CONDITION SPECTRUM

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Abstract. We define a new type of spectrum, called the ϵ-condition spectrum, of an element a in a complex unital Banach algebra A as

$$\sigma_\epsilon(a) := \left\{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible or } \|\lambda - a\| \left\|(\lambda - a)^{-1}\right\| \geq \frac{1}{\epsilon}\right\}.$$

This is expected to be useful in solving operator equations. We show that this is a particular case of the generalized spectrum defined by Ransford [10]. This ϵ-condition spectrum shares some properties of the usual spectrum such as non emptiness and compactness. But at the same time it has many properties that are different from the properties of the usual spectrum. For example, the ϵ-condition spectrum always has only a finite number of components. Also if a is not a scalar multiple of 1 then $\sigma_\epsilon(a)$ has no isolated points. Several examples are given to illustrate the main ideas.

1. Introduction

Let A be a complex algebra with unit 1. We shall identify $\lambda \cdot 1$ with λ. The spectrum of an element a in an algebra A is defined as

$$\sigma(a) := \{\lambda \in \mathbb{C} : a - \lambda \notin Inv(A)\}$$

where $Inv(A)$ is the set of invertible elements in A. There are several extensions and generalizations of the idea of the spectrum. Most notable among these are Ransford's generalized spectrum [10] and pseudospectra [6].

In this note we consider one more such extension in terms of the condition number. Let A be a complex Banach algebra with unit 1. The *condition number* of an invertible element a is defined as $\|a\| \left\|a^{-1}\right\|$ and denoted by $\kappa(a)$. It is convenient to make a convention that $\kappa(a) = \infty$ if a is not invertible. We shall use this convention throughout. The condition number is a very useful concept and arises naturally in solving systems of equations. Specifically it is a measure of the sensitivity of the answer to a problem to small changes in the initial data of the problem. (See [4], [8] for more information on condition number.) For a fixed $0 < \epsilon < 1$, define

$$\Omega_\epsilon = \left\{a \in Inv(A) : \kappa(a) < \frac{1}{\epsilon}\right\}$$

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and define the ϵ-*condition spectrum* for that fixed ϵ by

$$\sigma_\epsilon(a) := \{\lambda \in \mathbb{C} : a - \lambda \notin \Omega_\epsilon\}.$$

Suppose X is a Banach space, $T\colon X \to X$ is a bounded linear map and $y \in X$. Consider the operator equation

$$Tx - \lambda x = y. \tag{1}$$

Then

- $\lambda \notin \sigma(T)$ implies equation (1) is solvable;
- $\lambda \notin \sigma_\epsilon(T)$ implies equation (1) has a stable solution.

In view of this, the ϵ-condition spectrum is expected to be a useful tool in the numerical solution of operator equations.

Another well-known extension of the concept of spectrum is the idea of pseudospectrum, $\Lambda_\epsilon(a)$, defined as

$$\Lambda_\epsilon(a) := \left\{\lambda \in \mathbb{C} : \left\|(\lambda - a)^{-1}\right\| \geq \frac{1}{\epsilon}\right\},$$

with the convention that $\left\|(\lambda - a)^{-1}\right\| = \infty$ if $\lambda - a$ is not invertible. This is found to be a very useful concept in numerical computations, especially in those involving non-normal matrices. See [1], [5] and [11] for details. Also the website [6] contains a lot of information about pseudospectra.

In section 2, we consider some examples and prove some basic properties of the ϵ-condition spectrum and compare these properties with those of the usual spectrum. In particular, we prove that the ϵ-condition spectrum $\sigma_\epsilon(a)$ of a is a non empty compact set containing the spectrum $\sigma(a)$ (Theorem 2.7). We may note that since the ϵ-condition spectrum is a geometric concept as against the spectrum which is a purely algebraic concept, we should expect these two sets to have some different properties. In the final section 3, we prove some of these interesting geometric properties, namely that the ϵ-condition spectrum has no isolated points except in trivial situations (Theorem 3.1) and that the ϵ-condition spectrum always has only a finite number of components and moreover every component contains an element of the spectrum (Theorem 3.6).

Several questions still remain unanswered. Some of these are:

(1) Is there an analogue of the spectral radius formula and spectral mapping theorem?
(2) What is the relation between the ϵ-condition spectrum, pseudospectrum and numerical range?
(3) How can we compute the ϵ-condition spectrum?

We hope to deal with some of these questions in the future.

2. Basic properties

In this section, we prove some basic properties of the ϵ-condition spectrum. Assume that A is a complex Banach algebra with unit 1. First we review the Ransford's spectrum, generalization of the idea of the spectrum (in fact to normed linear spaces), as in [10].

Definition 2.1 (Ransford set)**.** An open subset Ω of A satisfying the following properties is called a Ransford set.

(1) $1 \in \Omega$,
(2) $0 \notin \Omega$,
(3) $z\Omega \subseteq \Omega$ for all $z \in \mathbb{C} \setminus \{0\}$.

Definition 2.2 (Ransford spectrum)**.** Let Ω be a Ransford set in A and let a be in A. Then the Ransford spectrum of a with respect to the Ransford set Ω is defined as follows:

$$\sigma^{\Omega}(a) := \{\lambda \in \mathbb{C} : a - \lambda \notin \Omega\}.$$

Note that $Inv(A)$ is a Ransford set and the usual spectrum $\sigma(a)$ is nothing but $\sigma^{Inv(A)}(a)$, that is, the Ransford spectrum with respect to the Ransford set $Inv(A)$, in this notation. For this spectrum, Ransford proved:

Theorem 2.3. *Let Ω be a Ransford set in A. Then the following holds.*

(1) $\sigma^{\Omega}(0) = \{0\}$ *and* $\sigma^{\Omega}(1) = \{1\}$
(2) *If for $a \in A$, then $\sigma^{\Omega}(a)$ is compact*
(3) *Let $E := \{a \in A : \sigma^{\Omega}(a) \neq \emptyset\}$. Then the map $a \to \sigma^{\Omega}(a)$ is an upper semicontinuous function from E to compact subsets of $\mathbb{C}$.*

We refer [10] for a proof of this theorem as well as for several properties of Ransford spectrum. For subsequent studies on Ransford spectrum see [2] and [9].

Definition 2.4. For a fixed $0 < \epsilon < 1$, define

$$\Omega_\epsilon := \left\{a \in Inv(A) : \kappa(a) < \frac{1}{\epsilon}\right\}.$$

As 0 is not invertible, $0 \notin \Omega_\epsilon$, also $1 \in \Omega_\epsilon$, since $\|1\| \left\|1^{-1}\right\|$=1. Note that

$$\|a\| \left\|a^{-1}\right\| = \|za\| \left\|(za)^{-1}\right\| , \forall z \in \mathbb{C} \setminus \{0\}$$

and this proves $z\Omega \subseteq \Omega$ for $z \in \mathbb{C} \setminus \{0\}$. The map $a \to \|a\| \left\|a^{-1}\right\|$ is continuous and hence Ω_ϵ is an open set. These observations prove that Ω_ϵ is a Ransford set.

Definition 2.5 (ϵ-condition spectrum)**.** Let $0 < \epsilon < 1$ and $a \in A$. The ϵ-*condition spectrum* of a for this ϵ is defined by

$$\sigma_\epsilon(a) := \{\lambda \in \mathbb{C} : \lambda - a \notin \Omega_\epsilon\} = \left\{\lambda \in \mathbb{C} : \kappa(\lambda - a) \geq \frac{1}{\epsilon}\right\}$$

with the convention that $\kappa(\lambda - a) = \infty$ when $\lambda - a$ is not invertible. The ϵ-condition spectral radius $r_\epsilon(a)$ is defined as

$$r_\epsilon(a) := \sup\{|z| : z \in \sigma_\epsilon(a)\}.$$

Recall that the usual spectral radius $r(a)$ is defined by

$$r(a) := \sup\{|z| : z \in \sigma(a)\}.$$

We use the following well known result often.

Lemma 2.6. *Let $a \in A$ and $|\lambda| > \|a\|$; then $\lambda - a$ is invertible,*

$$(\lambda - a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}} \quad \text{and} \quad \left\|(\lambda - a)^{-1}\right\| \leq \frac{1}{|\lambda| - \|a\|}.$$

In the next theorem, we give some properties of the ϵ-condition spectrum that follow in a straightforward manner from Definition 2.5 and by the fact that Ω_ϵ is a Ransford set.

Theorem 2.7. (1) *$\sigma_\epsilon(0) = \{0\}$ and $\sigma_\epsilon(1) = \{1\}$.*
(2) *If $0 < \epsilon_1 < \epsilon_2 < 1$ then $\sigma_{\epsilon_1}(a) \subseteq \sigma_{\epsilon_2}(a)$ for every $a \in A$*
(3) *$\sigma(a) \subseteq \sigma_\epsilon(a)$ for every $a \in A$. In fact*

$$\sigma(a) = \bigcap_{0<\epsilon<1} \sigma_\epsilon(a)$$

(4) *$\sigma_\epsilon(a)$ is a nonempty compact subset of $\mathbb{C}$ for every $a \in A$*
(5) *The map $a \to \sigma_\epsilon(a)$ is an upper semicontinuous function from A to compact subsets of $\mathbb{C}$.*
(6) *Suppose $a = sbs^{-1}$ for some $a, b, s \in A$; then $\sigma_\epsilon(a) \subseteq \sigma_{\kappa(s)^2\,\epsilon}(b)$*
(7) *$\sigma_\epsilon(\alpha + \beta a) = \alpha + \beta\sigma_\epsilon(a)$ for all $\alpha, \beta \in \mathbb{C}$*

Proof. 1, 2 and 3 follow from the definition of ϵ-condition spectrum. Since $\sigma(a)$ is nonempty 3 implies that $\sigma_\epsilon(a)$ is nonempty. Now 4 and 5 follow from Theorem 2.3 in this section. 6 follows from the inequality

$$\kappa(\lambda - a) \leq \kappa(s)^2\kappa(\lambda - b)$$

and 7 follows from

$$\kappa\left(z - (\alpha + \beta a)\right) = \kappa\left(\beta\frac{(z - \alpha)}{\beta} - \beta a\right) = \kappa\left(\frac{(z - \alpha)}{\beta} - a\right).$$

□

Remark 2.8. We have proved in (3) of the above theorem that $\sigma(a) \subseteq \sigma_\epsilon(a)$. The reverse inclusion (and hence equality) holds only in a very special case, when a is a scalar multiple of 1. This is proved in Corollary 3.5.

In the next theorem, we give some properties of the condition spectral radius $r_\epsilon(a)$.

Theorem 2.9. (1) *$r(a) \leq r_\epsilon(a) \leq \dfrac{1+\epsilon}{1-\epsilon}\,\|a\|$*
(2) *If $\|a^n\| \leq M < \frac{1}{\epsilon}$ for all $n \geq 0$, then $r_\epsilon(a) \leq \dfrac{1 + M^2\epsilon}{1 - M^2\epsilon}$.*

Proof of 1. Since $\sigma(a) \subseteq \sigma_\epsilon(a)$, we have $r(a) \leq r_\epsilon(a)$. To prove the remaining part of the inequality consider $\lambda \in \sigma_\epsilon(a)$. If $|\lambda| \leq \|a\|$, then clearly $|\lambda| \leq \dfrac{1+\epsilon}{1-\epsilon}\|a\|$.

Suppose $|\lambda| > \|a\|$ then $\lambda - a$ is invertible and $\left\|(\lambda - a)^{-1}\right\| \leq \dfrac{1}{|\lambda| - \|a\|}$. Hence

$$1 \leq \epsilon \left\|(\lambda - a)^{-1}\right\| \|\lambda - a\| \leq \epsilon \frac{|\lambda| + \|a\|}{|\lambda| - \|a\|}.$$

On simplification,

$$|\lambda| \leq \frac{1+\epsilon}{1-\epsilon}\|a\|.$$

Proof of 2. First note that

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} M^{\frac{1}{n}} = 1.$$

Let $\lambda \in \sigma_\epsilon(a)$. If $|\lambda| \leq 1$ then clearly $|\lambda| \leq \dfrac{1+M^2}{1-M\epsilon}$. If $|\lambda| > 1$, then $\lambda \notin \sigma(a)$. Hence $\lambda - a \in Inv(A)$ and

$$(\lambda - a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}.$$

Thus

$$\left\|(\lambda - a)^{-1}\right\| \leq \frac{1}{|\lambda|}\sum_{k=0}^{\infty} \frac{\left\|a^k\right\|}{|\lambda|^k} \leq \frac{M}{|\lambda|}\sum_{k=0}^{\infty} \frac{1}{|\lambda|^k} = \frac{M}{|\lambda|}\left(\frac{1}{1-\dfrac{1}{|\lambda|}}\right) = \frac{M}{|\lambda| - 1}.$$

Next since $\lambda \in \sigma_\epsilon(a)$, we have

$$\frac{1}{\epsilon} \leq \|\lambda - a\| \left\|(\lambda - a)^{-1}\right\| \leq (|\lambda| + M)\frac{M}{|\lambda| - 1}.$$

On simplification this becomes

$$|\lambda| \leq \frac{1+M^2\epsilon}{1-M\epsilon} \quad \text{provided} \quad M < \frac{1}{\epsilon}.$$

Hence

$$r_\epsilon(a) \leq \frac{1+M^2\epsilon}{1-M\epsilon}$$

□

The following example shows that inequality (1) of Theorem 2.9 can become equality.

Example 2.10. For the function $f \in C\big([-1,1]\big)$ defined by $f(x) = x$ in $[-1,1]$ this bound is attained. That is $\left(\dfrac{1+\epsilon}{1-\epsilon}\right)\|f\| \in \sigma_\epsilon(f)$.

Proof. Let $\lambda = \left(\dfrac{1+\epsilon}{1-\epsilon}\right)\|f\| = \left(\dfrac{1+\epsilon}{1-\epsilon}\right)$; then

$$\|f-\lambda\| = \frac{2}{1-\epsilon} \quad \text{and} \quad \left\|(f-\lambda)^{-1}\right\| = \frac{1-\epsilon}{2\epsilon}$$

□

Example 2.11 (A diagonal matrix)**.** Let $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1 \neq \lambda_2$ and let $P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Then

$$\|P-\lambda\| = \max\{|\lambda-\lambda_1|, |\lambda-\lambda_2|\},$$

$$\left\|(P-\lambda)^{-1}\right\| = \max\left\{\frac{1}{|\lambda-\lambda_1|}, \frac{1}{|\lambda-\lambda_2|}\right\}.$$

Hence

$$\sigma_\epsilon(P) = \left\{\lambda : \frac{|\lambda-\lambda_1|}{|\lambda-\lambda_2|} \geq \frac{1}{\epsilon}\right\} \cup \left\{\lambda : \frac{|\lambda-\lambda_2|}{|\lambda-\lambda_1|} \geq \frac{1}{\epsilon}\right\}.$$

Example 2.12 (A triangular matrix)**.** Let $R : \mathbb{C}^2 \to \mathbb{C}^2$ defined as $R(x,y) = (0,x)$ for all (x,y) in $\mathbb{C}^2$ (truncation of right shift operator). Considering R as an operator on $\mathbb{C}^2$ we get

$$\|R-\lambda\|_1 = \|R-\lambda\|_\infty = 1 + |\lambda|,$$

$$\left\|(R-\lambda)^{-1}\right\|_1 = \left\|(R-\lambda)^{-1}\right\|_\infty = \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}.$$

Hence in both $(\mathbb{C}^2, \| \ \|_1)$ and $(\mathbb{C}^2, \| \ \|_\infty)$

$$\sigma_\epsilon(R) = \left\{\lambda : |\lambda| \leq \frac{\sqrt{\epsilon}}{1-\sqrt{\epsilon}}\right\}.$$

Whereas in $(\mathbb{C}^2, \| \ \|_2)$

$$\sigma_\epsilon(R) = \left\{\lambda : \frac{\left(1+\sqrt{4|\lambda|^2+1}\right)^2}{2|\lambda|^2} \geq \frac{1}{\epsilon}\right\}.$$

Example 2.13 (Bilateral Shift)**.** Let $v : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ defined as

$$v(e_i) = e_{i+1}, \quad \forall i \in \mathbb{Z}$$

where the e_i's form the standard orthonormal basis. Let λ be a complex number. Consider $v-\lambda$, its defining function, (see [7])

$$a(t) = -\lambda + e^{it}.$$

It is known that $\|v-\lambda\| = \|a(t)\|_\infty$. Hence $\|v-\lambda\| = |\lambda|+1$.

When $|\lambda| \neq 1$, $v - \lambda$ is invertible and the defining function of $(v-1)^{-1}$ is $\dfrac{1}{a(t)}$. So,

$$\left\|(v-\lambda)^{-1}\right\| = \left\|\frac{1}{a(t)}\right\|_\infty = \frac{1}{\inf_{0\le t\le 2\pi}|a(t)|} = \begin{cases} \dfrac{1}{|\lambda|-1} & \text{for } |\lambda| > 1 \\ \dfrac{1}{1-|\lambda|} & \text{for } |\lambda| < 1 \end{cases}.$$

On combining them we get

$$\kappa(v-\lambda) = \|v-\lambda\| \left\|(v-\lambda)^{-1}\right\| = \begin{cases} \dfrac{|\lambda|+1}{|\lambda|-1} & \text{for } |\lambda| > 1 \\ \dfrac{1+|\lambda|}{1-|\lambda|} & \text{for } |\lambda| < 1 \end{cases}.$$

With this, $\|v-\lambda\| \left\|(v-\lambda)^{-1}\right\| \ge \dfrac{1}{\epsilon}$ if and only if either $\dfrac{1-\epsilon}{1+\epsilon} \le |\lambda| < 1$ or $1 < |\lambda| \le \dfrac{1+\epsilon}{1-\epsilon}$. It is known that $\sigma(a) = \{\lambda : |\lambda| = 1\}$. Hence we have

$$\sigma_\epsilon(v) = \left\{\lambda : \frac{1-\epsilon}{1+\epsilon} \le |\lambda| \le \frac{1+\epsilon}{1-\epsilon}\right\}.$$

Example 2.14 (Shift Operators)**.** Let R and L be respectively the right and left shift operators on l^p where $p = 1$ or ∞. One can get

$$\|R-\lambda\|_1 = \|R-\lambda\|_\infty = \|L-\lambda\|_1 = \|L-\lambda\|_\infty = |\lambda| + 1.$$

For $|\lambda| > 1$ both $(R-\lambda)^{-1}$ and $(L-\lambda)^{-1}$ exist and

$$\left\|(R-\lambda)^{-1}\right\|_1 = \left\|(R-\lambda)^{-1}\right\|_\infty = \left\|(L-\lambda)^{-1}\right\|_1 = \left\|(L-\lambda)^{-1}\right\|_\infty = \frac{1}{|\lambda|-1}.$$

With this, for $|\lambda| \ge 1$ and $p = 1$ or ∞, $\left\|(L-\lambda)^{-1}\right\|_p \|(L-\lambda)\|_p \ge \dfrac{1}{\epsilon}$ if and only if $|\lambda| \le \dfrac{1+\epsilon}{1-\epsilon}$. It is known that $\sigma(R) = \sigma(L) = \{\lambda : |\lambda| \le 1\}$ (see [7]). Hence we get

$$\sigma_\epsilon(R) = \sigma_\epsilon(L) = \left\{\lambda : |\lambda| \le \frac{1+\epsilon}{1-\epsilon}\right\}.$$

Example 2.15. Let $T : l^2 \to l^2$ be defined by

$$T\left(x_1, x_2, \ldots, x_n, \ldots\right) = \left(x_1, \frac{1}{2}x_2, \ldots, \frac{1}{n}x_n, \ldots\right).$$

It is known that $\sigma(T) = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ (see [7]). Let $\lambda = x + iy \in \mathbb{C}$. As

$$\|\lambda - T\| = \sup_{n\in\mathbb{N}} \left|\lambda - \frac{1}{n}\right|, \qquad \left\|(\lambda - T)^{-1}\right\| = \frac{1}{\inf_{n\in\mathbb{N}} \left|\lambda - \frac{1}{n}\right|},$$

we can see that

$$\|\lambda - T\| = \sup_{n\in\mathbb{N}} \left|\lambda - \frac{1}{n}\right| = \begin{cases} |\lambda - 1| & \text{if } x \leq \frac{1}{2} \\ |\lambda| & \text{if } x \geq \frac{1}{2} \end{cases} .$$

But $\left\|(\lambda - T)^{-1}\right\|$ depends on the distance between λ and the nearest element to it in $\sigma(a)$, so we consider $\lambda = x+iy$, and compute the ϵ-condition spectrum by dividing the real part of λ into a number of parts, that is, dividing the complex plane into a number of strips. In each strip we estimate $\|\lambda - T\|$ and $\left\|(\lambda - T)^{-1}\right\|$ and give those elements in the strip which are in the ϵ-condition spectrum.

Case: $x \leq 0$

$$\left\{x + iy : \left(x + \frac{\epsilon}{(1-\epsilon)}\right)^2 + y^2 \leq \frac{\epsilon}{(1-\epsilon)^2}\right\} \subseteq \sigma_\epsilon(T)$$

Case: $\frac{1}{n+1} \leq x \leq \frac{2n+1}{2n(n+1)}, \quad n \geq 2$

$$\left\{x + iy : \left(x - \frac{1-\epsilon(n+1)}{(n+1)(1-\epsilon)}\right)^2 + y^2 \leq \frac{\epsilon n^2}{(n+1)(1-\epsilon)^2}\right\} \subseteq \sigma_\epsilon(T).$$

Case: $\frac{2n+1}{2n(n+1)} \leq x \leq \frac{1}{n}, \quad n \geq 2$

$$\left\{x + iy : \left(x - \frac{1-n\epsilon}{n(1-\epsilon)}\right)^2 + y^2 \leq \frac{\epsilon(n-1)^2}{(n)(1-\epsilon)^2}\right\} \subseteq \sigma_\epsilon(T).$$

Case: $\frac{1}{2} \leq x \leq \frac{3}{4}$

$$\left\{x + iy : \left(x - \frac{1}{2(1-\epsilon)}\right)^2 + y^2 \leq \frac{\epsilon}{2(1-\epsilon)^2}\right\} \subseteq \sigma_\epsilon(T).$$

Case: $\frac{3}{4} \leq x \leq 1$

$$\left\{x + iy : \left(x - \frac{1}{1-\epsilon}\right)^2 + y^2 \leq \frac{\epsilon}{(1-\epsilon)^2}\right\} \subseteq \sigma_\epsilon(T).$$

Case:$x \geq 1$

$$\left\{x + iy : \left(x - \frac{1}{(1-\epsilon)}\right)^2 + y^2 \leq \frac{\epsilon}{(1-\epsilon)^2}\right\} \subseteq \sigma_\epsilon(T).$$

And hence $\sigma_\epsilon(T)$ contains these sets and nothing more. Note that, when n is such that $2n^2 - 1 \geq \frac{1}{\epsilon}$ then all those elements in the ϵ-condition spectrum with real part less than $\frac{1}{n}$ belong to a single component.

Next we consider some other ways of describing the ϵ-condition spectrum. Let A be a complex Banach algebra with unit 1. Let $Sing(A) = A \setminus Inv(A)$.

Suppose $a \in A$ and $r > 0$. We use the notation $D(a,r)$ for the open disc with center at a and radius r. Suppose $a \in Inv(A)$, then, since

$$D\left(a, \frac{1}{\|a^{-1}\|}\right) \subseteq Inv(A),$$

$$d(a, Sing(A)) := \inf\left\{\|a-b\| : b \in Sing(A)\right\} \geq \frac{1}{\|a^{-1}\|}.$$

Using this observation, we show that if b is sufficiently small and $\lambda \in \sigma(a+b)$, then $\lambda \in \sigma_\epsilon(a)$. The converse also holds if the Banach algebra A has an additional property.

Theorem 2.16. *Let A be a complex Banach algebra with 1. Let $0 < \epsilon < 1$ and $a \in A$ be such that a is not a scalar multiple of the identity. Suppose $\lambda \in \sigma(a+b)$ where $b \in A$ and $\|b\| \leq \epsilon \|\lambda - a\|$. Then $\lambda \in \sigma_\epsilon(a)$.*

Proof. If $\lambda \in \sigma(a)$ then the conclusion follows trivially as $\sigma(a) \subseteq \sigma_\epsilon(a)$. Suppose $\lambda \notin \sigma(a)$, then $\lambda - a$ is invertible and $\lambda - a - b$ is not invertible. Hence

$$\|b\| = \|(\lambda - a - b) - (\lambda - a)\| \geq \frac{1}{\|(\lambda - a)^{-1}\|}.$$

That is,

$$\frac{1}{\|(\lambda - a)^{-1}\|} \leq \|b\| \leq \epsilon \|\lambda - a\|.$$

Thus

$$\frac{1}{\epsilon} \leq \|\lambda - a\| \left\|(\lambda - a)^{-1}\right\|.$$

Hence $\lambda \in \sigma_\epsilon(a)$. □

Lemma 2.17. *Suppose A is a complex Banach algebra with the following property:*

$$\forall a \in Inv(A), \exists b \in Sing(A) \text{ such that } \|a - b\| = \frac{1}{\|a^{-1}\|}.$$

Then for every $a \in A$ such that a is not a scalar multiple of the identity and $\lambda \in \sigma_\epsilon(a)$, there exists an element $b \in Sing(A)$ such that

$$\|b\| \leq \epsilon \|\lambda - a\| \text{ and } \lambda \in \sigma(a+b).$$

Proof. If $\lambda \in \sigma(a)$, we can take $b = 0$.

Suppose $\lambda \in \sigma_\epsilon(a) \setminus \sigma(a)$. Then $\lambda - a \in Inv(A)$. Hence, by assumption, there exists an element $c \in Sing(A)$ such that

$$\|\lambda - a - c\| = \frac{1}{\|(\lambda - a)^{-1}\|}.$$

Let $b = \lambda - a - c$. Then

$$\|b\| = \frac{1}{\|(\lambda - a)^{-1}\|} \leq \epsilon \|\lambda - a\|.$$

Also $c = \lambda - a - b \in Sing(A)$, that is $\lambda \in \sigma(a+b)$. □

Example 2.18. The Banach algebra $\mathcal{C}(X)$, X, a compact Hausdorff space, satisfies the above property. Suppose $f \in \mathcal{C}(X)$ is invertible. Then there exists an $x_0 \in X$, such that

$$0 < |f(x_0)| \leq |f(x)|, \quad \forall x \in X.$$

Then $|f(x_0)| = \dfrac{1}{\|f^{-1}\|}$. Consider $g(x) = f(x_0)$ for all $x \in X$. Then

$$\|f - (f - g)\| = \|g\| = |f(x_0)| = \frac{1}{\|f^{-1}\|}.$$

Also $(f - g)(x_0) = 0$, hence $f - g$ is not invertible.

Example 2.19. Let $A = B(\mathcal{H})$ and $T \in Inv(A)$. Suppose there exists $x, y \in \mathcal{H}$ such that $\|x\| = 1 = \|y\|$ and

$$\left\|T^{-1}\right\| = \left|\langle T^{-1}x, y\rangle\right| = \langle T^{-1}x, y\rangle.$$

In particular, $T^{-1}x \neq 0$. (This will always happen if $\mathcal{H}$ is finite dimensional). Define $P : \mathcal{H} \to \mathcal{H}$ by

$$P(u) = \frac{1}{\|T^{-1}\|}\langle u, y\rangle x.$$

Then $\|P\| = \dfrac{1}{\|T^{-1}\|}$. Also

$$(T - P)(T^{-1}x) = x - \frac{1}{\|T^{-1}\|}\langle T^{-1}x, y\rangle x = 0.$$

Hence $T - P$ is not invertible. This shows that

$$d(T, Sing(A)) \leq \|P\| = \frac{1}{\|T^{-1}\|}$$

and this proves $d(T, Sing(A)) = \dfrac{1}{\|T^{-1}\|}$. Thus, in particular, if $\mathcal{H}$ is finite dimensional, then $B(H)$ satisfies the hypothesis of Theorem 2.17.

Example 2.20. The matrix algebra $\mathbb{C}^{n\times n}$ has this property.

Corollary 2.21. *Let A be a complex Banach algebra satisfying the hypothesis of Lemma 2.17 and $a \in A$ such that a is not a scalar multiple of the identity. Then $\lambda \in \sigma_\epsilon(a) \Leftrightarrow \exists b \in A$ with $\|b\| \leq \epsilon \|\lambda - a\|$ such that $\lambda \in \sigma(a + b)$.*

Proof. Proof follows from Theorem 2.16 and Lemma 2.17. □

3. Geometric properties

In this section. we prove some results that give a better geometric picture of the ϵ-condition spectrum and show that the ϵ-condition spectrum has several properties that are different from those of the usual spectrum. These results imply that except for a very special case, when a is a scalar multiple of 1, the ϵ-condition spectrum of a has no isolated points and it is strictly bigger than the usual spectrum.

Theorem 3.1. *Let A be a complex unital Banach algebra and $a \in A$ such that $a \neq \lambda$ for every $\lambda \in \mathbb{C}$. Then $\sigma_\epsilon(a)$ has no isolated points.*

Proof. Suppose $\lambda_0 \in \sigma_\epsilon(a)$ is an isolated point of $\sigma_\epsilon(a)$. Then there exists $r > 0$ such that for all λ with $0 < |\lambda - \lambda_0| < r$,

$$\left\|(\lambda - a)^{-1}\right\| \|\lambda - a\| < \epsilon^{-1}.$$

Case 1 : Suppose $\lambda_0 \in \sigma_\epsilon(a) \setminus \sigma(a)$. By the Hahn Banach Theorem there exists $\phi, \psi \in A'$(the dual space of A) such that

$$\begin{aligned} \phi(\lambda_0 - a) &= \|\lambda_0 - a\|, & & \|\phi\| = 1, \\ \psi((\lambda_0 - a)^{-1}) &= \left\|(\lambda_0 - a)^{-1}\right\| \quad \text{and} & & \|\psi\| = 1. \end{aligned}$$

Define $f : \mathbb{C} \setminus \sigma(a) \to \mathbb{C}$ by

$$f(z) = \phi(z - a)\psi((z - a)^{-1}).$$

Then f is analytic in $D(\lambda_0, r)$ Also, for all $\lambda \in D(\lambda_0, r)$ with $\lambda \neq \lambda_0$,

$$|f(\lambda)| \leq \left\|(\lambda - a)^{-1}\right\| \|\lambda - a\| < \epsilon^{-1}$$

and

$$\begin{aligned} f(\lambda_0) &= \phi(\lambda_0 - a)\psi((\lambda_0 - a)^{-1}) \\ &= \left\|(\lambda_0 - a)^{-1}\right\| \|\lambda_0 - a\| \\ &\geq \epsilon^{-1}. \end{aligned}$$

This contradicts the maximum modulus principle.
Case 2 : Suppose $\lambda_0 \in \sigma(a)$

Since $\lambda_0 \neq a$, $\|\lambda_0 - a\| > 0$. Since $\|\lambda - a\| \to \|\lambda_0 - a\|$ as $\lambda \to \lambda_0$, there exists $0 < \delta_1$ such that

$$|\lambda - \lambda_0| < \delta_1 \Rightarrow \|\lambda - a\| > \frac{\|\lambda_0 - a\|}{2}.$$

Also $\left\|(\lambda - a)^{-1}\right\| \to \infty$ as $\lambda \to \lambda_0$ (see [3]).

Hence there exists $0 < \delta_2 < r$ such that

$$|\lambda - \lambda_0| < \delta_2 \Rightarrow \left\|(\lambda - a)^{-1}\right\| > \frac{2}{\epsilon \|\lambda_0 - a\|}.$$

Consider λ satisfying

$$0 < |\lambda - \lambda_0| < min\{\delta_1, \delta_2, r\}$$

Then for this λ, we have $\left\|(\lambda - a)^{-1}\right\| \|\lambda - a\| > \dfrac{\|\lambda_0 - a\|}{2} \dfrac{2}{\epsilon \|\lambda_0 - a\|} > \epsilon^{-1}$, a contradiction again. □

Corollary 3.2. *If $\sigma_\epsilon(a) = \{\lambda_0\}$ for some $\lambda_0 \in \mathbb{C}$, then $a = \lambda_0$.*

Proof. By Theorem 3.1, $a = \lambda$ for some $\lambda \in \mathbb{C}$. Since $\emptyset \neq \sigma(a) \subseteq \sigma_\epsilon(a) = \{\lambda_0\}$ implies $\sigma(a) = \{\lambda_0\}$, it is easy to see that $\lambda = \lambda_0$. □

Remark 3.3. A very well known classical problem in operator theory, known as the "$T = I$?" problem, asks the following question. Let T be an operator on a Banach space. Suppose $\sigma(T) = \{1\}$. Under what additional conditions can we conclude $T = I$? A survey article [12] contains details of many classical results about this problem. From the previous corollary it follows that if $\sigma_\epsilon(T) = \{1\}$ then $T = I$. In other words if we replace the spectrum by the ϵ-condition spectrum in the "$T = I$" problem, then no additional conditions are required.

Corollary 3.4. *If a is not a scalar multiple of the identity, then for each $\lambda_0 \in \sigma(a)$, there exists $r > 0$ such that $D(\lambda_0, r) \subseteq \sigma_\epsilon(a)$. In particular, $\sigma_\epsilon(a)$ contains properly an open neighbourhood of $\sigma(a)$.*

Proof. Suppose for every $r > 0$, $D(\lambda_0, r) \nsubseteq \sigma_\epsilon(a)$ then there exists $\lambda_n \to \lambda_0$ such that $\lambda_n \notin \sigma_\epsilon(a)$. Since $m = \inf_{\lambda\in\mathbb{C}} \|a - \lambda\| > 0$ and $\lambda_n \notin \sigma_\epsilon(a)$, it follows that $\left\|(a - \lambda_n)^{-1}\right\| \leq \frac{1}{m\epsilon}$. On the other hand, since $\lambda_0 \in \sigma(a)$, we must have $\left\|(a - \lambda_n)^{-1}\right\| \to \infty$. A contradiction.

The last conclusion follows by taking λ_0 in the boundary of $\sigma(a)$. □

Corollary 3.5. *If for some $a \in A$, $\sigma_\epsilon(a) = \sigma(a)$ then $a = \lambda_0$ for some $\lambda_0 \in \mathbb{C}$.*

Proof. This follows from Corollary 3.4. □

The next theorem proves a special property of the ϵ-condition spectrum which is not true for the usual spectrum.

Theorem 3.6. *Let $a \in A$ and $0 < \epsilon < 1$. Then $\sigma_\epsilon(a)$ has a finite number of components and every component of $\sigma_\epsilon(a)$ contains an element from $\sigma(a)$.*

Proof. First we prove that $\sigma(a)$ is covered by a finite number of components of $\sigma_\epsilon(a)$. This is trivial if $\sigma(a)$ is a singleton set. Otherwise, by corollary 3.4, for each $\lambda \in \sigma(a)$ there exists $r_\lambda > 0$ such that $D(\lambda, r_\lambda) \subseteq \sigma_\epsilon(a)$. Hence $\{D(\lambda, r_\lambda) : \lambda \in \sigma(a)\}$ is an open cover for $\sigma(a)$. Since $\sigma(a)$ is compact,

$$\sigma(a) \subseteq \cup_{i=1}^{n} D(\lambda_i, r_i).$$

Since each $D(\lambda_i, r_i)$ is connected, it must be contained in some component C_i of $\sigma_\epsilon(a)$. Thus we get closed components $C_1, \ldots, C_m$ of $\sigma_\epsilon(a)$ such that

$$\sigma(a) \subseteq \cup_{i=1}^{m} C_i.$$

Claim: $\sigma_\epsilon(a) = \cup_{i=1}^{m} C_i$.

Suppose $z_0 \in \sigma_\epsilon(a) \setminus \cup_{i=1}^{n} C_i$; then $\kappa(z_0 - a) \geq \frac{1}{\epsilon}$. Let $r > \dfrac{1+\epsilon}{1-\epsilon} \|a\|$; then by part (1) of Theorem 2.9, $\sigma(a) \subseteq \sigma_\epsilon(a) \subseteq D(0, r)$. Let $S := D(0, r) \setminus \cup_{i=1}^{n} C_i$. Clearly S is an open set and $z_0 \in S$. Let S_0 be the component of S containing z_0. Note that S_0 is also an open set. Define $f \colon S_0 \subseteq \rho(a) \to \mathbb{R}$ as $f(z) = \kappa(z - a)$. By the Hahn-Banach Theorem, there exist ϕ and ψ such that

$$\phi(z_0 - a) = \|z_0 - a\|, \qquad \|\phi\| = 1,$$
$$\psi\big((z_0 - a)^{-1}\big) = \big\|(z_0 - a)^{-1}\big\|, \qquad \|\psi\| = 1$$

and define $g : S \to \mathbb{C}$ by

$$g(z) = \phi(z-a)\psi((z-a)^{-1}) \quad z \in S.$$

Observe that g is a holomorphic function on S_0 and

$$|g(z)| = \big|\phi(z-a)\psi\big((z-a)^{-1}\big)\big| \le \|z-a\| \, \big\|(z-a)^{-1}\big\| = f(z)$$

for all $z \in S_0 \subseteq \rho(a)$. Moreover, for all $z \in \partial D(0,r), f(z) < \frac{1}{\epsilon}$ and for all $z \in \cup_{i=1}^m \partial C_i, f(z) = \frac{1}{\epsilon}$ and hence

$$|g(z)| \le \frac{1}{\epsilon}, \quad z \in \partial S_0 \tag{2}$$

as

$$\partial S_0 \subseteq \partial D(0,r) \bigcup \cup_{i=1}^n \partial C_i.$$

We know S_0 is an open connected set and g is a holomorphic function on S_0 and satisfying inequality in (2) and z_0 is an interior point with $g(z_0) = f(z_0) \ge \frac{1}{\epsilon}$. Hence g must be constant by the maximum modulus principle so that for all $z \in S_0$,

$$f(z) \ge |g(z)| = g(z_0) \ge \frac{1}{\epsilon}.$$

Hence $S_0 \subseteq \sigma_\epsilon(a)$. Also by the continuity of g we get

$$|g(z)| \ge \frac{1}{\epsilon}, \quad \forall z \in \bar{S}_0.$$

If $\partial S_0 \cap \partial D(0,r) \ne \emptyset$ then this leads to a contradiction as

$$|g(z)| \le f(z) < \frac{1}{\epsilon}, \quad \forall z \in \partial D(0,r).$$

If $\partial S_0 \cap \partial C_i \ne \emptyset$ for some C_i then $S_0 \cup C_i$ is a connected subset of $\sigma_\epsilon(a)$. Hence $S_0 \cup C_i = C_i$ as C_i is a component. But then $S_0 \subseteq C_i$ a contradiction. □

Corollary 3.7. *If $M \in \mathbb{C}^{n\times n}$ and $\sigma_\epsilon(M)$ has n components, then M is diagonalizable.*

Proof. By Theorem 3.6 each of the n distinct components contains an eigenvalue. Thus M has n distinct eigenvalues. □

Note that for the matrix M in Example 2.12, $\sigma_\epsilon(M)$ has only one component. On the other hand for the matrix R of Example 2.11, $\sigma_\epsilon(R)$ has 2 components.

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