# BM4040 Mechanobiology

**Note: This text is only as a guide and may be incomplete and contain errors. If you find any error, please do let me know by email.**

# Navier Stokes Equations

## 1 Derivation of Navier-Stokes equations

In previous classes we have derived for incompressible flows

$$\nabla \cdot \underline{v} = 0 \tag{1}$$
$$\rho \underline{a} = \rho \underline{g} + \nabla \cdot \underline{\underline{\sigma}} \tag{2}$$
$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^{\mathsf{T}}. \tag{3}$$

There are two points which need to be appreciated,

1. First, in the derivation of the above relations we have not utilized any fluid-specific property. In other words, these equations are equally applicable to all the fluids. But our experience tells us that somewhere we also needs to account for the exact nature of the fluid, that is, we have to incorporate the information whether the fluid under consdieration is, for example, water or honey or blood or oil etc.
2. These equations show that we have $9$ unknowns ($6$ stress and $3$ velocity components) whereas we have only $4$ equations ($1$ continuity equations and $3$ linear momentum balance equation). This shows that it may not be possible to solve for the unknowns uniquely with these equations alone. We require something more.

Both of these issues are resolved by the *constitutive relations*.

### Constitutive equations

Constitutive equations are the emperically derived relations which provide information about the stress and deformation/flow of the material specific material. So, the constitutive equation of one material (say water) is different from another fluids (say honey). The most popular constitutive description for the fluids is the Newtonian fluid for which we have

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \eta \left( \nabla \underline{v} + (\nabla \underline{v})^{\mathsf{T}} \right) \tag{4}$$

where $p$ is the fluid pressure and $\eta$ is its viscosity.
We will come to this topic again in a later class.
Substitution of the Newtonian fluid constitutive relation to the linear momentum balance equation in the absence of any body force ($\underline{g} = 0$) gives

$$\rho \left( \frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\, \underline{v} \right) = -\nabla p + \eta \nabla^2 \underline{v} \tag{5}$$

This equation along with the mass balance equation are known as the *Navier-Stokes equations*.

## 2 Navier-Stokes equations in different coordinate systems

### Cartesian coordinates

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \quad \text{Continuity equation} \tag{6}$$
$$\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \eta \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) \tag{7}$$
$$\rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \eta \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) \tag{8}$$
$$\rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \eta \left( \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) \tag{9}$$

### Cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0 \quad \text{Continuity equation} \tag{10}$$

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\eta\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right) \tag{11}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\theta v_r}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\eta\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right) \tag{12}$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\eta\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right) \tag{13}$$

## 3 Non-dimensionalization

The purpose of the non-dimensionalization of any equation (say N-S in present conetxt) is two fold. First, appropriate choice of the characteristic quantities results in the reduction of independet system paramters, and second, it also provides an idea about the relative size of the different terms in the equation. The knowledge of the relative size/importance of different terms can help in reducing the equation to a simplified form. We will see both of these advantages in the following.

### Reynolds number

Navier-Stokes equation in the absence of any body forces is given by

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+(\nabla\underline{v})\,\underline{v}\right)=-\nabla p+\eta\nabla^2\underline{v} \tag{14}$$

We can use following characteristic quantities to non-dimensionalize the N-S equation

- Length: $L$
- Velocity: $U$
- Time: $L/U$
- Pressure: $\eta U/L$ (for viscous flows) or $\rho U^2$ (for inertial flows), these result in
- $\frac{\partial}{\partial t}=\frac{U}{L}\frac{\partial}{\partial \tilde{t}}$
- $\nabla=\frac{1}{L}\tilde{\nabla}$

as

$$\begin{aligned}
\rho\left(\frac{U^2}{L}\frac{\partial \tilde{\underline{v}}}{\partial \tilde{t}}+\frac{U^2}{L}\left(\tilde{\nabla}\tilde{\underline{v}}\right)\tilde{\underline{v}}\right)&=-\frac{1}{L}\frac{\eta U}{L}\tilde{\nabla}\tilde{p}+\eta\frac{U}{L^2}\tilde{\nabla}^2\tilde{\underline{v}}\\
\Rightarrow\frac{\rho UL}{\eta}\left(\frac{\partial \tilde{\underline{v}}}{\partial \tilde{t}}+\left(\tilde{\nabla}\tilde{\underline{v}}\right)\tilde{\underline{v}}\right)&=-\tilde{\nabla}\tilde{p}+\tilde{\nabla}^2\tilde{\underline{v}}\\
\Rightarrow Re\left(\frac{\partial \tilde{\underline{v}}}{\partial \tilde{t}}+\left(\tilde{\nabla}\tilde{\underline{v}}\right)\tilde{\underline{v}}\right)&=-\tilde{\nabla}\tilde{p}+\tilde{\nabla}^2\tilde{\underline{v}}
\end{aligned} \tag{15}$$

or

$$\begin{aligned}
\rho\left(\frac{U^2}{L}\frac{\partial \tilde{\underline{v}}}{\partial \tilde{t}}+\frac{U^2}{L}\left(\tilde{\nabla}\tilde{\underline{v}}\right)\tilde{\underline{v}}\right)&=-\frac{\rho U^2}{L}\tilde{\nabla}\tilde{p}+\eta\frac{U}{L^2}\tilde{\nabla}^2\tilde{\underline{v}}\\
\Rightarrow\frac{\rho UL}{\eta}\left(\frac{\partial \tilde{\underline{v}}}{\partial \tilde{t}}+\left(\tilde{\nabla}\tilde{\underline{v}}\right)\tilde{\underline{v}}\right)&=-\frac{\rho UL}{\eta}\tilde{\nabla}\tilde{p}+\tilde{\nabla}^2\tilde{\underline{v}}\\
\Rightarrow\frac{\partial \tilde{\underline{v}}}{\partial \tilde{t}}+\left(\tilde{\nabla}\tilde{\underline{v}}\right)\tilde{\underline{v}}&=-\tilde{\nabla}\tilde{p}+\frac{1}{Re}\tilde{\nabla}^2\tilde{\underline{v}}
\end{aligned} \tag{16}$$

where

$$Re=\frac{\rho UL}{\eta} \tag{17}$$

is the Reynolds number expressing the ratio of the inertial and viscous forces. It can be seen that the two system parameters $\rho$ and $\eta$ in the dimensional N-S equation can be replaced by single parameter *Re* in its non-dimensional form.

How does this help?: This reduction of the system parameters implies that for a thorough understanding of a particualr system we do not have to study it for all parameter values. If we study it for a wide range of *Re* it will provide us with the whole picture which can be utilized to predict the system response for different parameter values. As an example, We can study the N-S equation for any particular setup using analytical or numerical techniques for a range of *Re* and the results we will obtain from such an exercise can be utilized to study the flow of different fluids, say water or honey or any other Newtonian fluid as long as the *Re* is same.

Moreover, it can also be seen that for large values of *Re* (as shown by equation (16)) the N-S equation can be simplified to

$$\frac{\partial \tilde{\underline{v}}}{\partial \tilde{t}} + \left(\tilde{\nabla} \tilde{\underline{v}}\right) \tilde{\underline{v}} = -\tilde{\nabla} \tilde{p} \tag{18}$$

and, as a result, the flow does not depend on the fluid viscosity. This type of flow is known as the *Euler flow* and is a widely studied model for fluid flows at high velocities.

On the other hand, for small values of *Re* the N-S equation is simplified to (see equation (15)) the following *Stoke's flow* equation

$$\tilde{\nabla} \tilde{p} = \tilde{\nabla}^2 \tilde{\underline{v}} \tag{19}$$

which is a linear equation in velocity and pressure.

# 4 Some example flows

## Plane Couette flow

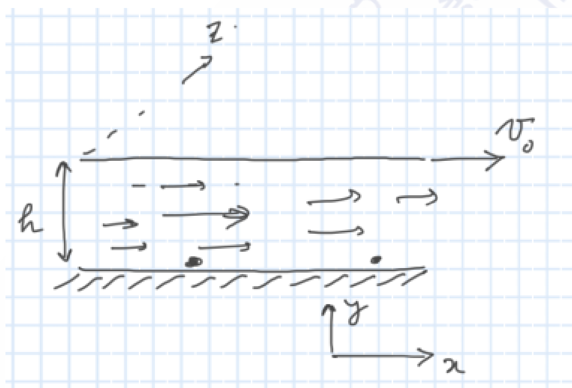


Figure 1:

It is a steady ($\frac{\partial \underline{v}}{\partial t} = 0$), unidirectional ($v_y = v_z = 0$) flow of an incompressible ($\nabla \cdot \underline{v}$) viscous ($\eta \neq 0$) fluid between two plates in the absence of any body force ($\underline{g} = 0$) and pressure gradient ($\nabla p = 0$) where one of the two plates is fixed and other plate is moving with a velocity $v_0$. The linear momentum balance in the $x$-direction results in

$$\begin{aligned}
\rho \left( \cancel{\frac{\partial v_x}{\partial t}} + \cancel{v_x \frac{\partial v_x}{\partial x}} + \cancel{v_y \frac{\partial v_x}{\partial y}} + \cancel{v_z \frac{\partial v_x}{\partial z}} \right) &= -\cancel{\frac{\partial p}{\partial x}} + \eta \left( \cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}} \right) \\
\Rightarrow \frac{\partial^2 v_x}{\partial y^2} &= 0 \\
\Rightarrow v_x &= c_2 y + c_1 \qquad (20)
\end{aligned}$$

Applying boundary conditions at $y = 0$ and $y = h$ gives

$$v_x = v_0 \frac{y}{h}. \tag{21}$$

## Flow between rotating cylinders

As the name suggests, this is an example of an incompressible fluid flow between two concentric cylinders of radii $a$ and $b$ ($a < b$) due to the rotation of the two cylinders. From the geometric setup of the problem it becomes apparent that it will be convenient to study this example in cylinderical coordinate system. We assume

- Incompressible flow $\nabla \cdot \underline{v} = 0$,
- steady flow,
- $v_z = 0$,
- Axisymmetric flow $\dfrac{\partial \underline{v}}{\partial \theta} = 0$,
- $\dfrac{\partial \underline{v}}{\partial z} = 0$,
- $\underline{v}(r = a) = a\omega_a \hat{e}_\theta$ and $\underline{v}(r = b) = b\omega_b \hat{e}_\theta$.

From incompressibility condition

$$
\begin{aligned}
\nabla \cdot \underline{v} &= 0 \\
\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}v_\theta + \frac{\partial}{\partial z}v_z &= 0 \\
\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rv_r) &= 0 \\
\Rightarrow \frac{\partial}{\partial r}(rv_r) &= 0 \\
\Rightarrow v_r &= \frac{c_1}{r}
\end{aligned}
\tag{22}
$$

Applying the boundary conditions give $c_1 = 0$, implying $v_r = 0$. From momentum balance equation, we get

$$
\begin{aligned}
\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r\frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_\theta v_r}{r}} + \cancel{v_z\frac{\partial v_\theta}{\partial z}}\right) &= \cancel{-\frac{1}{r}\frac{\partial p}{\partial \theta}} + \eta\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}\right) \\
\Rightarrow \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) &= 0 \\
\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) &= c_1 \\
\Rightarrow \frac{\partial}{\partial r}(rv_\theta) &= c_1 r \\
\Rightarrow rv_\theta &= c_1\frac{r^2}{2} + c_2 \\
\Rightarrow v_\theta &= c_1\frac{r}{2} + c_2\frac{1}{r}
\end{aligned}
\tag{23}
$$

and

$$
\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + \cancel{v_r\frac{\partial v_r}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} - \frac{v_\theta^2}{r} + \cancel{v_z\frac{\partial v_r}{\partial z}}\right) = -\frac{\partial p}{\partial r} + \eta\left(\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}}\right)
$$

$$
\Rightarrow \frac{\partial p}{\partial r} = \frac{v_\theta^2}{r} \tag{24}
$$

(25)

and

$$
\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r\frac{\partial v_z}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + \cancel{v_z\frac{\partial v_z}{\partial z}}\right) = \cancel{-\frac{\partial p}{\partial z}} + \eta\left(\cancel{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right) \tag{26}
$$

$\Rightarrow$ This equation is automatically satisfied.

We can obtain the values of constants $c_1$ and $c_2$ by applying the boundary conditions at $r = a$ and $r = b$ as

$$
c_1\frac{a}{2} + c_2\frac{1}{a} = \omega_a a \tag{27}
$$

$$
c_1\frac{b}{2} + c_2\frac{1}{b} = \omega_b b. \tag{28}
$$

This gives

$$c_1 = \frac{2\left(b^2\omega_b - a^2\omega_a\right)}{b^2 - a^2} \tag{29}$$

$$c_2 = \frac{\left(\omega_a - \omega_b\right)a^2b^2}{b^2 - a^2}. \tag{30}$$

Shear stress is given by

$$\begin{aligned}
\sigma_{r\theta} &= \eta\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}\right) \\
\Rightarrow \sigma_{r\theta} &= \eta\left(r\frac{\partial}{\partial r}\left(\frac{c_1}{2} + \frac{c_2}{r^2}\right)\right) \\
\Rightarrow \sigma_{r\theta} &= \eta\left(r\left(-2\frac{c_2}{r^3}\right)\right) \\
\Rightarrow \sigma_{r\theta} &= -2\eta\frac{c_2}{r^2} = -\frac{2\eta}{r^2}\left(\frac{\left(\omega_a - \omega_b\right)a^2b^2}{b^2 - a^2}\right)
\end{aligned} \tag{31}$$

Viscometer application: This analysis is used in the cylinderical viscometer for estimation of the viscoisty of any fluid. In a cylinderical viscometer the inner cylinder is rotated with a constant angular veolcity while keeping the outer cylinder fixed. The fluid is filled in the space between the two cylinders and the torque on the inner cylinder is measured. For this case, the torque is given by (setting $\omega_b = 0$)

$$T_a = \int_0^h \int_0^{2\pi} a\sigma_{r\theta} a d\theta dh = -\left(\frac{4\pi h\eta\omega_a a^2b^2}{b^2 - a^2}\right). \tag{32}$$

This gives the fluid viscosity as

$$\boxed{\eta = \frac{|T_a|}{\omega_a}\left(\frac{b^2 - a^2}{4\pi h a^2 b^2}\right)}. \tag{33}$$

## Flow in a cylindrical channel

This is an example of pressue driven incompressible flow in a cylindrical channel of radius $R$. It is assumed that flow is steady and fully developed. For simplicity it is assumed that the velocity is only along the channel axis. Similar to previous case, it is apparent that it is more convenient to consider this example in cylindrical coordinates. The assumptions are

- Incompressible flow $\nabla \cdot \underline{v} = 0$,
- steady ($\frac{\partial \underline{v}}{\partial t} = \underline{0}$) and fully developed ($\frac{\partial \underline{v}}{\partial z} = \underline{0}$) flow,
- $v_z \neq 0$, $v_r = v_\theta = 0$
- Axisymmetric flow $\frac{\partial \underline{v}}{\partial \theta} = \underline{v}$,
- no body forces.

It can be seen that with these assumptions the incompressibility condition is already satisfied.

The momentum balance equations give

$$\begin{aligned}
\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r\frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_\theta v_r}{r}} + \cancel{v_z\frac{\partial v_\theta}{\partial z}}\right) &= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \eta\left(\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}\right) \\
\Rightarrow \frac{\partial p}{\partial \theta} &= 0
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + \cancel{v_r\frac{\partial v_r}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} - \cancel{\frac{v_\theta^2}{r}} + \cancel{v_z\frac{\partial v_r}{\partial z}}\right) &= -\frac{\partial p}{\partial r} + \eta\left(\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}}\right) \\
\Rightarrow \frac{\partial p}{\partial r} &= 0
\end{aligned} \tag{35}$$

and

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}}+\cancel{v_r\frac{\partial v_z}{\partial r}}+\cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}+\cancel{v_z\frac{\partial v_z}{\partial z}}\right)=-\frac{\partial p}{\partial z}+\eta\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}+\cancel{\frac{\partial^2 v_z}{\partial z^2}}\right) \tag{36}$$

$$\Rightarrow -\frac{\partial p}{\partial z}+\eta\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right)=0$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=\frac{r}{\eta}\frac{\partial p}{\partial z}$$

$$\Rightarrow r\frac{\partial v_z}{\partial r}=\frac{r^2}{2\eta}\left(\frac{\partial p}{\partial z}\right)+c_1$$

$$\Rightarrow \frac{\partial v_z}{\partial r}=\frac{r}{2\eta}\left(\frac{\partial p}{\partial z}\right)+\frac{c_1}{r}$$

$$\Rightarrow v_z=\frac{r^2}{4\eta}\left(\frac{\partial p}{\partial z}\right)+c_1\ln r+c_2 \tag{37}$$

In order to keep $v_z$ finite at the center ($r=0$), we must have $c_1=0$. For $c_2$, we apply the no-slip boundary condition at the channel wall ($r=R$) to get

$$\frac{R^2}{4\eta}\left(\frac{\partial p}{\partial z}\right)+c_2=0\Rightarrow c_2=-\frac{R^2}{4\eta}\left(\frac{\partial p}{\partial z}\right). \tag{38}$$

This gives

$$v_z=-\frac{1}{4\eta}\left(\frac{\partial p}{\partial z}\right)\left(R^2-r^2\right). \tag{39}$$

This gives the total flow rate through the channel as

$$\begin{aligned}
Q&=\int_0^R\int_0^{2\pi} v_z r\,dr\,d\theta\\
&=-\frac{1}{4\eta}\left(\frac{\partial p}{\partial z}\right)\int_0^R\int_0^{2\pi}\left(rR^2-r^3\right)dr\,d\theta\\
&=-\frac{2\pi}{4\eta}\left(\frac{\partial p}{\partial z}\right)\int_0^R\left(rR^2-r^3\right)dr\\
&=-\frac{\pi}{2\eta}\left(\frac{\partial p}{\partial z}\right)\left(\frac{R^4}{2}-\frac{R^4}{4}\right)\\
&=-\frac{\pi}{8\eta}\left(\frac{\partial p}{\partial z}\right)R^4.
\end{aligned} \tag{40}$$

This shows that if the channel radius is doubled the total flow rate will grow by 16 times. Shear stress at the wall

$$\begin{aligned}
\sigma_{rz}&=\eta\left(\cancel{\frac{\partial v_r}{\partial z}}+\frac{\partial v_z}{\partial r}\right)\bigg|_{r=R}\\
&=\frac{R}{2}\left(\frac{\partial p}{\partial z}\right).
\end{aligned} \tag{41}$$

It is interesting that the shear stress at the wall does not depend on $\eta$.