BM2000 Control System

E. & O. E. Report of any error by email will be appreciated

First and second order systems

As mentioned previously, in the control system design, the two main characteristics of any system are

1. **Stability**: which we have looked at already and now, hopefully, understand the relationship between the system transfer function and its stability.
2. **Performance**: which has many metrics for evaluation and we are going to look at them now. Most of the performance characteristics are defined for first and second order systems and, therefore, now we will look at first and second order systems in some detail. It can be argued that the performance (at least the leading order dynamics) of any system (even higher order system) can also be described by the metrics defined for the first and second order systems.

1 First order system

The equation describing a first order system is given by

$$T\dot{y}(t) + y(t) = bx(t) \tag{1}$$

for which the transfer function is

$$G(s) = \frac{b}{Ts+1} \tag{2}$$

with poles $s = -1/T$. Therefore, the first order systems is stable if $T > 0$. Now we will look at this system's

1. *unit impulse response*: for which $X(s) = 1$ and we obtain

$$Y(s) = G(s) \Rightarrow y(t) = \frac{b}{T}e^{-t/T}. \tag{3}$$

2. *unit step response*: for which we have $X(s) = 1/s$ and we get

$$Y(s) = \frac{b}{s(Ts+1)} = \frac{b}{s} - \frac{bT}{sT+1} \Rightarrow y(t) = b\left(1 - e^{-t/T}\right). \tag{4}$$

For this we get $y(t \to \infty) = b$ for $x(t) = 1$. Therefore, b is also called the *steady-state gain* of the system.

Further, the system response at different time points can be summarized as

Time	System response	Approximate value
0	0	$= 0.00b$
T	$b(1-e^{-1})$	$\approx 0.63b$
$2T$	$b(1-e^{-2})$	$\approx 0.87b$
$3T$	$b(1-e^{-3})$	$\approx 0.95b$
$4T$	$b(1-e^{-4})$	$\approx 0.98b$
$5T$	$b(1-e^{-5})$	$\approx 0.99b$

Table 1: Response of a first order system at different time points.

which shows that the system reaches 98% of its final value in $t = 4T$ time. This time is another performance metric which is also known as *settling time*. One can easily see that the settling time increases with T.

3. *unit ramp response*: For this we have $x(t) = t$ or $X(s) = 1/s^2$ which gives us

$$Y(s) = \frac{b}{s^2(sT+1)} = \frac{b}{s^2} - \frac{bT}{s} + \frac{bT^2}{sT+1} \Rightarrow y(t) = b\left(t - T(1 - e^{-t/T})\right). \tag{5}$$

   This shows that the unit ramp response of a first order system increases indefinitely. This shows that the system output is unbounded and one may consider that this is a sign of an unstable system (in the sense of BIBO stability). This, however, is not correct. Recall that for BIBO stability the system input also has to be bounded which is not the case here.

Another interesting feature we can observe from these three system responses is that if we take a time derivative of ramp input we obtain a unit step and if we take a time derivative of the unit step we obtain unit impulse. Similarly, if we take time derivative of the ramp response we get step response and taking a time derivative of step response gives us impulse response. In other words, if $\tilde{y}(t)$ is the system response for input $\tilde{x}(t)$ then for system input $\dot{\tilde{x}}(t)$ the system output is $\dot{\tilde{y}}(t)$. This, however, is true only for the *linear time invariant* systems and does not hold for systems which are nonlinear or time dependent.

From the first order system we learnt about two system performance characteristics *steady state gain* and *settling time*. Based on these parameters one can compare the performance of two stable systems. Usually, the system with smaller settling time is considered to be better.

2 Second order system

A second order system (*not the most general one though*) can be described by

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = \omega_n^2 x(t) \tag{6}$$

where ω_n and ζ are called the *natural frequency* and *damping ratio*, respectively. Following the usual steps we obtain the transfer function for this to be

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{7}$$

which is also called the *standard form* of the second order system. It can be shown that this second order system is stable if $\omega_n > 0$ and $\zeta > 0$ (*This is a question in assignment 2*). We can write down the poles of this system as

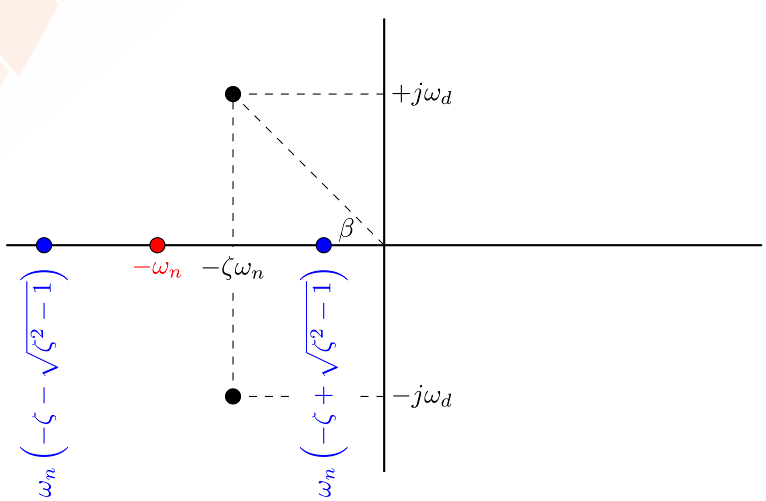
$$s = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right). \tag{8}$$

We can observe that this system shows three distinct behaviors

1. $0 < \zeta < 1$ or *Underdamped case*: In this case we get the poles to be

$$s == -\omega_n\zeta \pm j\omega_d \tag{9}$$

   where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is called the *damped natural frequency*. From the figure below one can also see that $\cos\beta = \zeta$.



2. $\zeta = 1$ or *Critically damped case*: In this case the poles are $-\omega_n$ and $-\omega_n$.

3. $\zeta > 1$ or *Overdamped case*: In this case the system has two distinct real poles at $\omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$ with $\omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right)$ being the dominant frequency (see the figure above).

Now let's focus on the underdamped case look at its unit step response. For this we have

$$\begin{aligned} Y(s) &= \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \\ &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \\ \Rightarrow y(t) &= 1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t). \end{aligned} \tag{10}$$

Before going into these performance characteristics it will be good to write down the system response in critically damped and overdamped case also. For the critically damped case, we get the unit step response to be

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t). \tag{11}$$

similarly, for the overdamped case we get

$$y(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{s_1 t}}{s_1} - \frac{e^{s_2 t}}{s_2}\right) \tag{12}$$

with s_1 and s_2 are the two poles of the system. Here the system response is comprised of two decaying exponential terms. From the expression of system response for underdamped case, we can define several performance characteristics which are summarized in the following figure and described below.

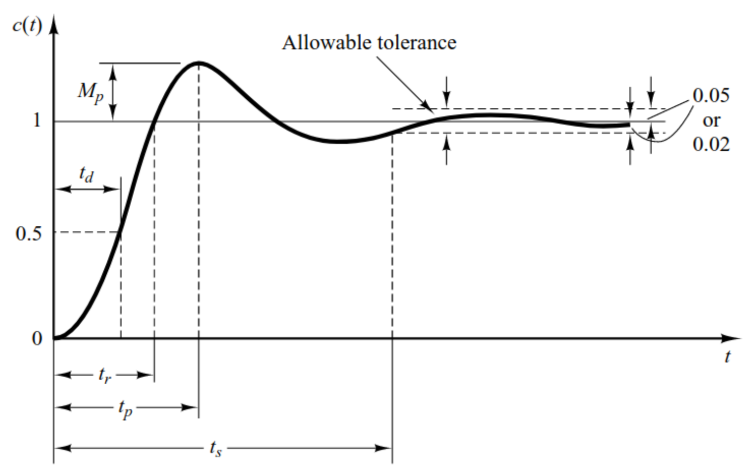


Figure 1: Unit step response of second order system showing different performance characteristics. This figure is taken from *Modern Control Engineering* by *Ogata*.

1. **Delay time t_d**: the time required for the system response to reach half of the steady state value the very first time.

2. **Rise time t_r**: For underdamped second-order systems, it is the time for the system to reach 100% of its steady state value for the first time. (*Note: For overdamped systems the 10% to 90% rise time is commonly used*). To obtain the

rise time we can solve $y(t) = 1$ which gives

$$\begin{aligned}
1 - e^{-\zeta\omega_n t_r}\cos(\omega_d t_r) - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t_r) &= 1 \\
\Rightarrow e^{-\zeta\omega_n t_r}\left(\cos(\omega_d t_r) - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r)\right) &= 0 \\
\Rightarrow \cos(\omega_d t_r) - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r) &= 0 \\
\Rightarrow \tan(\omega_d t_r) = -\frac{\sqrt{1-\zeta^2}}{\zeta} &= \frac{\omega_d}{\zeta\omega_n} \\
\Rightarrow t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right) &= \frac{\pi - \beta}{\omega_d}. \qquad (13)
\end{aligned}$$

We see from here that for large values of ζ the rise time is high.

3. **Peak time t_p**: the time required for the system response to reach the first peak of the overshoot. In order to calculate the peak time we need to solve $\frac{dy}{dt} = 0$ which gives

$$\zeta\omega_n e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right) + e^{-\zeta\omega_n t}\left(\omega_d\sin(\omega_d t) - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos(\omega_d t)\right) = 0 \qquad (14)$$

which simplifies to

$$\left.\frac{dy}{dt}\right|_{t=t_p} = \sin(\omega_d t_p)\frac{\omega_d}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} = 0. \qquad (15)$$

This gives

$$\sin(\omega_d t_p) = 0 \Rightarrow t_p = \frac{n\pi}{\omega_d}. \qquad (16)$$

Since the peak time corresponds to the first peak, we get $t_p = \frac{\pi}{\omega_d}$. This shows that peak time t_p increases with the damping ratio ζ.

4. **Maximum overshoot, M_p**: the maximum peak value of the response curve measured from its steady state value or

$$M_p = \frac{y(t_p) - y(t\to\infty)}{y(t\to\infty)} \times 100\%. \qquad (17)$$

We can use the value of t_p calculated above to obtain the maximum overshoot to be

$$M_p = (y(t_p) - 1)\times 100\% = e^{-(\zeta\omega_n/\omega_d)\pi}\times 100\% = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}\times 100\%. \qquad (18)$$

One can see that as ζ increases M_p decreases. This shows that rise time and peak overshoot are two conflicting requirements. For this, a general *thumb rule* is followedwhere we always try to keep ζ between 0.4 and 0.8.

5. **Settling time t_s**: the time required for the system response to reach and stay within a range (usually 2% or 5%) about the steady state value. In order to calculate the settling time we will rewrite the response of an underdamped system as

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \qquad (19)$$

which shows that the oscillatory response of the system remains within the envelop defined by $1 \pm \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$, as shown in the figure below.

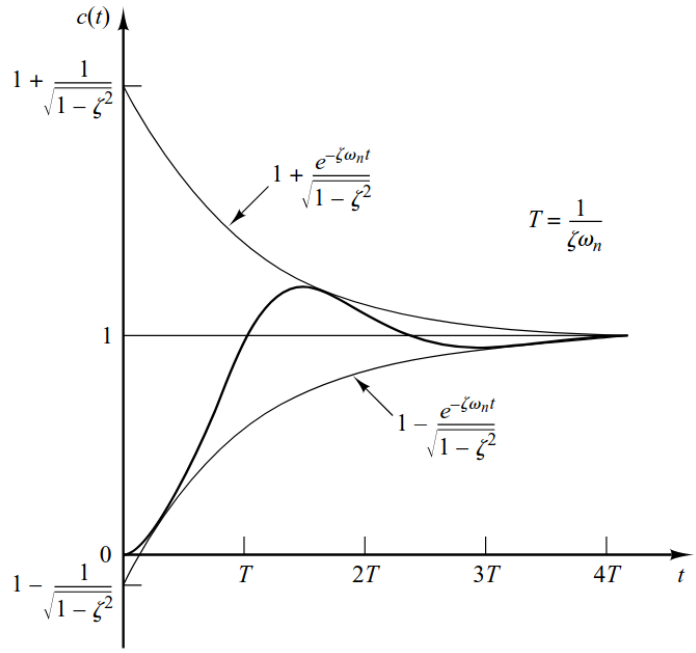


Figure 2: Envelop curves for the unit step response of an underdamped 2nd order system. This figure is taken from *Modern Control Engineering* by *Ogata*.

This means that the speed of decay of the transient response depends on the value of the time constant $T = 1/\zeta\omega_n$. Therefore the settling time can be measured in terms of this time constant, which gives

$$t_s = \begin{cases} 4T = \dfrac{4}{\zeta\omega_n}, & 2\% \text{ cutoff} \\ 3T = \dfrac{3}{\zeta\omega_n}, & 5\% \text{ cutoff} \end{cases} \tag{20}$$

In many application one may want to have as small a settling time as possible. But as you can see that for decreasing it one can increase the damping ratio ζ which can result in conflict with other performance characteristic, such as the peak time t_p.

These time-domain parameters are quite important and in control design system must be modified until the transient response is satisfactory. Next we are going to see how to use these metrics for control system design.