BM5163 Bayesian Inference in Bioengineering

Problem Set 2

Instructions

1. You are expected to work on these problems on your own and not submit the solutions.

Questions

1. A wearable biosensor measures blood lactate concentration μ (mmol/L). Suppose the prior distribution is given as $\mu \sim \mathcal{N}(2.0, 0.5^2)$. The sensor measurement model is given by $y|\mu \sim \mathcal{N}(\mu, 0.3^2)$. You perform one measurement and get $y = 2.4$.
   (a) Write down prior PDF.
   (b) Write down the likelihood function.
   (c) Derive the posterior PDF explicitly.
   (d) Compute posterior mean and variance.
   (e) Is the posterior variance larger or smaller than the prior variance? Explain.

2. A glucose sensor measures plasma glucose μ with known variance $\sigma^2 4$. Suppose prior distribution is given by $\mu \sim \mathcal{N}(100, 10^2)$, and following three measurements are recorded

$$y_1 = 110, y_2 = 95, y_3 = 105.$$

   Assume independent measurements.
   (a) Write likelihood function for all three measurements.
   (b) Show that the posterior is Normal.
   (c) Derive posterior mean and variance.
   (d) Express posterior mean as weighted average of prior and experimental data.
   (e) What happens as the number of measurements increases?

3. In a drug trial, blood pressure reduction μ (mmHg) is unknown. We have prior $\mu \sim \imath, \in \bigtriangledown$ and the measurement variance is $\sigma^2 = 9$. Suppose you measure BP reduction from four patients and the average value is $\bar{y} = 6$.
   (a) Write likelihood for sample mean.
   (b) Derive posterior distribution.
   (c) Compute posterior mean and variance.
   (d) Compare with a single observation case.
   (e) Interpret effect of sample size.

4. A biomedical sensor is used to measure the true concentration μ of a blood biomarker. However, the sensor suffers from an unknown calibration bias b. Therefore, the measurement model is $y = \mu + b + \epsilon$ where $\epsilon \sim \mathcal{N}(0, \sigma^2)$, $b \sim \mathcal{N}(0, \tau^2)$, and prior distribution is $\mu \sim \mathcal{N}(\mu_0, \sigma_0^2)$. You observe a single measurement y.
   (a) Write down the likelihood function $f(y|\mu, b)$.
   (b) Derive the marginal likelihood $f(y|\mu)$
   (c) Derive the posterior distribution
   (d) Compare the posterior variance with the case where calibration bias does not exist, i.e. $\tau^2 = 0$.
   (e) What is the physical interpretation of the effect of τ on inference?

(f) Extend this to multiple independent measurements.

5. A researcher studies a biomarker concentration in multiple patients. For ith patient, $y_i|\mu_i \sim \mathcal{N}(\mu_i, \sigma^2)$ where σ^2 (measurement variance) is known. However, true patient-specific biomarker levels μ_i vary across the population, and we have $\mu_i|\mu \sim \mathcal{N}(\mu, \tau^2)$ where μ is the population mean biomarker level and τ^2 is the biological variability between patients, and both are unknown. We now place a hyperprior on μ as $\mu \sim \mathcal{N}(\mu_0, \sigma_0^2)$ where μ_0 and σ_0^2 are known. You observe data from n patients. Analyze.