous function defined on K. Then with respect to the point wise multiplication of functions and with the norm
$$\|f\|_\infty = \sup_{t \in K} |f(t)|,$$
is a commutative Banach algebra.
(2) Let Ω be a locally compact Hausdorff space and let
$$\mathcal{A} = C_b(\Omega) := \{f \in \mathcal{C}(\Omega) : f \text{ is bounded}\}.$$
Then $\mathcal{A}$ is a commutative unital Banach algebra.
(3) Let $\mathcal{A} = \mathcal{C}_0(\Omega) := \{f \in \mathcal{C}(\Omega) : f \text{ vanishes at } \infty\}$
It can be verified easily that $\mathcal{A}$ is a non unital, normed algebra which is not a Banach algebra.
(4) Let
$$\begin{aligned} \mathcal{A} = \mathcal{C}_c(\Omega) : &= \{f \in \mathcal{C}(\Omega) : f \text{ has compact support}\} \\ &= \{f \in \mathcal{C}(\Omega) : \forall\, \epsilon > 0,\ \exists\, K_\epsilon \ni |f(t)| < \epsilon,\ \ \forall\, t \in K_\epsilon^c\}. \end{aligned}$$
Then $\mathcal{A}$ is a commutative, non unital, normed algebra which is not a Banach algebra.
(5) Let $X = [0,1]$. Then $C'[0,1] \subset C[0,1]$ is an algebra and $(C'[0,1], \|\cdot\|_\infty)$ is not complete. Now define a new norm on $C'[0,1]$ as
$$\|f\| = \|f\|_\infty + \|f'\|_\infty,\ f \in C'[0,1].$$
Then $(C'[0,1], \|\cdot\|)$ is a Banach algebra.
(6) Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Consider
$$\mathcal{A}(\mathbb{D}) := \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is analytic}\}.$$
Then $\mathcal{A}(\mathbb{D})$ is a closed subalgebra of $C(\bar{\mathbb{D}})$ and hence a Banach algebra and is known as the **disc algebra**.

(7) Let $S \neq \emptyset$ and $B(S) = \{f : S \to \mathbb{C} : f \text{ is bounded}\}$. For $f, g \in \mathcal{B}(S)$, define

$$(f+g)(s) = f(s) + g(s)$$
$$(fg)(s) = f(s)g(s)$$
$$(\alpha f)(s) = \alpha f(s), \text{for all } f, g \in B(S),\ \alpha \in \mathbb{C}.$$

Then $B(S)$ is an algebra with unit $f(s) = 1$ for all $s \in S$. (Note that all the above algebras fits into this framework). With the norm

$$\|f\|_\infty = \sup\{|f(s)| : s \in S\}$$

$B(S)$ is a commutative unital Banach Algebra.

Example 1.11 (Operator Algebras)**.** Observe for all the examples in this list, the addition is pointwise but the multiplication is composition of maps and hence in general all are non-commutative.

(1) Let X and Y are complex Banach spaces and $\mathcal{B}(X, Y)$ is the Banach space of bounded linear maps between X and Y. For $T, S \in \mathcal{B}(X, Y)$ and $x \in X$, addition and multications are difened as

$$(T+S)(x) = Tx + Sx$$
$$(TS)(x) = T(Sx).$$

Then $\mathcal{B}(X, Y)$ is a non-commutative Banacha algebra with respect to the operator norm defined by

$$\|T\| := \sup\{\|Tx\| : \|x\| \leq 1\}.$$

When $X = Y$ denote $\mathcal{B}(X, X)$ as $\mathcal{B}(X)$.

(2) The set of compact operators on X denoted by $\mathcal{K}(X)$ is a closed subalgebra of $\mathcal{B}(X)$. Hence it is a Banach algebra. Note that $\mathcal{K}(X)$ is an ideal in $\mathcal{B}(X)$.

(3) When the underlying space is a Hilber space H, the space $\mathcal{B}(H)$ and $\mathcal{K}(H)$ also non-commutative Banach algebras.

Definition 1.12. *A a non-empty subset I of $\mathcal{A}$ is called an **ideal** if*

(i) *I is a subspace i. e., if $a, b \in I$ and $\alpha \in \mathbb{F}$, then $\alpha a + b \in I$*

(ii) *I is an ideal in the ring i.e., $a \in \mathcal{A}$ and $c \in \mathcal{A}$ implies that $ac, ca \in I$.*

*An ideal I is said to be **maximal** if $I \neq \mathcal{A}$ and if J is any ideal of $\mathcal{A}$ such that $I \subseteq J$, then $J = \mathcal{A}$.*

Remark 1.13. Every ideal is a subalgebra but a subalgebra need not be an ideal.

Examples

(1) Let H be a complex Hilbert space, then $\mathcal{K}(H)$ is an ideal of $\mathcal{B}(H)$

(2) Let K be a compact, Hausdorff space and F be a closed subset of K. Then
$$I_F := \{f \in C(K) : f|_F = 0\}$$
is an ideal. In fact, these are the only ideals in $C(K)$. .

(3) The set of all $n \times n$ upper/lower triangular matrices is a subalgebra but not an ideal.

(4) Let $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$ and $\mathcal{D} = \{(a_{ij}) \in \mathcal{A} : a_{ij} = 0,\ i \neq j\}$. Then $\mathcal{D}$ is a subalgebra but not an ideal.

Exercise 1.14. *Show that I_F is maximal if and only if F is a singleton set*

Definition 1.15. *Let $\mathcal{A}, \mathcal{B}$ be two algebras, a map $\phi : \mathcal{A} \to \mathcal{B}$ is said to be a homomorphism if*

(i) *ϕ is linear and*

(ii) *ϕ is multiplicative i.e., $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$*

If ϕ is one-to-one, then ϕ is called an isomorphism.

Exercise 1.16. *If $\phi : \mathcal{A} \to \mathcal{B}$ is an homomorphism, then ker(ϕ) is an ideal. Suppose $\mathcal{A}$ is an algebra with unit 1 and $\mathcal{B} = \mathbb{C}$, $\phi : \mathcal{A} \to \mathbb{C}$ be a homomorphism. Then*

(i) *$\phi(1) = 1$ and*

(ii) *ϕ maps invertible elements of $\mathcal{A}$ into invertible elements of $\mathbb{C}$*

2. Invertibility

Throughout we assume that $\mathcal{A}$ is a complex unital Banach algebra. Recall that $G(\mathcal{A}) := \{a \in \mathcal{A} :$ a is invertible in $\mathcal{A}\}$. In this section we discuss the properties of this set. We have seen that $G(\mathcal{A})$ is a multiplicative group. We know that if $z \in \mathbb{C}$ with $|z| < 1$, then $(1 - z)$ is invertible and
$$(1 - z)^{-1} = \sum_{n=0}^{\infty} z^n.$$
This result can be generalized to elements in a Banach algebra.

Lemma 2.1. *If $a \in \mathcal{A}$ with $\|a\| < 1$. Then $1 - a \in G(\mathcal{A})$ and*
$$(1 - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$
Further more, $\|(1 - a)^{-1}\| \leq \dfrac{1}{1 - \|a\|}$.

Proof. Let $s_n := 1 + a + a^2 + \cdots + a^n$. Then $\|s_n\| \leq \sum_{j=0}^{n} \|a\|^j$. Since $\|a\| < 1$, the sequence s_n is convergent. This shows that s_n is absolutely converegent. Since $\mathcal{A}$ is a Banach algebra, the series $\sum_{n=0}^{\infty} a^n$ is convergent. Let $b = \sum_{n=0}^{\infty} a^n$.

Consider $(1-a)b = \lim_{n\to\infty}(1-a)s_n = \lim\limits_{n\to\infty}(1-a^{n+1}) = 1$. Similarly we can show that $b(1-a) = 1$. Hence $a^{-1} = b$. To get the bound consider

$$\|b\| = \lim_{n\to\infty} \|s_n\| \leq \lim_{n\to\infty} \sum_{k=0}^{n} \|a\|^k = \sum_{n=0}^{\infty} \|a\|^k = \frac{1}{1-\|a\|}.$$

□

Corollary 2.2. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$.*

(1) *Let $\lambda \in \mathbb{C}$ such that $\|a\| < |\lambda|$. Then $(\lambda.1 - a)^{-1} \in G(\mathcal{A})$ and $(\lambda.1-a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}$. Furthermore $\|(a-\lambda)^{-1}\| \leq \frac{1}{|\lambda| - \|a\|}$*

(2) *Let $a \in \mathcal{A}$ be such that $\|1-a\| < 1$, then $a \in \mathcal{A}$ and*

$$a^{-1} = \sum_{n=0}^{\infty}(1-a)^n.$$

Proof. To prove (1), take $a_\lambda = \frac{a}{\lambda}$ and apply Lemma 2.1. To prove (2), substitute $1-a$ in Lemma 2.1. □

Proposition 2.3. *The set $G(\mathcal{A})$ is open in $\mathcal{A}$.*

Proof. Let $a \in G(\mathcal{A})$. Let $D = \{b \in \mathcal{A} : \|a-b\| < \frac{1}{\|a^{-1}\|}\}$. Note that $a^{-1}(a-b) = 1 - a^{-1}b$. So $\|1-a^{-1}b\| \leq \|a^{-1}\|\|a-b\| < 1$. Hence $a^{-1}b \in G(\mathcal{A})$. So $b = a(a^{-1}b) \in G(\mathcal{A})$. This shows that $D \subset G(\mathcal{A})$. □

Proposition 2.4. *The map $a \mapsto a^{-1}$ is continuous on $G(\mathcal{A})$.*

Let $(a_n) \subseteq G(\mathcal{A})$ be such that $a_n \to a$. Our aim is to show that $a_n^{-1} \to a^{-1}$. For this consider

$$\|a_n^{-1} - a^{-1}\| = \|a^{-1}(a_n - a)a_n^{-1}\| \leq \|a^{-1}\|\|a_n - a\|\|a_n^{-1}\|. \tag{1}$$

Hence if we can show $\|a_n^{-1}\|$ is bounded by a fixed constant, we are done. As $a_n \to a$, there exists $n_0 \in \mathbb{N}$ such that $\|a_n - a\| < \frac{1}{2\|a^{-1}\|}$ for all $n \geq n_0$. Thus $\|a^{-1}a_n - 1\| < \frac{1}{2}$. Hence by Lemma 2.1, $a^{-1}a_n \in G(\mathcal{A})$ for all $n \geq n_0$.

$$\|aa_n^{-1}\| = \|(a^{-1}a_n)^{-1}\| = \sum_{k=0}^{\infty} \| \left((a^{-1}a_n)^{-1} - 1\right)^k \| < \sum_{k=0}^{\infty} \frac{1}{2^k} < 2.$$

Therefore $\|a_n^{-1}\| \leq \|a_n^{-1}a\|\,\|a^{-1}\| < 2\,\|a^{-1}\|$. Now by Equation 1, it follows that $a_n^{-1} \to a^{-1}$.

Example 2.5. (a) Let $\mathcal{A} = \mathbb{C}$. Then Then $G(\mathcal{A}) = \{z \in \mathbb{C} : z \neq 0\}$

(b) Let $\mathcal{A} = C(K)$, where K is compact, Hausdorff space. Then

$$G(\mathcal{A}) = \{f \in \mathcal{A} : f(t) \neq 0 \text{ for each } t \in K\}.$$

(c) Let $\mathcal{A} = M_n(\mathbb{C})$. Then $G(\mathcal{A}) = \{A \in M_n(\mathbb{C}) : det(A) \neq 0\}$.
(c) Let $\mathcal{A} = \mathcal{B}(H)$. Then $G(\mathcal{A}) = \{A \in \mathcal{B}(H) : A^{-1} \in \mathcal{B}(H)\}$

Exercise 2.6. *In which of the above cases $G(\mathcal{A})$ is dense? What about in general?*

3. The spectrum

In this section we define the concept of spectrum of an element in a Banach algebra.

Definition 3.1. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. The resolvent $\rho_{\mathcal{A}}(a)$ of a with respect to $\mathcal{A}$ is defined by*

$$\rho_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C} : a - \lambda 1 \in G(\mathcal{A})\}.$$

The spectrum $\sigma_{\mathcal{A}}(a)$ of a with respect to $\mathcal{A}$ is defined by $\sigma_{\mathcal{A}}(a) = \mathbb{C} \setminus \rho_{\mathcal{A}}(a)$. That is same as saying

$$\sigma_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C} : a - \lambda 1 \textit{ is not invertible in } \mathcal{A}\}.$$

If $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$ such that $1 \in \mathcal{B}$. If $a \in \mathcal{B}$, then once discuss the invertibilty of a in $\mathcal{B}$ and in $\mathcal{A}$. In such cases we write $\rho_{\mathcal{B}}(a)$ and $\rho_{\mathcal{A}}(a)$. Similar convention hold for the spectrum. But if we want discuss the spectrum in one algebra, then we omit the prefix.

Example 3.2.
(1) Let $z \in \mathbb{C}$. Then $\sigma(z) = \{z\}$.
(2) Let $A \in M_n(\mathbb{C})$. Then $\sigma(A) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigen value of } A\}$
(3) Let $f \in C(K)$ for some compact Hausdorff K. Then $\sigma(f) = \text{range}(f)$

First we need to show that the spectrum of an element in a Banach algebra is a non empty compact set.

Theorem 3.3. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Then $\sigma(a) \neq \emptyset$.*

Proof. If a is not invertible, then $0 \in \sigma(a)$. Assume that a is invertible. Then $0 \notin \sigma(a)$. Assume that $\sigma(a) = \emptyset$. Then $\rho(a) = \mathbb{C}$. Let $\phi \in A^*$. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(\lambda) = \phi((a - \lambda 1)^{-1})$. Let $\lambda_0 \in \mathbb{C}$. Then consider

$$\begin{aligned}
\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} &= \frac{\phi((a - \lambda 1)^{-1}) - \phi((a - \lambda_0 1)^{-1})}{\lambda - \lambda_0} \\
&= \frac{\phi\Big((a - \lambda 1)^{-1} - (a - \lambda_0 1)^{-1}\Big)}{\lambda - \lambda_0} \\
&= \frac{\phi\Big((a - \lambda_0 1)^{-1}(\lambda - \lambda_0)(a - \lambda 1)^{-1}\Big)}{\lambda - \lambda_0} \\
&= \phi\Big((a - \lambda_0 1)^{-1}(a - \lambda 1)^{-1}\Big).
\end{aligned}$$

Hence $\lim\limits_{\lambda \to \lambda_0} \dfrac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = \phi(\lambda_0 1 - a)^2$. As λ_0 is arbitrary, f is entire.

Note that $|f(\lambda)| \leq \|\phi\| \, \|(\lambda 1 - a)^{-1}\| \leq \|\phi\| \frac{1}{|\lambda| - \|a\|}$ for each $|\lambda| > \|a\|$. Hence $|f(\lambda)| \to 0$ as $|\lambda| \to \infty$. Hence f is bounded. By Liouville's theorem, f must be constant. AS $|f(\lambda)| \to 0$, $f = 0$. That is $\phi(\lambda 1 - a)^{-1} = 0$ for all $\phi \in \mathcal{A}^*$. Hence by Hahn-Banach theorem, $(\lambda 1 - a)^{-1} = 0$, a contradiction. Therefore $\sigma(a)$ is non empty. □

Exercise 3.4. *What about the case when $|\lambda| \leq \|a\|$? in the last proof.*

Remark 3.5. (a) We know by Lemma 2.1 that if $\lambda \in \mathbb{C}$ such that $|\lambda| > \|a\|$, then $(a - \lambda 1) \in G(\mathcal{A})$. Hence

$$\sigma(a) \subseteq \{z \in \mathbb{C} : |z| \leq \|a\|\}.$$

Hence $\sigma(a)$ is bounded subset of $\mathbb{C}$.

Exercise 3.6. *Let $\phi \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Define $g : \mathbb{C} \setminus \sigma(a) \to \mathbb{C}$ by $g(\lambda) = \phi((\lambda.1 - a)^{-1})$. Show that g is analytic in $\mathbb{C} \setminus \sigma(a)$.*

Theorem 3.7. *$\sigma(a)$ is a compact set.*

Proof. Note by Theorem 3.3 and Exercise 3.6, it follows that $\sigma(a)$ is a compact subset of $\mathbb{C}$. □

Theorem 3.8 (Gelfand-Mazur theorem)**.** *Every Banach division algebra is isometrically isomorphic to $\mathbb{C}$.*

Proof. Let $a \in \mathcal{A}$. Then $\sigma(a) \neq \emptyset$. Let $\lambda \in \sigma(\mathcal{A})$. Then $a - \lambda 1 \notin G(\mathcal{A})$. As $\mathcal{A}$ is a divison algebra, $a - \lambda 1 = 0$. Hence $a = \lambda \cdot 1$. Now define a map $\eta : \mathbb{C} \to \mathcal{A}$ by $\eta(\lambda) = \lambda \cdot 1$. It can be checked easily that η is an isometric isomorphism. □

Proposition 3.9. *Let $\mathcal{A}$ be a unital Banach algebra and $a, b \in \mathcal{A}$. Then*

$$1 - ab \in G(\mathcal{A}) \Leftrightarrow 1 - ba \in G(\mathcal{A}).$$

Proof. Assume that $1 - ab \in G(\mathcal{A})$. Let $c := b(1 - ab)^{-1}a$. It can be checked easily that $c(1 - ba) = 1 = (1 - ba)c$. □

Corollary 3.10. *Let $\mathcal{A}$ be a unital Banach algebra and $a, b \in \mathcal{A}$. Then $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$.*

Definition 3.11 (Spectral Radius)**.** *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Then the spectral radius of a is defined by*

$$r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Note that $0 \leq r(a) \leq \|a\|$.

Example 3.12. (1) Let $\mathcal{A} = C(K)$ and $f \in \mathcal{A}$. Then

$$r(f) = \sup\{|\lambda| : \lambda \in \text{ range}(f)\} = \|f\|_\infty$$

(2) Let $T \in \mathcal{B}(H)$ be normal. Then $r(T) = \sup\{\lambda \in \sigma(T)\} = \|T\|$

(3) Let $\mathcal{A} = M_n(\mathbb{C})$. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Since A is nilpotent, $\sigma(A) = \{0\}$ and hence $r(A) = 0$. But $\|A\| = 1$.

(4)

Exercise 3.13. *Let $\mathcal{A}$ be a unital complex Banach algebra and $a \in \mathcal{A}$. Show that*

(a) *if a is invertible, then $\sigma(a^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(a)\}$*
(b) $\sigma(a+1) = \{\lambda + 1 : \lambda \in \sigma(a)\}$
(c) *$r(a^n) = r(a)^n$ for all $n \in \mathbb{N}$*
(d) *if $b \in \mathcal{A}$, then $r(ab) = r(ba)$*

Theorem 3.14 (Spectral radius formula)**.**

$$r(a) = \lim \|a^n\|^{\frac{1}{n}} = \inf \|a^n\|^{\frac{1}{n}}.$$