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BM5163 Bayesian Inference in Bioengineering

Problem Set 4

Instructions

1. You are expected to work on these problems on your own and not submit the solutions.

Questions

1. Suppose X_i's are i.i.d

$$X_i \sim \text{Bernoulli}(p).$$

Assume the prior

$$f(p) = p^{\alpha-1}\,(1-p)^{\beta-1}$$

   (a) Write the likelihood function for the data.
   (b) Obtain posterior.
   (c) Does posterior belong to the same family as prior?
   (d) Derive the posterior predictive probability that the next observation equals 1.

2. A hospital reports the number of new infections detected per day over a period of four days to be 4, 2, 5, 3. The hospital administration wants to estimate the average infection rate per day.
   (a) Propose a probability model for the counts.
   (b) Choose a conjugate prior for the model parameter.
   (c) Derive the posterior distribution.
   (d) Compute the posterior predictive distribution for tomorrow's count.
   (e) Explain how the result would change if the prior implied a strong belief in low infection rates.

3. A biomedical sensor measures blood oxygen level. Four readings are recorded 96, 97, 95, 98. Assume measurement noise is approximately Gaussian.
   (a) Propose a Bayesian model for the data and identify a conjugate prior for the unknown parameter(s).
   (b) Derive the posterior distribution.

4. Patients entering an ICU are classified into three categories- mild, moderate, and severe. In one week, the counts for each category are 10, 6, and 4, respectively.
   (a) Propose a probabilistic model for the category probabilities.
   (b) Choose a conjugate prior distribution.
   (c) Derive the posterior distribution.
   (d) Compute the posterior predictive probability that the next patient is severe.
   (e) Discuss how the inference changes if prior knowledge suggests severe cases are rare.

5. Suppose $X \sim \text{Binomial}(n, p)$.
   (a) Derive the Fisher information for p.
   (b) Derive the Jeffreys prior for p.
   (c) Show that the prior is a Beta distribution and identify its parameters.
   (d) Suppose $n = 10$, and $x = 7$. Compute the posterior distribution.

(e) Compare the posterior mean under Jeffreys and uniform priors. Discuss the differences when n is small.

6. Suppose a random variable follows $X \sim \text{Exp}(\lambda)$.
   (a) Derive the Jeffreys prior for λ.
   (b) Let $\theta = \log \lambda$. Compute the Jeffreys prior directly in terms of θ.
   (c) Show that transforming the Jeffreys prior obtained in part (a) gives the same result.

7. Suppose you want to construct a prior distribution for a positive parameter θ representing a physiological rate (for example, a metabolic rate constant). You know very little about θ, except that its expected value is known to be m, that is, $\mathbb{E}(\theta) = m$. Since you do not have any other information, you want to choose a prior distribution that represents maximum uncertainty subject to this constraint. The uncertainty of a continuous probability distribution can be quantified using Shannon entropy

$$H(p) = -\int_0^\infty p(\theta) \log p(\theta) d\theta$$

- Check if a distribution with larger entropy represents less prior information?

Now, you want to find the density $p(\theta)$ that maximizes entropy $H(p)$ under constraints

$$\int_0^\infty p(\theta) d\theta = 1$$

$$\int_0^\infty \theta p(\theta) d\theta = m.$$

Read about the Euler-Lagrange equation, and use it to obtain the prior that maximizes the entropy. This prior is also known as *maximum entropy prior*. We use maximum entropy when you know certain expectation constraints and want the least informative distribution consistent with those constraints.