BM4040 Mechanobiology

Note: This text is only as a guide and may be incomplete and contain errors. If you find any error, please do let me know by email.

Balance Laws

1 Balance Laws

Balance of Mass

Consider a fixed volume in space V in the flow. If we assume that there is no production of fluid inside this region, then we can see that the rate of change of the mass of fluid inside this region should be equal to the rate of inflow/outflow of mass through its ounding surface S. In other words

$$\frac{\partial}{\partial t}\int\limits_V \rho dV = \int\limits_S \left(-\rho \underline{v}\cdot\underline{n}\right) dS. \tag{1}$$

Since, we have fixed the volume V in space, it does not change and we can write the LHS as

$$\frac{\partial}{\partial t}\int\limits_V \rho dV = \int\limits_V \frac{\partial \rho}{\partial t} dV \tag{2}$$

For the RHS of the equation we are going to use the divergence theorem to obtain

$$\int\limits_S \left(-\rho \underline{v}\cdot\underline{n}\right) dS = \int\limits_S \nabla\cdot\left(-\rho\underline{v}\right) dV. \tag{3}$$

This gives,

$$\begin{aligned}
\int\limits_V \frac{\partial \rho}{\partial t} dV &= \int\limits_S \nabla\cdot\left(-\rho\underline{v}\right) dV \\
\Rightarrow \int\limits_V \frac{\partial \rho}{\partial t} dV + \int\limits_S \nabla\cdot\left(\rho\underline{v}\right) dV &= 0 \\
\Rightarrow \int\limits_V \left(\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\underline{v}\right)\right) dV &= 0
\end{aligned} \tag{4}$$

or

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\underline{v}\right) = 0} \tag{5}$$

If the fluid is incompressible, that is it density is constant, then conservation of mass leads to

$$\boxed{\nabla\cdot\underline{v} = 0} \tag{6}$$

Balance of Linear Momentum

We will derive the balance of linear momentum by application of Newton's second law of $\underline{F} = m\underline{a}$ to a small volume of fluid. Consider a small differential cuboidal element of fluid with side lengths Δx, Δy and Δz. This volume of fluid can experience two types of forces, *body forces*-which act on each point of the fluid, and *surface forces*-which act only through the the surface. We consider that the body force perunit mass is $\underline{g}$. The surface forces are described in terms of the stress tensor $\underline{\underline{\sigma}}$. The figure below shows the surface forces acting on the fces of the cubioidal fluid element.

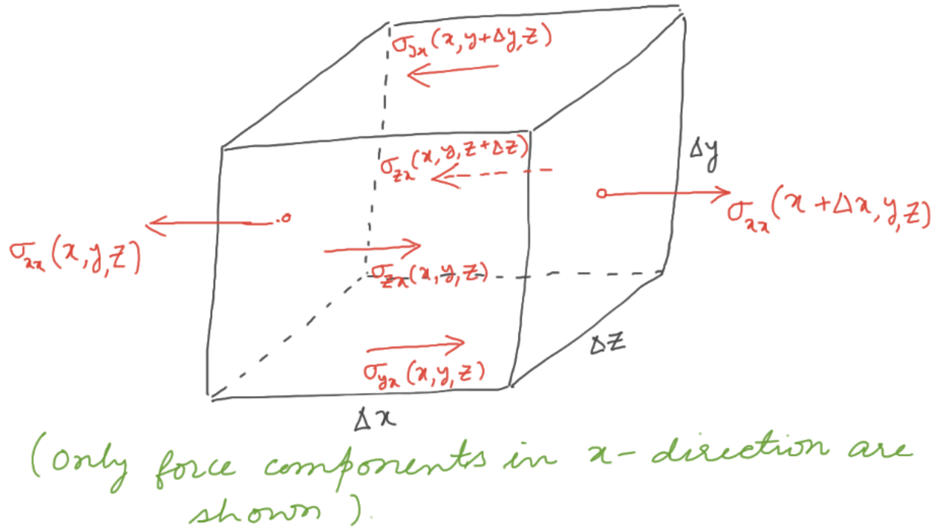

Figure 1:

Applying Newton's second law in the x-drection gives

$$\begin{aligned}
\rho \Delta x \Delta y \Delta z a_x = \rho \Delta x \Delta y \Delta z g_x &+ \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\Delta x - \sigma_{xx}\right)\Delta y \Delta z \\
&+ \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\Delta y - \sigma_{yx}\right)\Delta x \Delta z \\
&+ \left(\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}\Delta z - \sigma_{zx}\right)\Delta x \Delta y
\end{aligned} \tag{7}$$

or

$$\begin{aligned}
\rho \Delta x \Delta y \Delta z a_x &= \rho \Delta x \Delta y \Delta z g_x + \left(\frac{\partial \sigma_{xx}}{\partial x}\Delta x\right)\Delta y \Delta z + \left(\frac{\partial \sigma_{yx}}{\partial y}\Delta y\right)\Delta x \Delta z + \left(\frac{\partial \sigma_{zx}}{\partial z}\Delta z\right)\Delta x \Delta y \\
\Rightarrow (\rho a_x)\,\Delta x \Delta y \Delta z &= \left(\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right)\Delta x \Delta y \Delta z \\
\Rightarrow \rho a_x &= \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}
\end{aligned} \tag{8}$$

Similarly, we can also do the same thing for y- and z- directions to obtain

$$\rho a_x = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \tag{9}$$

$$\rho a_y = \rho g_y + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} \tag{10}$$

$$\rho a_z = \rho g_z + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \tag{11}$$

In direct notation this can be written as

$$\boxed{\rho \underline{a} = \rho \underline{g} + \nabla \cdot \underline{\underline{\sigma}}} \tag{12}$$

Balance of Angular Momentum

In the absence of any microscopic force couples (as in magnetic fluids), the balance of angular moment results in $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^{\mathsf{T}}$.

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