Name: _____

Roll No.: _____

BM2053 Mathematical Models & Systems Biology

Exam 1 Solution sketch

Duration: 45mins **Total marks: 40**

1. Write the answers very clearly and within the answer sheets provided. **No additional sheet of paper will be accepted.**
2. Do not use a pencil to write answers or sketch the curves.

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1. To engineer a stable microbial cell factory for bio-based chemical production, synthetic biologists have designed a system. The dynamics of that system can be described by the following equations

$$\frac{dE}{dt} = V_0 - k_d E\left(1 + \frac{P}{P_0}\right)$$
$$\frac{dP}{dt} = \frac{k_{cat} EP}{K_m + P} - k_e P$$

where $E(t)$ and $P(t)$ are the concentrations of enzyme and protein product.

(a) How many free parameters are there in this system? **(10)**

Choose characteristic quantities $t_c = 1/k_d$, $P_c = P_0$, and $E_c = V_0/k_d$ to get

$$\dot{e} = 1 - e\,(1 + p) \tag{1}$$
$$\dot{p} = \Lambda \frac{ep}{\alpha + p} - \beta p = p\left(\Lambda \frac{e}{\alpha + p} - \beta\right) \tag{2}$$

This gives three free parameters Λ, α and β.

(b) How many fixed points does this system have? Identify them. **(10)**

This has one trivial fixed point $(e^*, p^*) = (1, 0)$. The equation for the other fixed point gives

$$p^2 + (\alpha + 1)p + \alpha = \frac{\Lambda}{\beta} \tag{3}$$

which has only one positive fixed point only if $(\alpha + 1)^2 > 4(\alpha - \Lambda/\beta)$. Therefore, this system has one or two relevant fixed points.

2. Consider the following equation of motion of a pendulum

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta).$$

where g is the gravitational acceleration and l is the length of the pendulum. Identify the fixed points for this system and their stability. You can use the Jacobian or phase plane method to answer this question. **(5 + 15)**

We can convert this system into two first order equations by defining $\dot{\theta} = \omega$ to get

$$\dot{\theta} = \omega \tag{4}$$
$$\dot{\omega} = -\frac{g}{l}\sin\theta \tag{5}$$

This gives the fixed points to be $(\theta^*, \omega^*) = (n\pi, 0)$ for $n \in \mathbb{N}$. Calculating Jacobian gives

$$J = \begin{bmatrix} 0 & 1 \\ -(g/l)\cos\theta & 0 \end{bmatrix} \tag{6}$$

with eigenvalues $\lambda = -(g/l)\cos(n\pi)$ which are positive (unstable) and negative (stable) for odd and even values of n.