BM2053 Mathematical Models & Systems Biology

Problem Set 1

Instructions

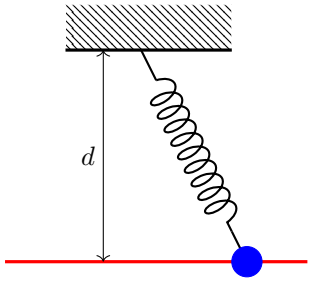
1. You are not supposed to submit the solutions to these questions.
2. To get the most out of these, solve all the problems on your own.

Questions

1. Solve the following systems of equations (show all the steps) and sketch the curves for x, or x and y as a function of time.
   (a) $\frac{dx}{dt} = -x + x^2$ with initial conditions $x(0) = -1/2$.
   (b) $\frac{dx}{dt} = -x + 2y$ and $\frac{dy}{dt} = x - y$ with initial conditions $x(0) = 1$ and $y(0) = 1/2$.
   (c) $\frac{dx}{dt} = -x - y$ and $\frac{dy}{dt} = 2x - y$ with initial conditions $x(0) = 0$ and $y(0) = 1$.
2. For each of the following systems, find the fixed points and sketch the nullclines and the vector field
   (a) $\dot{x} = -1 + x + y$, $\dot{y} = \alpha e^{-x}$
   (b) $\dot{x} = x - x^3$, $\dot{y} = -y$
   (c) $\dot{x} = x(x - y)$, $\dot{y} = y(2x - y)$
   (d) $\dot{x} = x^2 - y$, $\dot{y} = x - y$
3. For the following systems of equations identify the fixed points and the nature of their stability.
   (a)
   $$\ddot{\epsilon} = -\frac{\epsilon}{1 - \left(\frac{\epsilon}{\epsilon_0}\right)^2}$$
   where $\epsilon_0 > 0$.
   (b)
   $$\dot{x} = -\frac{xy}{1+x} + \frac{y}{1+y}$$
   $$\dot{y} = -\frac{x(1-y)}{1+x}$$
   (c)
   $$\dot{a} = -a + \frac{\alpha b^2}{1+b^2}$$
   $$\dot{b} = -b + \frac{\alpha a^2}{1+a^2}$$
   where $\alpha > 0$.

4. Sketch the phase portraits for the equations/systems of equations in Question-1 and 2.

5. Consider the following contraption where one end of a spring (undeformed length l_0, spring constant k) is connected to a fixed wall, and the other end is connected to a bead of mass m (shown in blue color). The bead is constrained to move along the red color line.



   (a) Write down the equation of motion for the bead.
   (b) Identify the fixed points in the system for $l_0 > d$ and $l_0 < d$.
   (c) Draw the phase diagram for this system.
   (d) Identify the nature of the stability of the fixed points.
   (e) Plot the location of the stable point(s) (with a solid line) and unstable fixed point(s) (with a dashed line) as a function of spring length l_0.

6. Consider an ecosystem of rabbits and sheep. Suppose that both species survive on the same food supply (grass) and the amount of grass available is limited. If we ignore all other factors and focus on only two effects

   - Each species would grow to its maximum limit in the absence of the other. This is also known as logistic growth.
   - , However, when the two species come in contact with each other, the sheep nudges the rabbit aside and does not let it eat the grass. We assume that this conflict occurs at a rate proportional to the size of each population. Such conflict would reduce the rate of growth of both, but the effect is more severe for the rabbits.

   By combining all of these effects, we can write down the equations describing the change in the number of both species

$$\dot{r} = r(3 - r - 2s)$$
$$\dot{s} = s(2 - r - s)$$

   where r and s represent the number of rabbits and sheep, respectively.

   - Identify the fixed points of the system.
   - Sketch the phase diagram.
   - What is the nature of the stability of the FPs?

7. Write down the equation of motion of a simple pendulum and analyze it using the phase diagram method.