# ALMOST MULTIPLICATIVE FUNCTIONS ON COMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. Let $A$ be a complex commutative Banach algebra with unit 1 and $\delta > 0$. A linear map $\phi : A \to \mathbb{C}$ is said to be *$\delta$-almost multiplicative* if

$$|\phi(ab) - \phi(a)\phi(b)| \leq \delta \|a\| \|b\| \quad \text{for all} \quad a, b \in A.$$

Let $0 < \epsilon < 1$. The *$\epsilon$-condition spectrum of an element $a$* in $A$ is defined by

$$\sigma_\epsilon(a) := \left\{ \lambda \in \mathbb{C} : \|\lambda - a\| \left\| (\lambda - a)^{-1} \right\| \geq \frac{1}{\epsilon} \right\}.$$

In this note, we prove following results connecting these two notions for commutative Banach algebras.

(1) If $\phi(1) = 1$ and $\phi$ is $\delta$-almost multiplicative, then $\phi(a) \in \sigma_\delta(a)$ for all $a$ in $A$.
(2) If $\phi$ is linear and $\phi(a) \in \sigma_\epsilon(a)$ for all $a$ in $A$, then $\phi$ is $\delta$-almost multiplicative for some $\delta$.

The first result is analogous to the Gelfand theory and the last result is analogous to the classical Gleason-Kahane-Zelazko theorem.

## 1. INTRODUCTION

Let $A$ be a complex commutative Banach algebra with unit 1. The classical Gelfand theory implies that the usual spectrum of an element $a$ in $A$, denoted by $\sigma(a)$, consists of the values $\phi(a)$ where $\phi$ is a non-zero multiplicative linear functional (a character) on $A$. The set of all characters of $A$, denoted by $Car(A)$, is called the carrier space of $A$. In this note, we study a possible similar relation between the condition spectrum $\sigma_\epsilon(a)$ and almost multiplicative linear functionals. Let $Inv(A)$ and $Sing(A)$ denote respectively the set of all invertible and singular elements of $A$.

**Definition 1** (Almost multiplicative function)**.** *Let $\delta > 0$. A linear map $\phi : A \to \mathbb{C}$ is said to be $\delta$-almost multiplicative, if*

$$|\phi(ab) - \phi(a)\phi(b)| \leq \delta \|a\| \|b\| \quad \textit{for all} \quad a, b \in A.$$

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The study of almost multiplicative linear functions originated with the study of deformation theory of Banach algebras. The multiplicative functions and almost multiplicative functions on certain algebras have interesting properties and applications. There is an almost multiplicative functional near to every multiplicative functional. The investigation of the converse part leads to the study of a class of Banach algebras, known as AMNM algebras (See [2, 3, 6, 5]).

One of the other notion used to prove the main theorem of this article is *condition spectrum*. Condition spectrum is a generalization of the spectrum (similar to pseudospectrum), recently studied by the authors in [4]. Though it can be defined in a wider context, we define it here for Banach algebras.

**Definition 2** (Condition spectrum)**.** *Let $A$ be a Banach algebra. For $0 < \epsilon < 1$, the $\epsilon$-condition spectrum of an element $a$ in $A$ is defined by,*

$$\sigma_\epsilon(a) := \left\{ \lambda \in \mathbb{C} : \|\lambda - a\| \left\|(\lambda - a)^{-1}\right\| \geq \frac{1}{\epsilon} \right\}$$

*with the convention that $\|\lambda - a\| \left\|(\lambda - a)^{-1}\right\| = \infty$ when $\lambda - a$ is not invertible.*

Since the condition spectrum is a special case of the spectrum defined by Ransford [7] it shares some of the properties of the usual spectrum like, non-emptiness, compactness etc.,. On the other hand, it has some properties that are different from those of the usual spectrum, such as: having no isolated points and having a finite number of connected components.

The following two simple properties, mentioned without proof, are necessary to establish the results that follow. The proofs are given in [4].

(1) For every $a \in A$ and for every $\epsilon > 0$, $\sigma(a) \subseteq \sigma_\epsilon(a)$. The two sets coincide if and only if $a$ is a scalar multiple of the identity. Hence, to avoid trivial situations, from now on, in all following results, by $a$ we mean an element which is not a scalar multiple of the identity.
(2) If $\lambda \in \sigma_\epsilon(a)$ then $|\lambda| \leq \dfrac{1+\epsilon}{1-\epsilon} \|a\|$.

## 2. Main Results

It is known that, for every multiplicative functional $\phi$, the value of $\phi$, at any element of $A$, belongs to the spectrum of the corresponding element.

Similarly, the value of an almost multiplicative functional at an element belongs to the condition spectrum of the corresponding element.

**Theorem 3.** *Let $A$ be complex Banach algebra with unit 1 and let $\phi$ be a $\delta$-almost multiplicative linear functional on $A$ and $\phi(1) = 1$. Then $\phi(a) \in \sigma_\delta(a)$ for every element $a$ in $A$.*

*Proof.* Let $a \in A$ and $\phi(a) = \lambda$. If $\lambda - a$ is not invertible, then $\lambda \in \sigma(a) \subseteq \sigma_\delta(a)$. Thus the conclusion follows from property (1).

Next assume that $\lambda - a$ is invertible. Then

$$1 = |\phi(1)| = \left|\phi(1) - \phi(\lambda - a)\phi\Big((\lambda - a)^{-1}\Big)\right| \leq \delta \, \|\lambda - a\| \left\|(\lambda - a)^{-1}\right\|.$$

That is,

$$\|\lambda - a\| \left\|(\lambda - a)^{-1}\right\| \geq \frac{1}{\delta},$$

which implies $\lambda\big( = \phi(a)\big) \in \sigma_\delta(a)$. □

The following lemma gives a sufficient condition for a linear function to be almost multiplicative from its behaviour on the unit sphere. The main idea of the proof can be found in [3].

**Lemma 4.** *Let $A$ be a commutative Banach algebra and $\phi : A \to \mathbb{C}$ be a linear map. If*

$$|\phi(a^2) - (\phi(a))^2| \leq \delta_1 \quad \text{for all } a \in A \text{ with } \|a\| = 1$$

*then $\phi$ is $2\delta_1$-almost multiplicative.*

*Proof.* Noting that the inequality holds trivially if $a = 0$ and replacing non-zero $a$ by $a/\|a\|$, we obtain

$$|\phi(a^2) - (\phi(a))^2| \leq \delta_1 \|a\|^2, \text{ for all } a \in A.$$

Next note that for all $a, b \in A$, we have

$$4(\phi(ab) - \phi(a)\phi(b)) = \phi((a+b)^2) - (\phi(a+b))^2 - \phi((a-b)^2) + (\phi(a-b))^2$$

Hence for all $a, b \in A$ with $\|a\| = 1 = \|b\|$, we get

$$\begin{aligned} |(\phi(ab) - \phi(a)\phi(b))| &\leq \frac{1}{4}\delta_1(\|a+b\|^2 + \|a-b\|^2) \\ &\leq \frac{1}{4}\delta_1(4+4) = 2\delta_1 \end{aligned}$$

Hence for arbitrary $a, b \in A$, we get

$$|(\phi(ab) - \phi(a)\phi(b))| \leq 2\delta_1 \|a\| \|b\|.$$

□

The following theorem that connects condition spectrum and almost multiplicative linear functionals can be considered as an approximate version of the Gleason-Kahane-Zelazko Theorem. This proof is similar to the proof of Theorem 8.7 in [3].

**Theorem 5.** *Let $A$ be a complex commutative Banach algebra with unit $1$, $0 < \epsilon < 1/3$ and $\phi : A \to \mathbb{C}$ be a linear function. If $\phi(a) \in \sigma_\epsilon(a)$ for every $a$ in $A$, then $\phi$ is $\delta$-almost multiplicative, where*

$$\delta = \frac{4}{\ln(\frac{1}{\epsilon})}\left(1 + \frac{2}{(\ln 2/3)^2}\right).$$

*Proof.* Note that, since $\sigma_\epsilon(1) = \{1\}$, we have $\phi(1) = 1$. Also, it follows from the property (2), that $\phi$ is continuous and

$$\|\phi\| \leq \frac{1+\epsilon}{1-\epsilon}.$$

Next, let $a \in A$ with $\|a\| = 1$. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) := \phi(\exp(za)), \quad \forall z \in \mathbb{C}. \tag{1}$$

Then $f$ is an entire function. Also, since for all $z \in \mathbb{C}$

$$|f(z)| \leq \|\phi\| \|\exp(za)\| \leq \frac{1+\epsilon}{1-\epsilon}\exp(|z|\|a\|) \leq \frac{1+\epsilon}{1-\epsilon}\exp(|z|),$$

the function $f$ is entire and of the exponential type of order less than or equal to one.

From equation (1), by linearity and continuity of $\phi$, $f$ also has the form,

$$f(z) = \sum_{n=0}^{\infty} \frac{\phi(a^n)z^n}{n!}. \tag{2}$$

Let $\alpha_j, j = 1, 2, \ldots$ denote the zeros of $f$ arranged in such a way that

$$|\alpha_1| \leq |\alpha_2| \leq \ldots.$$

**Claim.**

$$\phi(a^2) - (\phi(a))^2 = -\sum_j \frac{1}{\alpha_j^2} \tag{3}$$

*The right hand side above becomes a finite sum if the number of zeros of $f$ is finite and reduces to $0$ if $f$ has no zero.*

By Hadamard's theorem [1], the genus of $f$ is zero or one as the order is one.

**Case 1: Genus of $\mathbf{f(z)}$ is 1.** By the Hadamard factorization theorem [1], there exists a polynomial $g$ of degree less then or equal to 1, such that,

$$f(z) = \exp\left(g(z)\right) \prod_j \left(1 - \frac{z}{\alpha_j}\right) \exp\left(\frac{z}{\alpha_j}\right), \quad \forall z \in \mathbb{C}. \tag{4}$$

Since $f(0) = \phi(1) = 1$, we have $g(0) = 0$. Hence we may assume $g(z) = \beta z$ for some $\beta \in \mathbb{C}$. Next, each term in the product can be written as:

$$\left(1 - \frac{z}{\alpha_j}\right) \exp\left(\frac{z}{\alpha_j}\right) = \left(1 - \frac{z}{\alpha_j}\right)\left(1 + \frac{z}{\alpha_j} + \frac{1}{2}\frac{z^2}{\alpha_j^2} + \dots\right) = 1 - \frac{1}{2}\frac{z^2}{\alpha_j^2} + \dots$$

Thus,

$$f(z) = \left(1 + \beta z + \frac{1}{2}\beta^2 z^2 + \dots\right) \prod_j \left(1 - \frac{1}{2}\frac{z^2}{\alpha_j^2} + \dots\right). \tag{5}$$

Comparing coefficients of $z$ and $z^2$ in two expressions, (1) and (5), of $f(z)$, we get

$$\phi(a) = \beta, \quad \frac{1}{2}\phi(a^2) = \frac{1}{2}\beta^2 - \frac{1}{2}\sum_j \frac{1}{\alpha_j^2}.$$

Thus,

$$\phi(a^2) - (\phi(a))^2 = -\sum_j \frac{1}{\alpha_j^2}. \tag{6}$$

**Case 2: Genus of $\mathbf{f(z)}$ is 0.** By the Hadamard factorization theorem [1], there exists a polynomial $g$ of degree zero, such that,

$$f(z) = \exp\left(g(z)\right) \prod_j \left(1 - \frac{z}{\alpha_j}\right), \quad \forall z \in \mathbb{C}. \tag{7}$$

Since $f(0) = \phi(1) = 1$, we have $g \equiv 0$. Thus,

$$f(z) = \prod_j \left(1 - \frac{z}{\alpha_j}\right). \tag{8}$$

Comparing coefficients of $z$ and $z^2$ in two expressions, (2) and (8), of $f(z)$, we get

$$\phi(a) = -\sum_j \frac{1}{\alpha_j}, \quad \frac{1}{2}\phi(a^2) = \sum_{i<j} \frac{1}{\alpha_i \alpha_j}.$$

Thus,

$$\phi(a^2) - (\phi(a))^2 = -\sum_j \frac{1}{\alpha_j^2}. \tag{9}$$

Thus we end up with the same expression as (6) and that proves the claim.

To get the bound for the right hand side of (3), we estimate $|\alpha_j|$ in two ways. First, since $0 = f(\alpha_j) = \phi(\exp(\alpha_j a))$, we have $0 \in \sigma_\epsilon(\exp(\alpha_j a))$. Hence

$$\frac{1}{\epsilon} \leq \|\exp(\alpha_j a)\| \|\exp(-\alpha_j a)\| \leq \exp(2|\alpha_j| \|a\|).$$

Since $\|a\| = 1$, we obtain

$$|\alpha_j| \geq \frac{1}{2} \ln\left(\frac{1}{\epsilon}\right)$$

The second estimate is obtained from Jensen's formula [8]. For this, let $r > 0$, $n(r)$ denote the number of zeros of $f$ in the closed disc with the centre at the origin and radius $r$ and

$$M(r) := \sup\{|f(r\exp(i\theta))| : 0 \leq \theta < 2\pi\} \leq \frac{1+\epsilon}{1-\epsilon} \exp(r).$$

Then by Jensen's formula,

$$n(r)\ln(2) \leq \ln(M(2r)) \leq \ln\left(\frac{1+\epsilon}{1-\epsilon}\right) + 2r.$$

Putting $r = |\alpha_j|$ in the above inequality, we get

$$j \ln(2) \leq \ln\left(\frac{1+\epsilon}{1-\epsilon}\right) + 2|\alpha_j|$$

Suppose $\epsilon$ is such that $\ln(\frac{1+\epsilon}{1-\epsilon}) \leq \frac{1}{2}\ln(1/\epsilon)$. (This is satisfied if $0 < \epsilon < 1/3$.) Then by using $|\alpha_j| \geq \frac{1}{2}\ln(1/\epsilon)$, we get $j\ln(2) \leq 3|\alpha_j|$ that is $|\alpha_j| \geq \frac{\ln(2)}{3} j$.

Let $\gamma := \frac{1}{2}\ln(1/\epsilon)$ and $\eta := \frac{\ln(2)}{3}$. Then $|\alpha_j| \geq \gamma$ as well as $|\alpha_j| \geq \eta j$ for all $j$. Consider $k = [\gamma]$, the integral part of $\gamma$. Then $k \leq \gamma \leq k+1$. Recall,

$$\phi(a^2) - (\phi(a))^2 = -\sum_j \frac{1}{\alpha_j^2}.$$

To estimate $\left|\phi(a^2) - (\phi(a))^2\right|$, we split the right hand side sum into two parts and use the first inequality for $1 \leq j \leq k$ and the second inequality for $j > k$. Hence

$$\begin{aligned}
|\phi(a^2) - (\phi(a))^2| &\le \sum_j |\frac{1}{\alpha_j^2}| \\
&\le \sum_{j=1}^{k} |\frac{1}{\alpha_j^2}| + \sum_{j=k+1}^{\infty} |\frac{1}{\alpha_j^2}| \\
&\le \frac{k}{\gamma^2} + \frac{1}{\eta^2} \sum_{j=k+1}^{\infty} \frac{1}{j^2} \\
&\le \frac{1}{\gamma} + \frac{1}{\eta^2} \left( \frac{1}{(k+1)^2} + \int_{k+1}^{\infty} \frac{1}{x^2} dx \right) \\
&\le \frac{1}{\gamma} + \frac{1}{\eta^2} \left( \frac{1}{(k+1)^2} + \frac{1}{k+1} \right) \\
&\le \frac{1}{\gamma} + \frac{1}{\eta^2} \left( \frac{1}{\gamma^2} + \frac{1}{\gamma} \right) \\
&\le \frac{1}{\gamma} \left( 1 + \frac{2}{\eta^2} \right)
\end{aligned}$$

We have proved that

$$|\phi(a^2) - (\phi(a))^2| \le \delta_1 \quad \text{for all } a \in A \text{ with } \|a\| = 1.$$

where $\delta_1 := \frac{1}{\gamma}\left(1 + \frac{2}{\eta^2}\right)$. Thus the conclusion follows from Lemma 4 with

$$\delta := 2\delta_1 = \frac{2}{\gamma}\left(1 + \frac{2}{\eta^2}\right) = \frac{4}{\ln(\frac{1}{\epsilon})}\left(1 + \frac{2}{(\ln 2/3)^2}\right).$$

□

Using Theorem 5, we can deduce the the classical Gleason-Kahane-Zelazko Theorem for commutative Banach algebras.

**Corollary 6** (GKZ Theorem). *Let $A$ be a complex commutative unital Banach algebra and $\phi : A \to \mathbb{C}$ be a linear function. If $\phi(a) \in \sigma(a)$ for every $a$ in $A$, then $\phi$ is multiplicative.*

*Proof.* Since $\sigma(a) \subseteq \sigma_\epsilon(a)$ for every $0 < \epsilon < 1$ (by property (1)),

$$\phi(a) \in \sigma_\epsilon(a), \qquad \forall\, a \in A, \quad 0 < \epsilon < 1/3.$$

Applying Theorem 5, we get that $\phi$ is $\delta$-almost multiplicative. Note that $\delta \to 0^+$ as $\epsilon \to 0^+$. Hence $\phi$ is multiplicative. □

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## REFERENCES


1. Lars V. Ahlfors, *Complex analysis*, third ed., McGraw-Hill Book Co., New York, 1978, An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics. MR MR510197 (80c:30001)
2. Krzysztof Jarosz, *Almost multiplicative functionals*, Studia Math. **124** (1997), no. 1, 37–58. MR MR1444808 (98d:46051)
3. B. E. Johnson, *Approximately multiplicative functionals*, J. London Math. Soc. (2) **34** (1986), no. 3, 489–510. MR MR864452 (87k:46105)
4. S. H. Kulkarni and D. Sukumar, *The condition spectrum*, Acta Sci. Math. (Szeged) **74** (2008), no. 3-4, 625–641. MR MR2487937
5. Huaxin Lin, *Almost multiplicative morphisms and some applications*, J. Operator Theory **37** (1997), no. 1, 121–154. MR MR1438204 (98b:46091)
6. Chun-Gil Park, *On an approximate automorphism on a $C^*$-algebra*, Proc. Amer. Math. Soc. **132** (2004), no. 6, 1739–1745 (electronic). MR MR2051135 (2005a:46129)
7. T. J. Ransford, *Generalised spectra and analytic multivalued functions*, J. London Math. Soc. (2) **29** (1984), no. 2, 306–322. MR MR744102 (85f:46091)
8. Walter Rudin, *Functional analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991. MR MR1157815 (92k:46001)



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