Improved bounds for the sunflower lemma †

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† *Improved bounds for the sunflower lemma*, Ryan Alweiss, Shachar Lovett, Kewen Wu, Jiapeng Zhang, STOC 2020.

Sunflower

Definition

A collection of sets $S_1, S_2, \ldots, S_r$ is an *r-sunflower* if

$$S_i \cap S_j = S_1 \cap S_2 \cap \cdots \cap S_r, \ \forall i \neq j.$$

$K := S_1 \cap S_2 \cap \cdots \cap S_r$ is the *kernel/core*.
$S_1 \setminus K, \ldots, S_r \setminus K$ are the *petals*.

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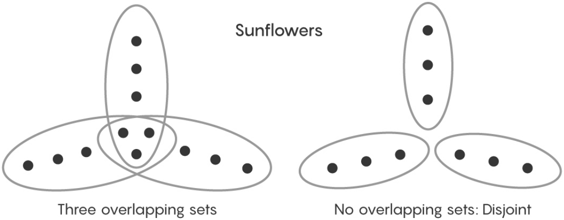

Figure: Examples of 3-sunflowers

Sunflower Lemma

w-set system : all the sets in the set system (or family) are of size at most w

Lemma (Erdos and Rado, 1960)

Let $\mathcal{F}$ be a w-set system with $|\mathcal{F}| > w!(r-1)^w$. Then, $\mathcal{F}$ contains an r-sunflower.

We know of a w-set system with $(r-1)^w$ sets that does not contain an r-sunflower.

Proof

Given: A w-set system $\mathcal{F}$ with $|\mathcal{F}| > w!(r-1)^w$.

Notation: for an element x, $\mathcal{F}_x = \{S \in \mathcal{F} \ : \ x \in S\}$.

Proof.

Proof by induction on w. True for $w = 1$.

Case 1 There are r pairwise disjoint sets in $\mathcal{F}$:

We are done.

Case 2 No. of pairwise disjoint sets is at most $r-1$:

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Union of any subcollection of $r - 1$ sets form a *hitting set* for $\mathcal{F}$. Let H denote this hitting set.

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Union of any subcollection of $r-1$ sets form a *hitting set* for $\mathcal{F}$. Let H denote this hitting set.

$|H| \leq (r-1)w$. Thus, by an averaging argument $\exists x \in H$ such that $|\mathcal{F}_x| \geq \frac{|\mathcal{F}|}{(r-1)w} > (w-1)!(r-1)^{w-1}$.

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Remove x from every set in $\mathcal{F}_x$. By induction hypothesis, $\mathcal{F}_x$ contains an r-sunflower. □

Known results

General Bound	Fixed r	Citation
$w!(r-1)^w$	$w^{w(1+o(1))}$	[Erdos, Rado, 1960]
for $r = 3$ only $\rightarrow$	$w^{w(3/4+o(1))}$	[Fukuyama, 2018]
$\mathbf{(cr^3 \log w \cdot \log\log w)^w}$,	$\mathbf{(\log w)^{w(1+o(1))}}$	**[Alweiss et al., 2020]**
$(cr\log(wr))^w$	$(\log w)^{w(1+o(1))}$	[Rao, 2020]

Table: Lower bounds for $|\mathcal{F}|$ that guarantee an r-sunflower. Here, $o(1)$ depends on r and c is a constant.

Conjecture (Sunflower Conjecture, Erdos and Rado, 1960)

For a fixed r, if $|\mathcal{F}| > c^w$, then $\mathcal{F}$ contains an r-sunflower, where $c = c(r)$.

Trivia

Q. Sunflower is named after the star ‘sun’. Name another flower that is named after a star?

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ANSWER:

Rajanikanth (Rajanigandha, Water Lilly)

Kamal (Lotus)

Revisiting the proof of sunflower lemma

Link of $\mathcal{F}$ at T

Definition

Given a family $\mathcal{F}$ and a set T, the **link of $\mathcal{F}$ at T**, denoted by $\mathcal{F}_T$, is defined as

$$\mathcal{F}_T = \{S \setminus T \ : \ S \in \mathcal{F}, T \subseteq S\}$$

Example

$\mathcal{F} = \{\{1,2,3,4\},\{1,2,3,6,7\},\{2,3\},\{7,8,9\},\{1,2,4,6,7\}\}$

$T = \{2,3\},\ \mathcal{F}_T = \{\{1,4\},\{1,6,7\},\emptyset\}$

$T = \{1,2\},\ \mathcal{F}_T = \{\{3,4\},\{3,6,7\},\{4,6,7\}\}$

Proof revisited

Given: A w-set system $\mathcal{F}$ with $|\mathcal{F}| > w!(r-1)^w$.

Proof.

Proof by induction on w. True for $w = 1$.

Case 1 There are r pairwise disjoint sets in $\mathcal{F}$:

We are done.

Case 2 No. of pairwise disjoint sets is at most $r-1$:

Any subcollection of $r-1$ sets form a *hitting set* for $\mathcal{F}$. Let H denote this hitting set.

$|H| \leq (r-1)w$. Thus, by an averaging argument $\exists x \in H$ such that $|\mathcal{F}_x| \geq \frac{|\mathcal{F}|}{(r-1)w} > (w-1)!(r-1)^{w-1}$.

Remove x from every set in $\mathcal{F}_x$. By induction hypothesis, $\mathcal{F}_x$ contains an r-sunflower. □

Proof in the language of links

Given: A w-set system $\mathcal{F}$ with $|\mathcal{F}| > w!(r-1)^w$.

Proof.

Proof by induction on w. True for $w = 1$.

Case 1 For some x, $|\mathcal{F}_x| > (w-1)!(r-1)^{w-1}$:

By induction hypothesis, $\mathcal{F}_x$ contains an r-sunflower.

Case 2 For every x, $|\mathcal{F}_x| \leq (w-1)!(r-1)^{w-1}$:

This implies no hitting set of size $(r-1)w$ for $\mathcal{F}$.

This implies there are r pairwise disjoint sets in $\mathcal{F}$ □

Generalizing the above approach

Let $w, r \in \mathbb{N}$. Let $\kappa = \kappa(r, w)$ be a monotone non-decreasing function over w for any fixed r.

Theorem

Let $\mathcal{F}$ be a w-set system with $|\mathcal{F}| > \kappa^w$. Then, $\mathcal{F}$ contains an r-sunflower.

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Let $\mathcal{F}$ be a w-set system with $|\mathcal{F}| > \kappa^w$. Then, $\mathcal{F}$ contains an r-sunflower.

Proof.

Let X be the universe, i.e., every set in $\mathcal{F}$ is a subset of X.

Proof by induction on w.

Case 1 For some $T \subseteq X$, $1 \leq |T| < w$, $|\mathcal{F}_T| > \kappa^{w-|T|}$:

By induction hypothesis, $\mathcal{F}_T$ contains an r-sunflower.

Case 2 For every $T \subseteq X$, $1 \leq |T| < w$, $|\mathcal{F}_T| \leq \kappa^{w-|T|}$:

To show: there are r pairwise disjoint sets in $\mathcal{F}$

□

κ-spread family

Bound in [Alweiss et al., 2020]: $|\mathcal{F}| > (cr^3 log\ w \cdot loglog\ w)^w$, then r-sunflower exists

Bound we show: $|\mathcal{F}| > (\mathbf{64r^4 \log^4 w})^{\mathbf{w}}$, then r-sunflower exists

Throughout the talk, let $\kappa = \kappa(w, r) = 64r^4 \log^4 w$.

Definition

A w-set system $\mathcal{F}$ is κ-**spread** if

- $|\mathcal{F}| > \kappa^w$, and
- for every set T with $|T| = t < w$, $|\mathcal{F}_T| \leq \kappa^{w-t}$.

Outline of the proof

Theorem

Let $\kappa = 64 r^4 \log^4 w$. Let $\mathcal{F}$ be a w-set system with $|\mathcal{F}| > \kappa^w$. Then, $\mathcal{F}$ contains an r-sunflower.

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Proof.

Proof by induction on w.

Case 1 $\mathcal{F}$ is not κ-spread:

follows from induction hypothesis.

Case 2 $\mathcal{F}$ is κ-spread:

To show: there are r pairwise disjoint sets in $\mathcal{F}$

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(α, β)-satisfying family

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Let $0 < \alpha, \beta < 1$. Let $W \sim \mathcal{U}(X, \alpha)$. A family $\mathcal{F}$ of subsets of X is (α, β)-**satisfying** if

$$Pr[\exists S \in \mathcal{F}, S \subseteq W] > 1 - \beta$$

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$\mathcal{F} = \{\{1, 2, 3\}, \{1, 3, 4\}, \{3, 5\}\}$.

DNF formula corresponding to $\mathcal{F}$:

$(x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_3 \wedge x_4) \vee (x_3 \wedge x_5)$

$(1/3, 1/3)$-satisfying families

Lemma

Let $\mathcal{F}$ be a family of subsets of X that is $(1/3, 1/3)$-satisfying.
Then, $\mathcal{F}$ contains 3 pairwise disjoint sets.

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For each $x \in X$, independently and uniformly at random assign a color from the set $\{$red, blue, green$\}$.

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Let E_R denote the event that $\mathcal{F}$ contains a set all whose elements got red color. Similarly, E_B, E_G.

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Since $\mathcal{F}$ is $(1/3, 1/3)$-satisfying, we have $Pr[E_R] > 2/3$. Same true for E_B, E_G.

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$$
\begin{aligned}
Pr[E_R \wedge E_B \wedge E_G] &= 1 - Pr[\overline{E_R} \vee \overline{E_B} \vee \overline{E_G}] \\
&\geq 1 - (Pr[\overline{E_R}] + Pr[\overline{E_B}] + Pr[\overline{E_G}]) \\
&> 1 - (\frac{1}{3} + \frac{1}{3} + \frac{1}{3}) = 0
\end{aligned}
$$

$(1/r, 1/r)$-satisfying families

Lemma

Let $\mathcal{F}$ be a family of subsets of X that is $(1/r, 1/r)$-satisfying. Then, $\mathcal{F}$ contains r pairwise disjoint sets.

Proof.

Same way as above. □

Outline of the proof

Theorem

Let $\kappa = 64r^4 \log^4 w$. Let $\mathcal{F}$ be a w-set system with $|\mathcal{F}| > \kappa^w$. Then, $\mathcal{F}$ contains an r-sunflower.

Proof.

Proof by induction on w.

Case 1 $\mathcal{F}$ is not κ-spread:

follows from induction hypothesis.

Case 2 $\mathcal{F}$ is κ-spread:

To show: $\mathcal{F}$ **is** $(1/r, 1/r)$**-satisfying**.

□

Proving a weaker bound

Lemma

Let $\kappa = 10wr\log r$. If $\mathcal{F}$ is κ-spread, then $\mathcal{F}$ is $(1/r, 1/r)$-satisfying.

Apply Janson's Inequality to get a weak bound similar to that in Sunflower Lemma:

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Let $W \sim \mathcal{U}(X, 1/r)$.

For each set $S_i \in \mathcal{F}$, let Z_i be the indicator RV for $S_i \subseteq W$.

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Let $W \sim \mathcal{U}(X, 1/r)$.

For each set $S_i \in \mathcal{F}$, let Z_i be the indicator RV for $S_i \subseteq W$.

Find $\mu = \sum_i E[Z_i]$ and $\Delta = \sum_{i \sim j} E[Z_i Z_j]$.

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Find $\mu = \sum_i E[Z_i]$ and $\Delta = \sum_{i \sim j} E[Z_i Z_j]$.

By Janson's Inequality,

$$Pr[\forall i,\ Z_i = 0] \leq e^{-\frac{\mu^2}{2\Delta}}.$$

Set $e^{-\frac{\mu^2}{2\Delta}} \leq 1/r$ and find an appropriate κ that satisfies it.

What is left to be proven

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Recalling the definitions...

Definition

A w-set system $\mathcal{F}$ is κ-**spread** if

- $|\mathcal{F}| > \kappa^w$, and
- for every set T with $|T| = t < w$, $|\mathcal{F}_T| \leq \kappa^{w-t}$.

Definition

Let $0 < \alpha, \beta < 1$. A family $\mathcal{F}$ of subsets of X is (α, β)-**satisfying** if

$$Pr_{W \sim \mathcal{U}(X,\alpha)}[\exists S \in \mathcal{F}, S \subseteq W] > 1 - \beta$$

Bad (W, S) pairs

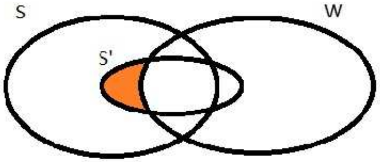
$\mathcal{F}$ is a κ-spread w-set system of subsets of X.
Let $w' < w$. Let $W \sim \mathcal{U}(X, p)$.

Definition

For an $S \in \mathcal{F}$, the pair (W, S) is **good** if there exists a set S' (could be equal to S) in $\mathcal{F}$ that satisfies:

- $S' \subseteq S \cup W$, and
- $|S' \setminus W| \leq w'$

Otherwise, (W, S) is a **bad pair**.



Pseudo-spread set systems

Let $\kappa = 64r^4 \log^4 w$ (basically, a function that is monotone non-decreasing over w for a fixed r).

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Definition

Let $w_1 \leq w,\ 0 < \delta$. A w_1-set system $\mathcal{F}$ is (κ, w, δ)-**nearly-spread** if

- $|\mathcal{F}| > (1-\delta)\kappa^w$, and
- for every set T with $|T| = t < w_1$, $|\mathcal{F}_T| \leq \kappa^{w-t}$.

A key lemma

Lemma 1

Let $w_2 < w_1 \leq w,\ 0 < \delta, \Delta$. Let $\mathcal{F}_1$ be a (κ, w, Δ)-nearly-spread w_1-set system. If every $(\kappa, w, \Delta + \delta)$-nearly-spread w_2-set system is (α_2, β_2)-satisfying, then, for any $0 < p < 1$, $\mathcal{F}_1$ is (α_1, β_1)-satisfying, where

$$\alpha_1 = p + (1 - p)\alpha_2, \quad \beta_1 = \beta_2 + \frac{(4/p)^{w_1}}{\delta(1 - \Delta)\kappa^{w_2}}$$

Proof.

Given a $W \sim \mathcal{U}(X, p)$, we construct $\mathcal{F}_2$ from $\mathcal{F}_1$ in the following way:

1. Initialize $\mathcal{F}_2 = \{\}$.
2. For each $S \in \mathcal{F}_1$:

   if (W, S) is **good**, then by definition $\exists\ S' \in \mathcal{F}_1$ with $S' \subseteq S \cup W$ such that $|S' \setminus W| \leq w_2$. Set $\mathcal{F}_2 = \mathcal{F}_2 \cup \{S' \setminus W\}$.

A key lemma contd...

The lemma follows from the following claim:

Claim 1:
$pr[\mathcal{F}_2$ is not $(\kappa, w, \Delta + \delta)$-nearly-spread w_2-set system$] \leq \frac{(4/p)^{w_1}}{\delta(1-\Delta)\kappa^{w_2}}$

□

Lemma (Lemma 1 restated)

Let $w_2 < w_1 \leq w$, $0 < \delta, \Delta$. Let $\mathcal{F}_1$ be a (κ, w, Δ)-nearly-spread w_1-set system. If every $(\kappa, w, \Delta + \delta)$-nearly-spread w_2-set system is (α_2, β_2)-satisfying, then, for any $0 < p < 1$, $\mathcal{F}_1$ is (α_1, β_1)-satisfying, where

$$\alpha_1 = p + (1 - p)\alpha_2, \quad \beta_1 = \beta_2 + \frac{(4/p)^{w_1}}{\delta(1 - \Delta)\kappa^{w_2}}$$

Proving Claim 1: counting bad pairs (W, S)

Let $|X| = n$. Assume $|W|$ is pn-sized subset of X chosen uniformly at random.

Claim 1.1: Let $B(W) = \{S \in \mathcal{F}_1 \ : \ (W, S) \text{ is bad}\}$. Then, $E_W[|B(W)|] \leq (4/p)^{w_1} \kappa^{w - w_2}$.

1. No. of choices for $W \cup S$: $\sum_{i=0}^{w_1} \binom{n}{pn+i} \leq p^{-w_1} \binom{n}{pn}$

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2. Let S' be the first set in $\mathcal{F}$ such that $S' \subseteq W \cup S$. Let $A = S \cap S'$. No. of choices of A: 2^{w_1}

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2. Let S' be the first set in $\mathcal{F}$ such that $S' \subseteq W \cup S$. Let $A = S \cap S'$. No. of choices of A: 2^{w_1}
3. Since (W, S) bad, $|A| > w_2$. Further, $|\mathcal{F}_A| \leq \kappa^{w-w_2}$. Thus, no. of choices of S given A: κ^{w-w_2}

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4. No. of choices of $S \cap W$: 2^{w_1}

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Claim 1.1: Let $B(W) = \{S \in \mathcal{F}_1 \ : \ (W, S) \text{ is bad}\}$. Then, $E_W[|B(W)|] \le (4/p)^{w_1} \kappa^{w-w_2}$.

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3. Since (W, S) bad, $|A| > w_2$. Further, $|\mathcal{F}_A| \le \kappa^{w-w_2}$. Thus, no. of choices of S given A: κ^{w-w_2}
4. No. of choices of $S \cap W$: 2^{w_1}
5. Thus, the no. of bad pairs is: $(4/p)^{w_1} \kappa^{w-w_2} \binom{n}{pn}$

Proving Claim 1: W is δ-bad

Definition

For a $\delta > 0$, we say W is δ-bad for a w_1-set system $\mathcal{F}_1$ if $|B(W)| > \delta|\mathcal{F}_1|$.

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Applying Markov's Inequality and Claim 1.1, we get

$$Pr[W \text{ is } \delta\text{-bad for} \mathcal{F}_1] \leq \frac{E_W[|B(W)|]}{\delta|\mathcal{F}_1|} \leq \frac{(4/p)^{w_1}}{\delta(1-\Delta)\kappa^{w_2}}$$

This gives Claim 1 (restated below)

Claim 1:
$pr[\mathcal{F}_2$ is not $(\kappa, w, \Delta + \delta)$-nearly-spread w_2-set system$] \leq \frac{(4/p)^{w_1}}{\delta(1-\Delta)\kappa^{w_2}}$

How Lemma 1 helps

Let $\mathcal{F}_0 := \mathcal{F}, w_0 = w,\ \Delta_0 = 0$. For $1 \le i \le \log w$,
$w_i = w/2^i,\ \gamma_i = \frac{(4/p)^{w_{i-1}}}{\kappa^{w_i}},\ \delta_i = \sqrt{\gamma_i},\ p = \frac{1}{r \log w},\ \Delta_i = \delta_1 + \cdots + \delta_i < 1/2$.

Apply Lemma 1 repeatedly for $\log w$ times...

Lemma

Let $\mathcal{F}_{i-1}$ be a $(\kappa, w, \Delta_{i-1})$-nearly-spread w_{i-1}-set system. If every $(\kappa, w, \Delta_{i-1} + \delta_i)$-nearly-spread w_i-set system is (α_i, β_i)-satisfying, then, for any $0 < p < 1$, $\mathcal{F}_{i-1}$ is $(\alpha_{i-1}, \beta_{i-1})$-satisfying, where

$$\alpha_{i-1} = p + (1-p)\alpha_i \le p + \alpha_i$$

$$\begin{aligned} \beta_{i-1} &= \beta_i + \frac{(4/p)^{w_{i-1}}}{\delta_i (1 - \Delta_{i-1}) \kappa^{w_i}} \\ &\le \beta_i + \frac{\sqrt{\gamma_i}}{(1 - \Delta_{i-1})} \end{aligned}$$

How Lemma 1 helps...

Thus,

$$\begin{aligned} \alpha_0 &\leq p \log w \\ &= 1/r \\ \beta_0 &\leq \frac{\sqrt{\gamma_1}}{(1-\Delta_0)} + \cdots + \frac{\sqrt{\gamma_i}}{(1-\Delta_{i-1})} + \cdots \\ &\leq 2 \log w \sqrt{\gamma_{\log w}} \\ &\leq 1/r. \end{aligned}$$

We thus proved..

Theorem

Let $\mathcal{F}$ be a w-set system. If $|\mathcal{F}| > (64 r^4 \log^4 w)^w$, then r-sunflower exists.

Concluding remarks

- The paper also shows construction of a w-set system of size $(\log w)^{w(1-o(1))}$, where $o(1)$ is a function of r, which is not $(1/r, 1/r)$-satisfying.

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- (Cavalar et al., 2020) Improves lower bound known for size of a monotone circuit computing an explicit n-variate monotone Boolean function from $exp(n^{1/3-o(1)})$ to $exp(n^{1/2-o(1)})$.
- (Frankston et al., 2020) uses the technique here to solve a conjecture of Talagrand in random graphs

Thank You