Ideal Lattices in Ring Learning with Errors (Ring-LWE)

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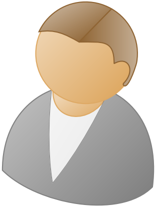
June 10, 2020

Overview

Public Key Cryptosystems

Key ingredients : A one-way function (do they exist?) and a public key K.
RSA:$K = (N, e)$

$$c = m^e \bmod N \longrightarrow$$



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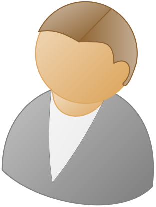

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RSA breaks when you have quantum computers!

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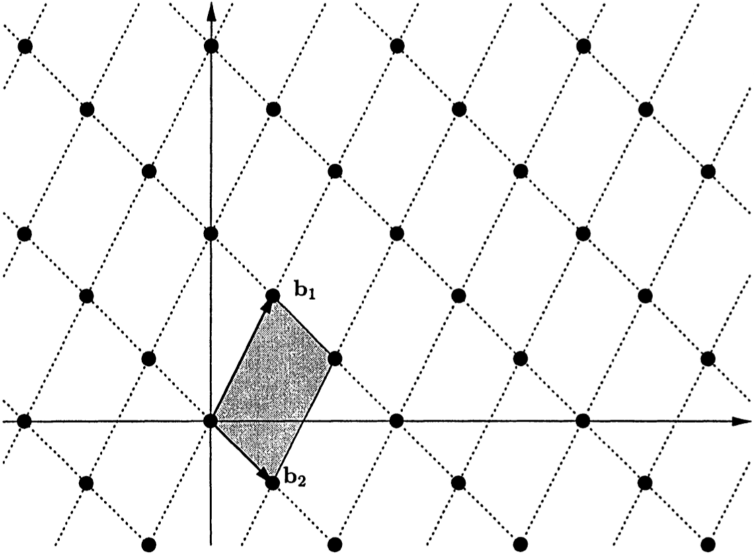
- Fully Homomorphic Encryption and many other "exotic" schemes!

Integer Lattices – Two Dimensional Example



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- The lattice can be written as,

$$\mathcal{L}(\mathbf{B}) = \{\mathbf{Bx} : \mathbf{x} \in \mathbb{Z}^n\}.$$

One Lattice, Many Bases

The basis vectors of the previous example is :

$$\mathbf{b_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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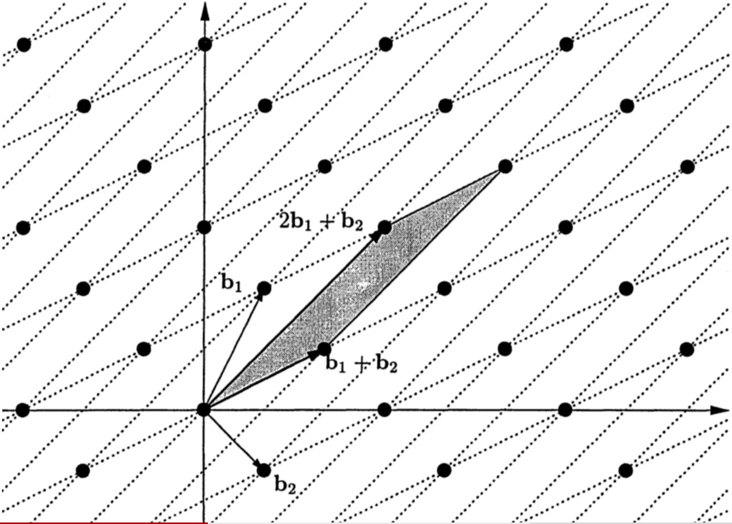
. The following vectors also generate the same lattice, $\mathcal{L}(\mathbf{b_1}, \mathbf{b_2})$

$$\mathbf{b_1}' = \mathbf{b_1} + \mathbf{b_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b_2}' = 2\mathbf{b_1} + \mathbf{b_2} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

.

One Lattice, Many Bases

The grids are different, the intersection points are the same.



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  - $\lambda_i(\Lambda)$ is the i-th successive minima defined as

$$\lambda_i(\Lambda) := \min_S(\max_{v \in S} ||v||),$$

where S runs over all l.i. sets $S \subset \Lambda$ with $|S| = i$.

Computational Lattice Problems

1. Shortest Vector Problem (SVP) : Find a shortest nonzero vector $v \in \Lambda$.
2. Shortest Independent Vector Problem (SIVP) : Find l.i. vectors $v_1, \ldots, v_n$ in Λ such that $\max_i ||v_i|| = \lambda_n(\Lambda)$.
3. Closest Vector Problem (CVP): given any target vector $w \in \mathbb{R}^n$ find the closest lattice point $v \in \Lambda$ to w.

Computational Lattice Problems

- There are approximation variants, SVP$_\gamma$, CVP$_\gamma$, SIVP$_\gamma$.
  Let $\gamma \geq 1$, SVP$_\gamma$: find a vector v with $||v|| \leq \gamma\lambda_1(\Lambda)$.

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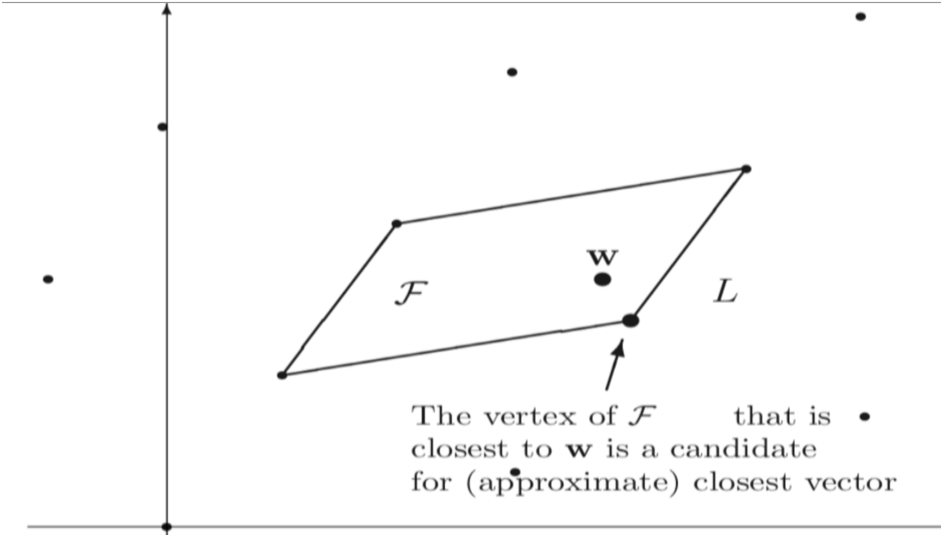
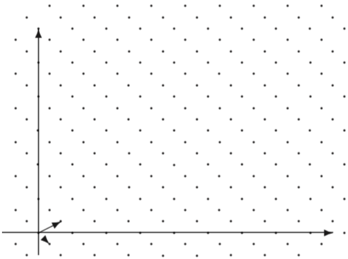
- Decision SVP : Given Λ and length d, decide if the shortest vector is shorter than d or not.
- GapSVP$_\gamma$: approximation version of the decision SVP, decide if the shortest vector is shorter than d or if it is longer than $\gamma \cdot d$.

Using basis to solve CVP



A trapdoor for lattice-based cryptosystems



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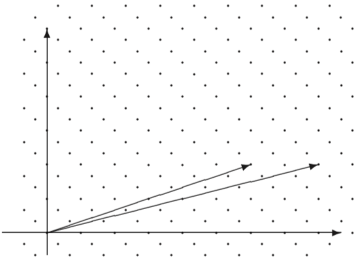

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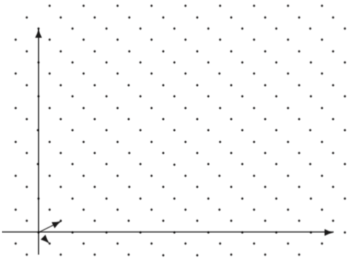

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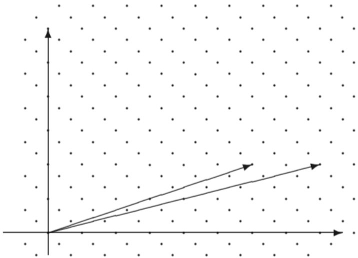

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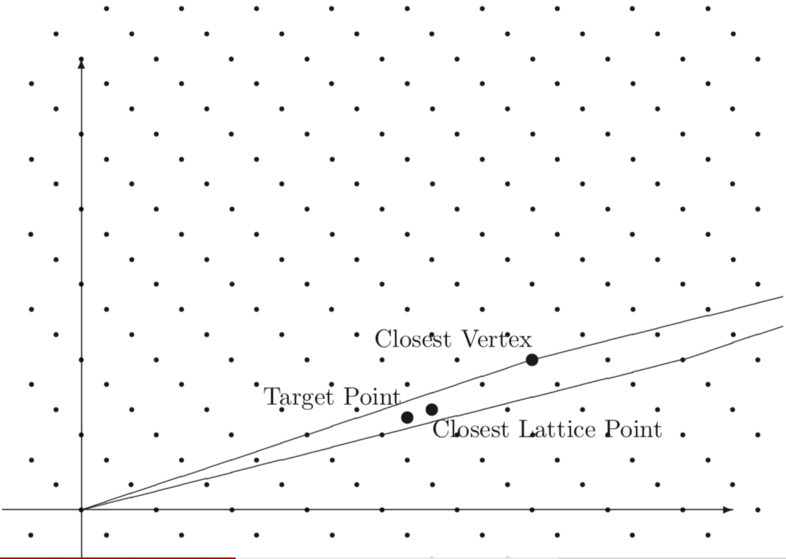
Good bases : nearly orthogonal and short

A bad basis and CVP



Algorithms for Lattice Problems

- For $n = 2$, problem is very easy!
- For higher dimensions, LLL algorithm (1982) - runs in $poly(n)$ time, but the vector returned is an exponential multiple of the actual shortest vector.

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Result

For $\gamma = poly(n)$, solving for very short vectors in high dimensions require $2^{\Omega(n)}$ time and space.

Lattice-based cryptography - Milestones

- Ajtai introduces SIS (1996) : first average case/worst case lattice problem reduction.
- Ajtai-Dwork : a PKC based on SIS
- J. Hoffstein, J. Pipher, J. H. Silverman : NTRU (1996)
- Regev (2005) : Learning with Errors problem. An efficient LWE solver implies an efficient quantum algorithm for SIVP.
- Micciancio, Lyubashevsky, (2002, 2006) : Ideal Lattices and their applications in collision resistant hash functions and digital signatures.
- Peikert, Lyubashevsky, Regev(2009,2010) : Ring-LWE
- Gentry (2009) : Fully Homomorphic Encryption

Learning With Errors [Regev '05]

- Parameters: n: dimension , q : an integer of $poly(n)$, χ : error distribution on $\mathbb{Z}$, vectors $\mathbf{a_i} \in {\mathbb{Z}_q}^n$ chosen uniformly at random.

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$$\langle \mathbf{a_1}, \mathbf{s} \rangle + e_1 = b_1 \pmod{q}$$
$$\langle \mathbf{a_2}, \mathbf{s} \rangle + e_2 = b_2 \pmod{q}$$
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In matrix notation,

$$\mathbf{As} + \mathbf{e} = \mathbf{b}.$$

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- Solving Search-LWE solves Decision-LWE. We will show that they are equivalent for q is a prime.

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- The typical choice for χ is discrete Gaussian - better security but sampling in practice is non-trivial.

Discrete Gaussian

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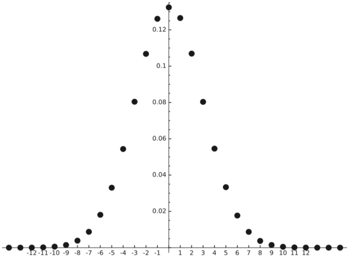

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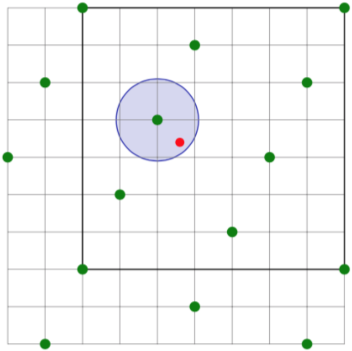
- LWE is a CVP problem on $\mathcal{L}(\mathbf{A})$: given $\mathbf{b} \approx \mathbf{v} = A\mathbf{s} \in \mathcal{L}(\mathbf{A})$, find $\mathbf{v}$.



Hardness Results of LWE [Regev'05,'09]

Theorem

Solving the LWE decision problem is at least as hard as quantumly solving SIVP$_{\gamma=poly(n)/\alpha}$ (and GapSVP$_\gamma$) on arbitrary n-dimensional lattices.

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Larger the error rate, smaller your gap!

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- Completely classical reductions under weaker parameters – (Peikert, '09).
- The result works for $q > 2\sqrt{n}$. Open question : for smaller values of q. When q is very large ($\approx 2^{2n}$) there are attacks.

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- For a pair $(\mathbf{a}, b)$ choose a fresh $k \in \mathbb{Z}_q$.
- Invoke $\mathcal{D}$ on pairs,

$$(\mathbf{a} + (l, 0, \ldots, 0), b + l \cdot k),$$

$l \in \mathbb{Z}_q$ chosen uniformly at random.

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- If $k \neq s_1$, then since q is prime b is uniform $\Rightarrow \mathcal{D}$ rejects.
- Since $q = poly(n)$ we can try all these possibilities for k.
- q need not be prime or $poly(n)$ - (Peikert '09)

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- Getting one $b_i \in \mathbb{Z}_q$ requires an n-dimensional mod q inner product.
- Typically $O(n^2)$ work.

$$\begin{pmatrix} \cdots \mathbf{a_i} \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ \mathbf{s} \\ \vdots \end{pmatrix} + e = b \in \mathbb{Z}_q$$

- Another issue – Rather large keys!

$$pk = \begin{pmatrix} \cdots \mathbf{a_i} \cdots \end{pmatrix}, \begin{pmatrix} \vdots \\ \mathbf{b} \\ \vdots \end{pmatrix}$$

Ring-Learning With Errors [Peikert, Lyubashevsky, Regev('09)]

Let $R = \mathbb{Z}[x]/\langle x^n + 1\rangle$ for n a power of 2.
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- Same ring structures used in NTRU cryptosystems.

Ring-LWE

- Search : find secret ring element $s(x) \in R_q$ given

$$a_1 \cdot s + e_1 = b_1 \in R_q$$
$$a_2 \cdot s + e_2 = b_2 \in R_q$$
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- Decision : distinguish (a_i, b_i) from uniform $(a_i, b_i) \in R_q \times R_q$.

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- Ideals in $\mathbb{Z}[x]/\langle f \rangle$ are ideal lattices.

Hardness Results in Ideal Lattices

There is a quantum reduction from a worst case lattice problem SVP$_{\gamma=poly(n)}$ on arbitrary ideal lattices to search Ring-LWE.
There is a classical reduction from search Ring-LWE to decision Ring-LWE for any ideal lattice in cyclotomic R.

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Results are w.r.t. ideal lattices that have more structure. But no significant difference in security proofs versus general n-dim lattices.

- Decision Ring-LWE is needed for crypto – if you can break the crypto scheme then you can distinguish (a_i, b_i) from (a_i, b_i), etc, etc.

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    - Now we have an embedding that is $+$ and $\cdot$ coordinate-wise.

$$f(x) \longmapsto (f(z^1), f(z^3), \ldots, f(z^{2n-1}))$$

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- With the new embedding we now have coordinate multiplication - $a \cdot s = (a_1 s_1, \cdots, a_n s_n)$.

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long elements relative to $q = 13$.
- We have error distributions that depend on q in very complicated ways.

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- τ preserves norms in the coefficient embedding – $||\tau_k(p(x))|| = ||p(x^k)|| = ||p(x)||$

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    - Let $v_j \in R_q$ be $(0, 0, \ldots, 1, 0, \ldots, 0)$, jth position has 1,
    - $\alpha_l \in R_q$ be chosen uniformly random,
    - and k be our guess for s_i of s.

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- Replace Ring-LWE samples (a_l, b_l) by

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  - The error distribution should be preserved.
  - Other alternatives – Polynomial-LWE (Stehle, et.al 2009).

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- Many second round lattice-crypto entrants at the NiST PQC standardization contest.

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  - Have a characterization for multivariate ideal lattices based on coefficient mapping using Gröbner basis.
  - How to extend it to build Ring-LWE? How to define the canonical embedding?

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- Story of resilience?

References

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