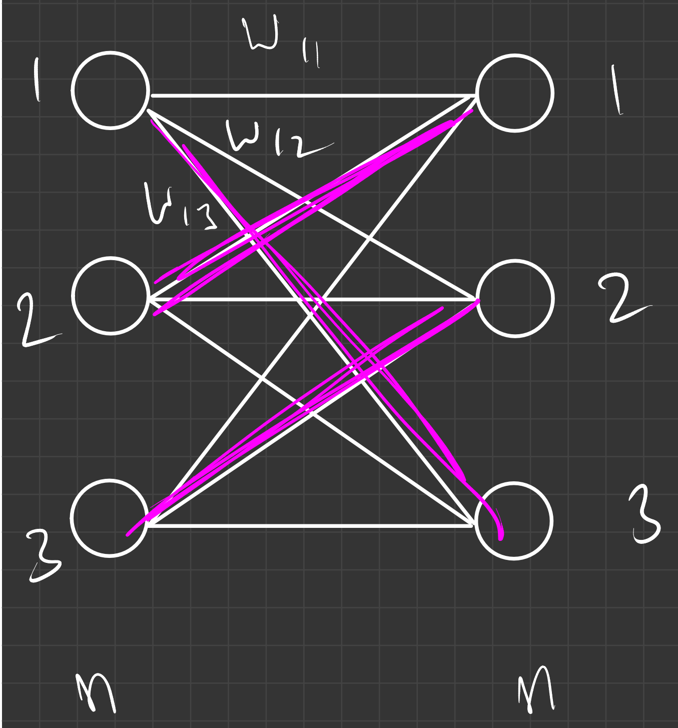
Convex Sets

Motivating example: Bipartite max-wt matching



$(1,3)\ (3,1)\ (3,2)$

w_{ij}

A perfect matching is a subgraph of this complete bipartite graph s.t each vertex has degree 1.

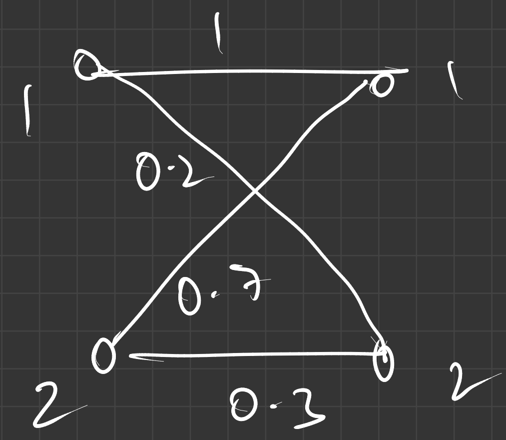
OR

A perfect matching is a permutation on $\{1, 2, \ldots n\}$

$$W_{pm} = \sum_{(i,j) \in pm} w_{ij}$$

Given : $\{w_{ij} : i,j \in [n]\} \rightarrow W \in \mathbb{R}^{n \times n}$

$$W = \begin{bmatrix} 1 & 0.2 \\ 0.7 & 0.3 \end{bmatrix}$$



$(1,1)$ $(2,2)$

Variables : $x_{ij} \in \{0,1\}$ represents the connections

$$x_{ij} = \begin{cases} 1 & \text{if } i \,\&\, j \text{ are connected in PM} \\ 0 & \text{else} \end{cases}$$

$$f(X) = \sum_{i,j} w_{ij} x_{ij}$$

<u>Problem</u> : $\max f(X)$

X : permutation matrix

$$= \max \sum_{ij} w_{ij} x_{ij}$$

$$\forall i, \quad \sum_{j=1}^{n} x_{ij} = 1$$

$$\forall j, \quad \sum_{i=1}^{n} x_{ij} = 1$$

$$x_{ij} \in \mathbb{Z}_{\geq 0}$$

LP Relaxation

Find :

$$\max \sum_{ij} w_{ij} x_{ij}$$

$$\forall i, \sum_{j=1}^{n} x_{ij} = 1$$

$$\forall j, \sum_{i=1}^{n} x_{ij} = 1$$

$$x_{ij} \geq 0 \quad \forall i, j$$

Linear program

General LP :

$$\max \quad \underline{a}^T \underline{x}$$

$$A\underline{x} \leq \underline{b}$$

elementwise

General LP:

$$\underline{a}_i^T \underline{x} \geq b_i$$

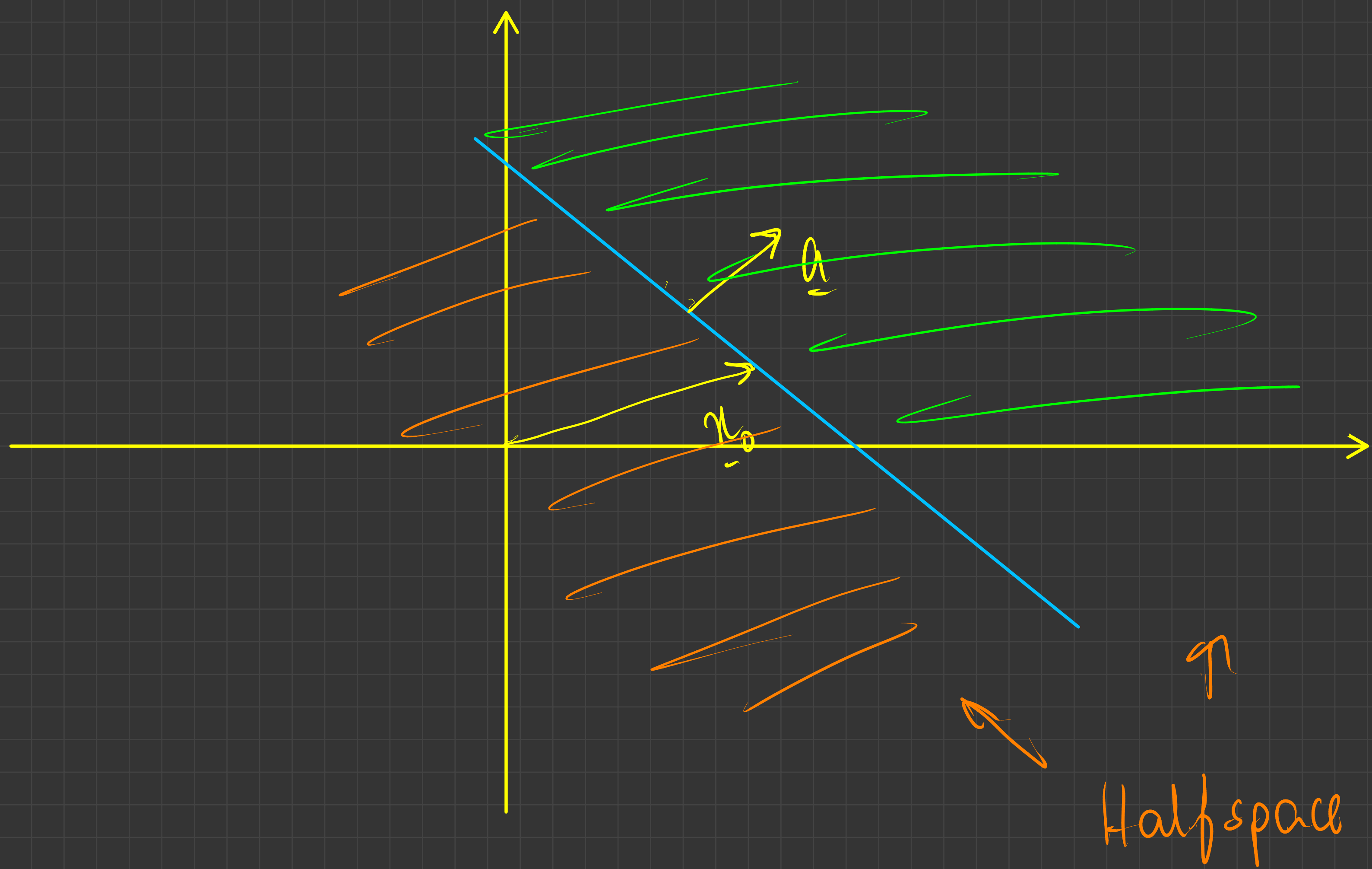
<u>Hyperplane</u>: $\{\underline{x} \in \mathbb{R}^n : \underline{a}^T \underline{x} = b\} = \mathcal{H}$

$$\dim(\mathcal{H}) = n-1$$

$$\mathcal{H} = \{\underline{x} : a^T(\underline{x} - \underline{x}_0) = 0\}$$

$$= \{\underline{y} + \underline{x}_0 : \underline{a}^T \underline{y} = 0\}$$

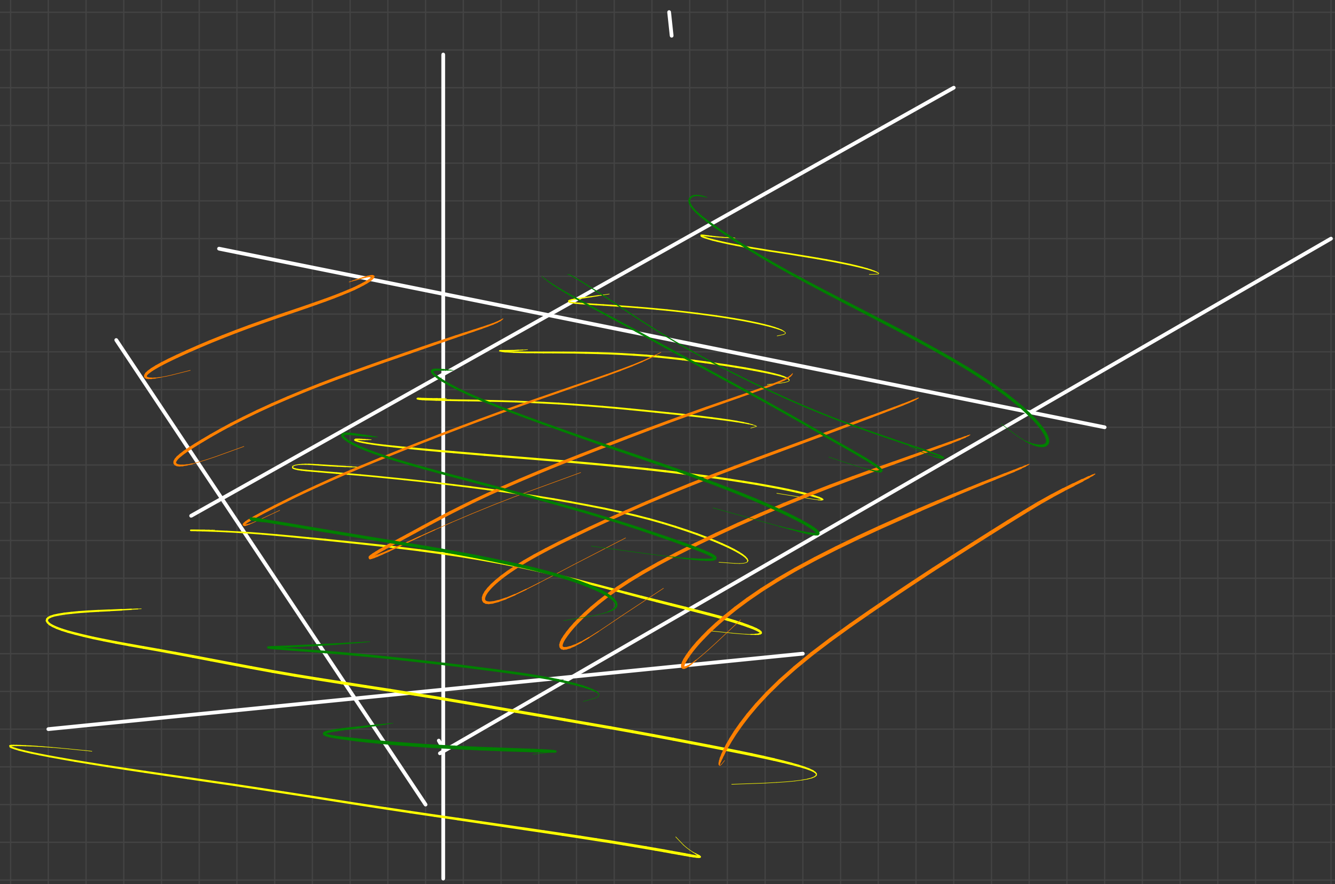
$\{y : \underline{a}^T \underline{y} = 0\}$ is a subspace of dim $n-1$



$\underline{a}$
$\underline{x_0}$
Halfspace

$A\underline{x} \geq \underline{b} \rightarrow m$ linear constraints

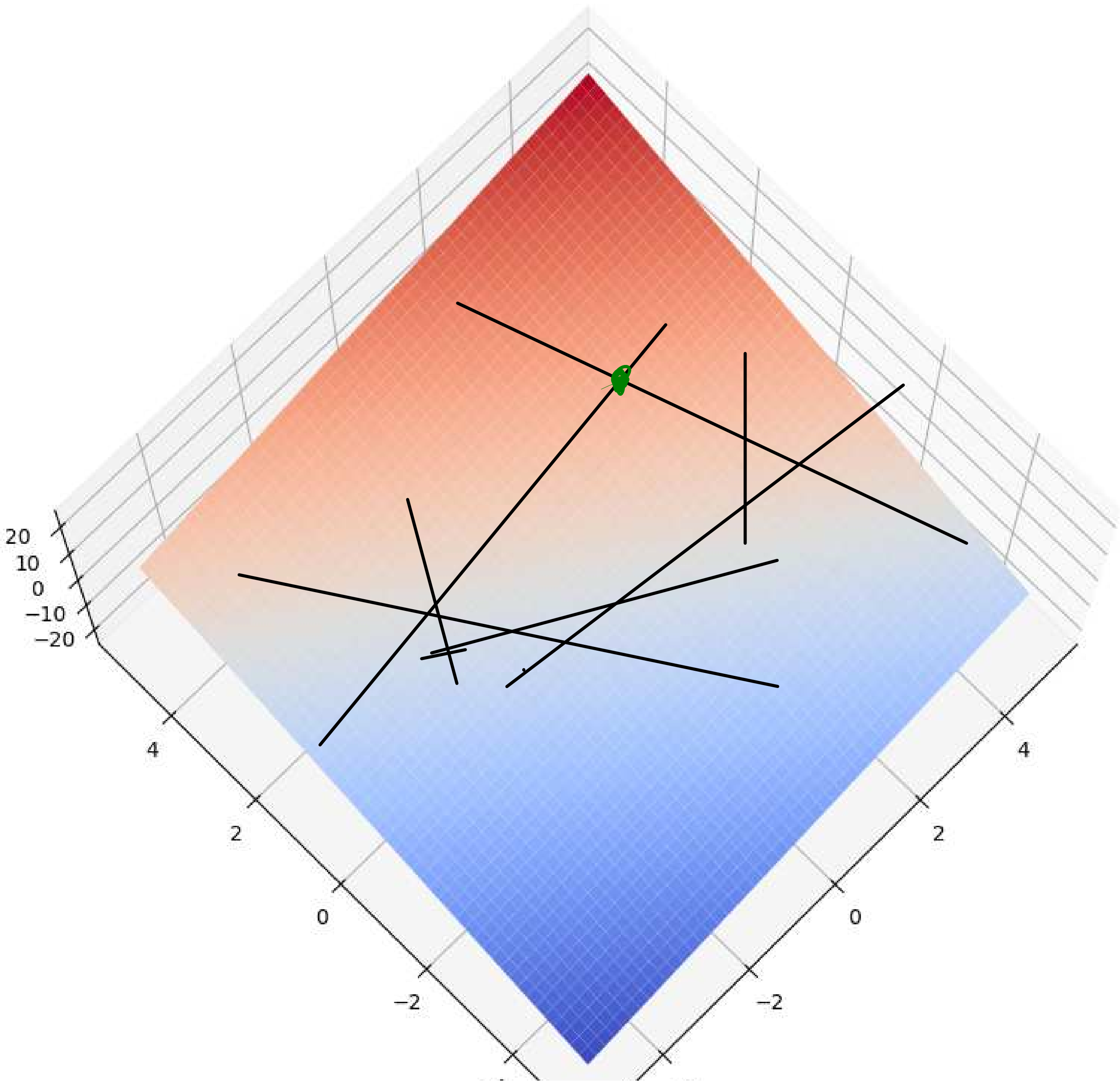
$m \times n$



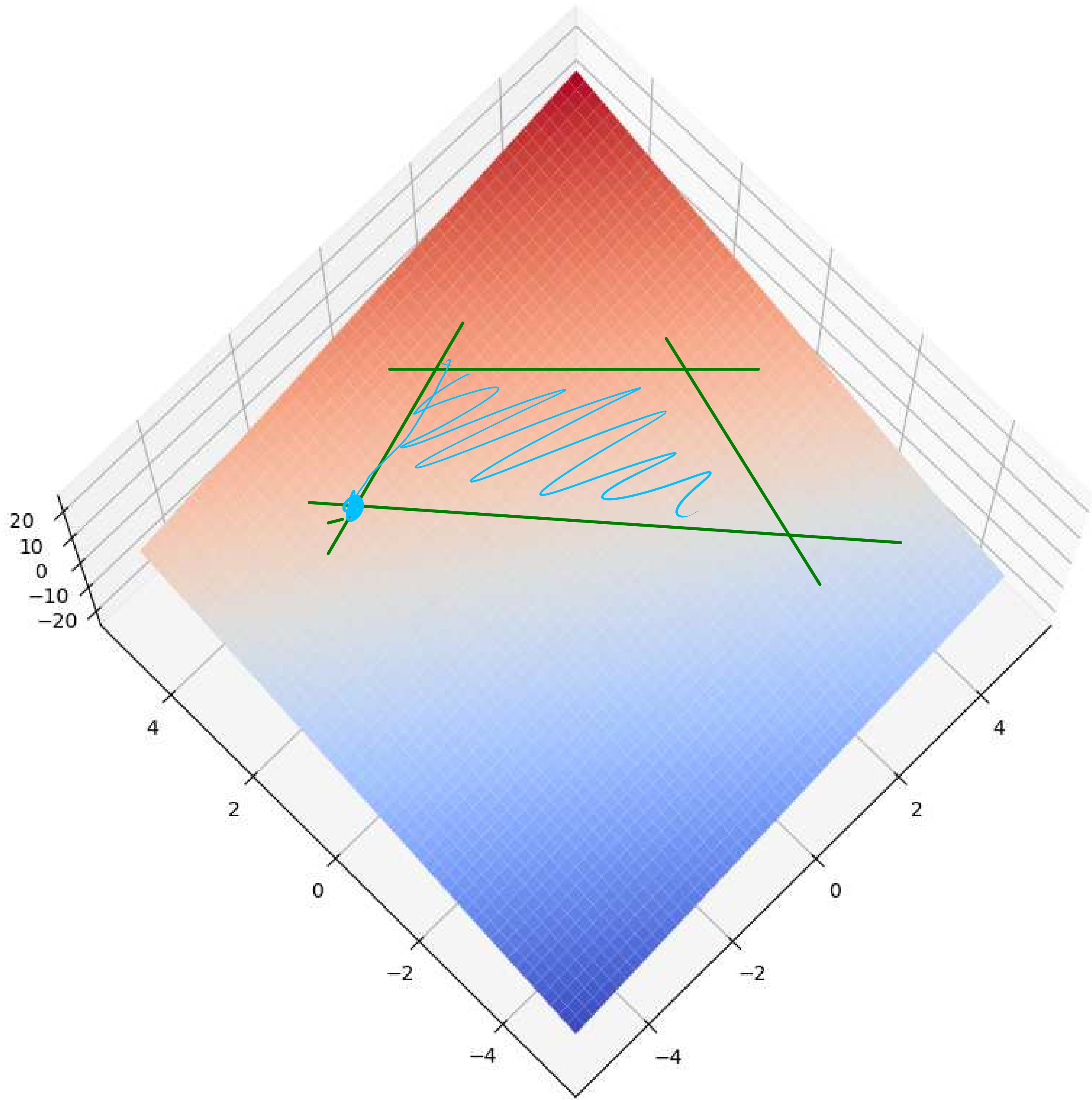
<u>Polyhedron</u> : any set that is an intersection of finitely many halfspaces.

(may be unbounded)

Polytope : Bounded polyhedron



20
10
0
−10
−20
4
2
0
−2
4
2
0
−2


20
10
0
−10
−20
4
2
0
−2
−4
4
2
0
−2
−4

Fact: For any LP, the optimum is reached at a vertex.

$$P_n = \left\{ X \in \mathbb{R}^{n\times n} : \begin{array}{l} x_{ij} \geq 0 \ \forall i,j \\ \sum_i x_{ij} = 1 \\ \sum_j x_{ij} = 1 \end{array} \right\}$$

Doubly stochastic matrices

Birkhoff – von Neumann theorem – Vertex set of P_n is the set of permutation matrices

Recap

- Straight lines and line segments
- Convex sets
- Hyperplanes
- Halfspaces
- Linear programming

Affine sets

We say that S is affine if $\forall\, x_1, x_2 \in S$,

$$\alpha x_1 + (1-\alpha)x_2 \in S \quad \forall \alpha \in \mathbb{R}$$

$$S \text{ is affine} \;\; \overset{\checkmark}{\Rightarrow} \;\; S \text{ is convex}$$

$$\not\Leftarrow$$

$$\not\Leftarrow$$

$$\not\Leftarrow$$

Affine hull and convex hull

Given any set S,

The affine hull, $\mathrm{aff}(S)$ = smallest affine set containing S

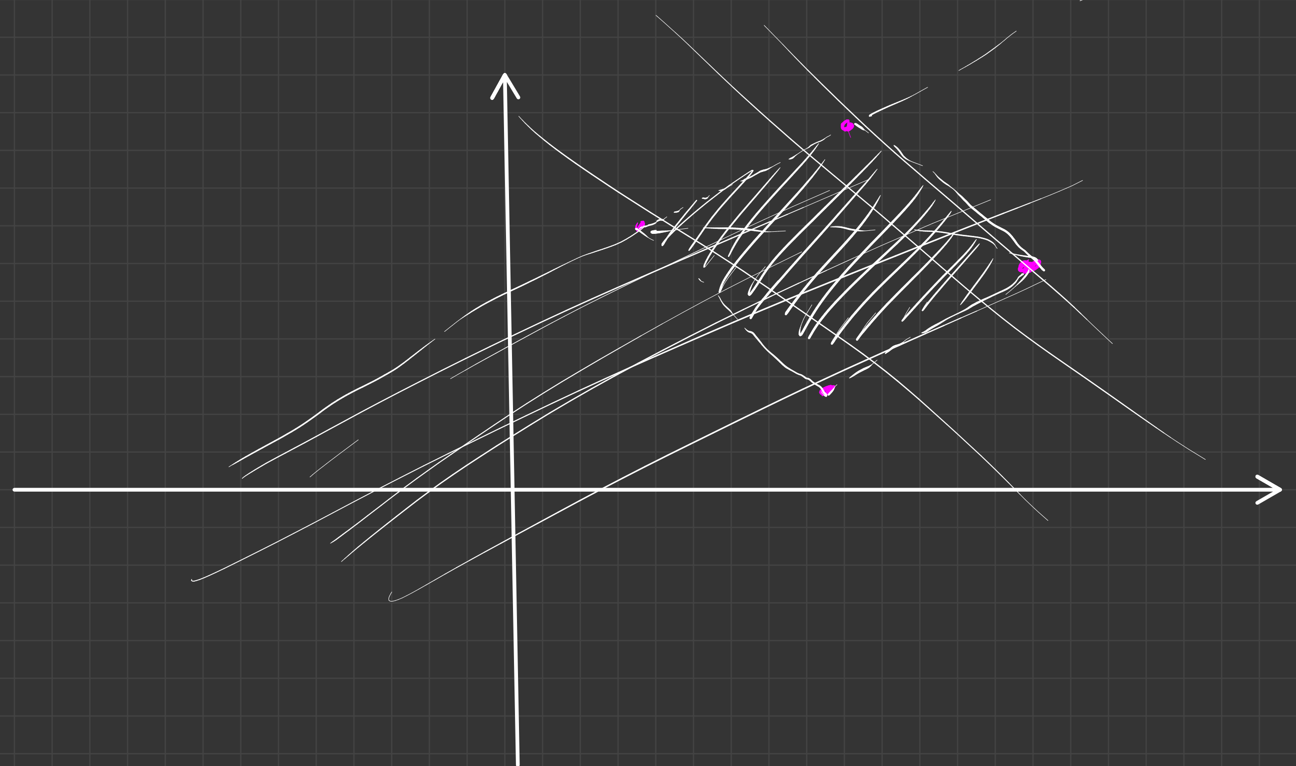
$\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m,$

$$\underbrace{\begin{matrix} \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m \\ \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1 \end{matrix}}_{\text{affine combination}}$$

$$\mathrm{aff}(S) = \left\{ \underline{x} = \sum_{i=1}^{m} \alpha_i \underline{x}_i \; : \; \begin{matrix} \alpha_i \in \mathbb{R} & \underline{x}_i \in S \\ \sum_{j=1}^{m} \alpha_j = 1 & \forall m \end{matrix} \right\}$$

Convex hull(S) is the smallest convex set that contains S

$$\operatorname{conv}(S) = \left\{ \underline{x} = \sum_{i=1}^{m} \alpha_i x_i \; : \; \begin{array}{l} \alpha_i \in [0,1] \\ \sum_i \alpha_i = 1 \end{array} \quad \begin{array}{l} \underline{x}_i \in S \\ \forall m \end{array} \right\}$$



Characterizing affine sets; Affine dimension

Suppose S is affine

$\underline{x}_0 \in S$

Define : $S' = S - \underline{x}_0 = \{\underline{x} - \underline{x}_0 : \underline{x} \in S\}$

<u>Claim</u> : S' is a subspace

$\underline{x}_1', \underline{x}_2' \in S'$

$\alpha x_1' + \beta x_2' \in S$

$\forall \alpha, \beta$

$\alpha x_1' + \beta x_2' \quad \in S'$

$$= \alpha x_1 - \alpha x_0 + \beta x_1 - \beta x_0$$

$$= \alpha x_1 + \beta x_2 + (-\alpha-\beta)x_0$$

We know,

$x_1' = x_1 - x_0 \qquad x_1, x_2 \in S$

$x_2' = x_2 - x_0$

$$= \alpha x_1 + \beta x_2 + (-\alpha-\beta)x_0 + x_0 - x_0$$

$$= \underbrace{\alpha x_1 + \beta x_2 + (1-\alpha-\beta)x_0}_{\text{affine combination of } x_1, x_2, x_0 \in S} - x_0 \qquad = x - x_0$$

$\in S \qquad\qquad x \in S$

$\therefore$ Affine sets are shifts of vector subspaces.

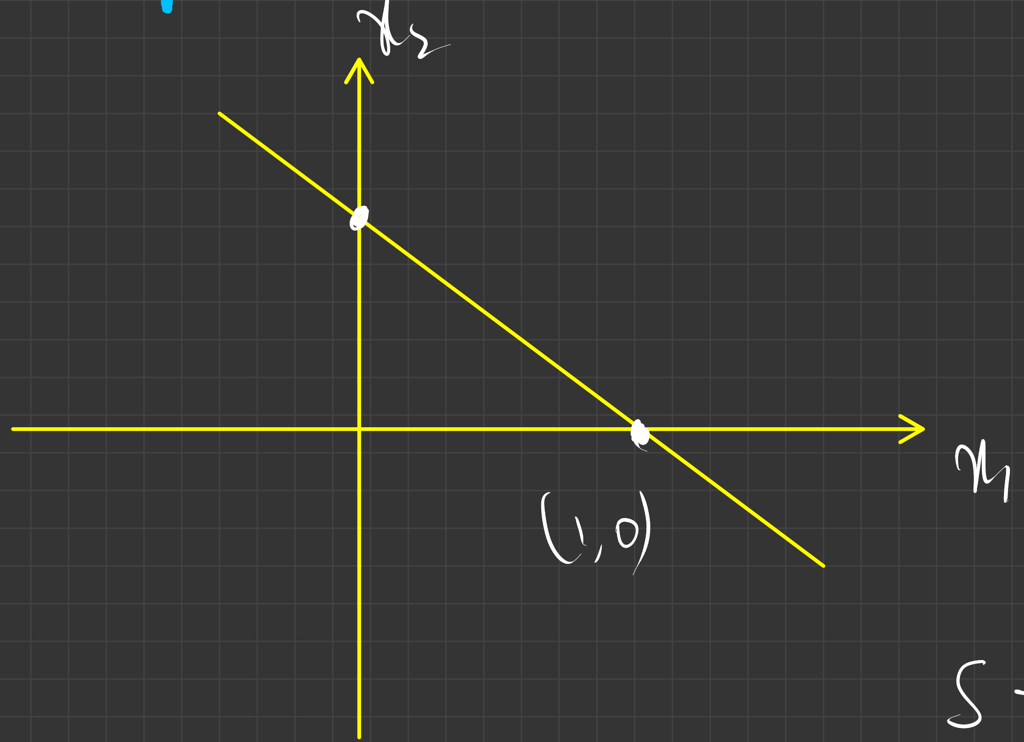
Affine dimension (S) = $\dim(S - x_0)$

for any S,

$$\operatorname{aff\,dim}(S) = \dim\left(\operatorname{aff}(S) - x_0\right)$$

$$x_0 \in S$$

Examples



$$S = \{ \underline{x} : x_1 + x_2 = 1 \}$$

$$x_1 \in \mathbb{R}$$
$$x_2 \in \mathbb{R}$$

$$x_1' = x_1 - 1, \quad x_2' = x_2$$

$$S - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \{ \underline{x}' : \quad x_1' \in \mathbb{R} - 1$$
$$x_2' \in \mathbb{R}$$

$$x_1' + x_2' = 0$$

$$x_1'' = x_1$$

$$x_2'' = x_2 - 1$$

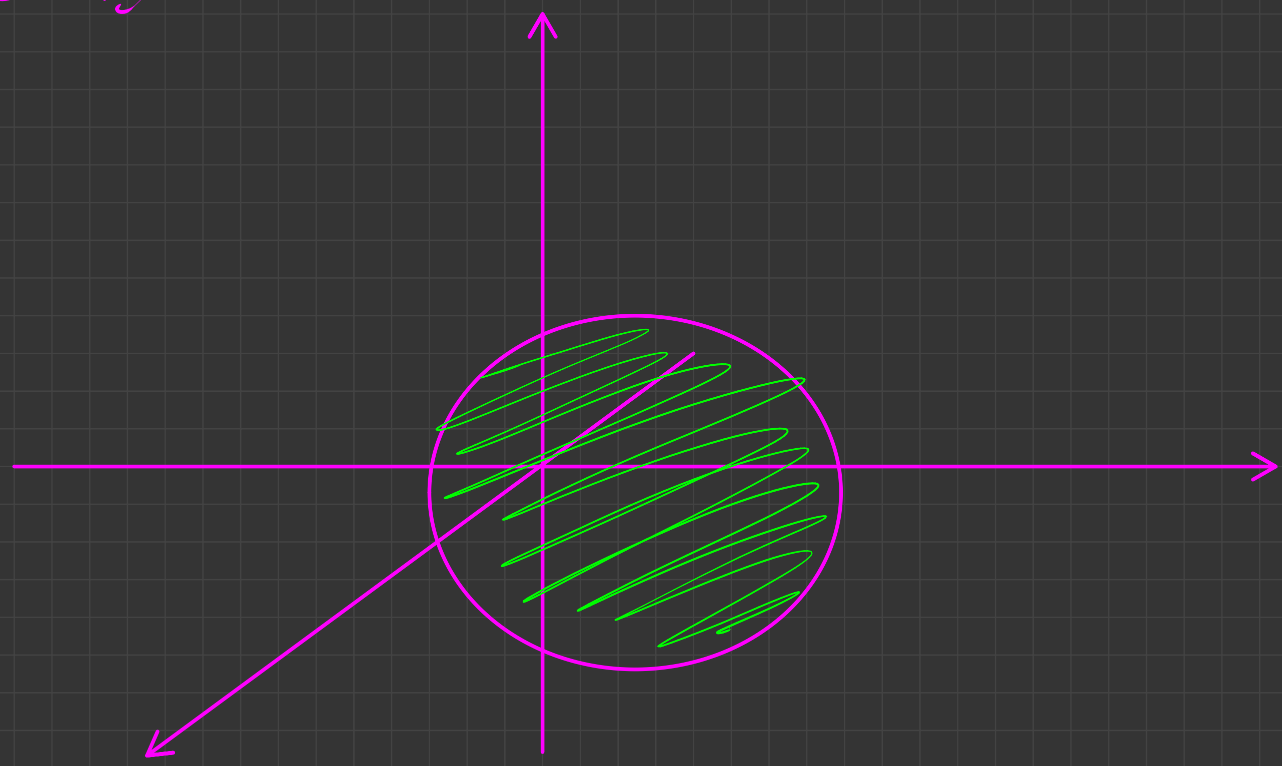
$\mathbb{R}^2$: $\dim(2) \longrightarrow \mathbb{R}^2 \to$ affine

$\dim = 1$ all straight lines

$\dim = 0$ all points / singleton sets

$\mathbb{R}^3$:


x_1
S
x_2
$S - x_0$



Limit points, closure and interior

Given any set S, consider any convergent sequence of points in S

$x_1, x_2 \ldots$ $\qquad x_i \in S$

For each $\epsilon > 0$, $\exists N_\epsilon$

$$|x_{i_1} - x_{i_2}| < \epsilon \qquad \forall\ i_1, i_2 > N_\epsilon$$

The limit of such a sequence is a limit pt

<u>Eg:</u> $S = (0, 1)$

limit points also include 0 & 1

Closure(S) = set of all limit pts of S
(smallest closed set containing S)

$(0,1) \cup (1,2)$

Closure$(\mathbb{Q}) = \mathbb{R}$

Closure$(\mathbb{N}) = \mathbb{N}$

Closure$\left(\left\{\frac{1}{2^i} : i \in \{1, 2, \dots\}\right\}\right)$

$= \left\{\frac{1}{2^i} : i \in \{1, 2, \dots\}\right\} \cup \{0\}$

Interior (S) : Largest open set contained in S

$\underline{x} \in S$ is an interior point if $\exists\ \epsilon > 0$

st $B(\underline{x}, \epsilon) \subseteq S$

Open ball of radius ϵ & center $\underline{x}$

$\{ y : \| \underline{x} - y \| \leq \epsilon \}$

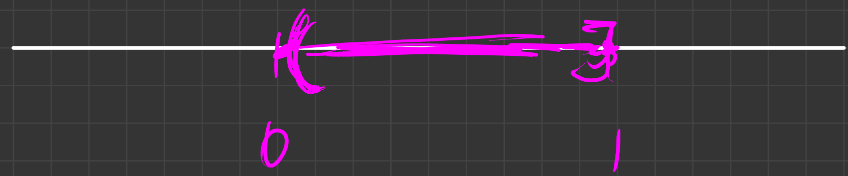
Boundary(S) = closure(S) \ Interior(S)

Eg: $S = (0, 1]$

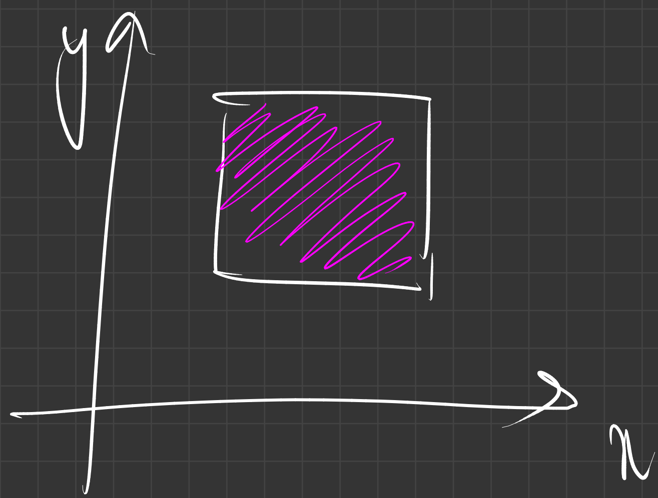
$\text{Closure}(S) = [0, 1]$

$\text{Interior}(S) = (0, 1)$

$\text{Boundary}(S) = \{0, 1\}$



②

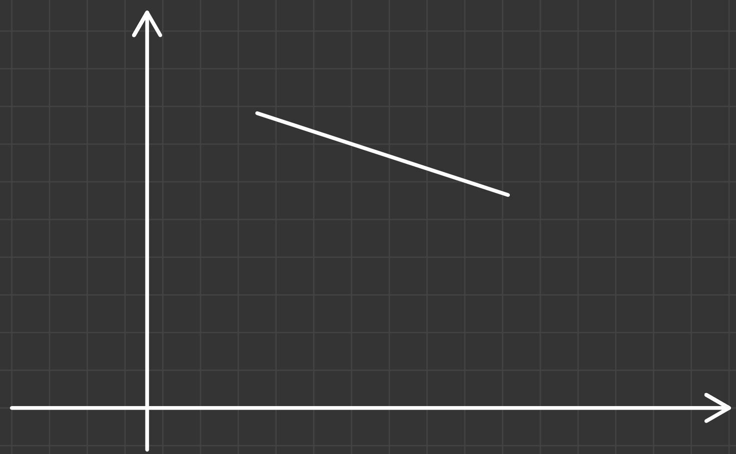


③ $S = \mathbb{Q}$

$\text{Closure}(S) = \mathbb{R}$

$\text{Int}(S) = \phi$

④



$2x + 3y = 1$

$x \leq 2$

$y \leq 5$

$\text{Closure}(S) = S$

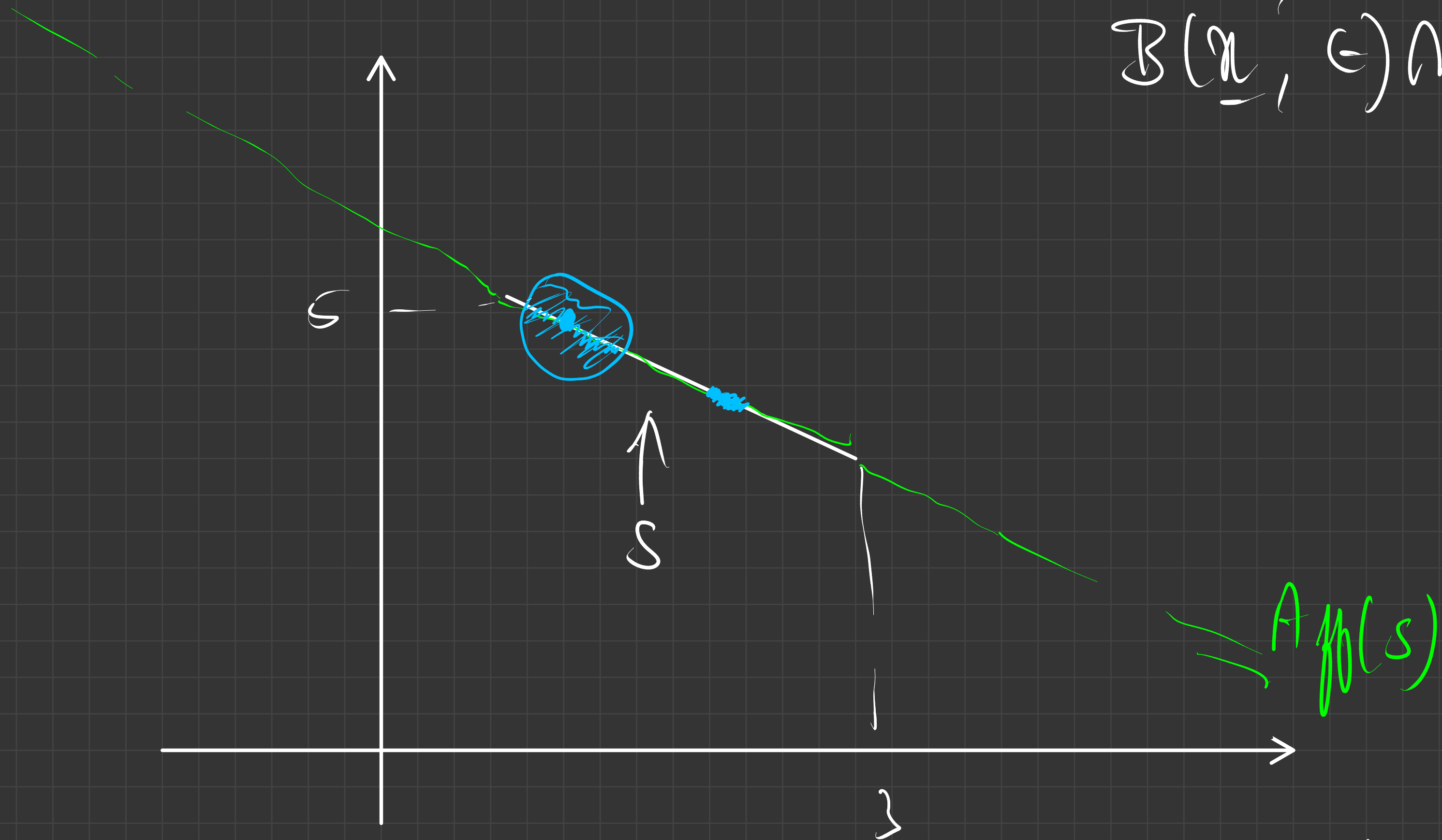
$\text{Int}(S) = \phi$

$\text{Bd}(S) = S$

Relative interior and relative boundary

Relative interior of S

$$\mathrm{Relint}(S) = \{ \underline{x} \in S : \exists \epsilon > 0,\ B(\underline{x}, \epsilon) \cap \mathrm{Aff}(S) \subseteq S \}$$



$$x + 2y = 1$$
$$x \le 3$$
$$y \le 5$$

$$\mathrm{Bd}(S) = \{ (3, -1), (-9, 5) \}$$

$$\mathrm{Relint}(S) = \left\{ (x, y) : \begin{array}{l} x + 2y = 1 \\ x < 3 \\ y < 5 \end{array} \right\}$$

Find the affine dimension, closure, int, relint, boundary

① $S \subseteq \mathbb{R}^2$ $\quad S = \{ \underline{x} : \|\underline{x}\|_2 \leq 1 \}$

$\text{Closure}(S) = S$

$\text{Int}(S) = \{ \underline{x} : \|\underline{x}\|_2 < 1 \}$

$\text{RelInt}(S) = \text{Int}(S)$

$\text{Bd}(S) = \{ \underline{x} : \|\underline{x}\|_2 = 1 \}$

(2) $S \subseteq \mathbb{R}^2$: $S = \{(1,2), (2,1), (3,1)\}$

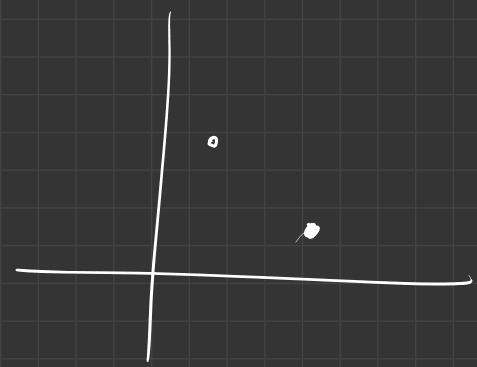
$$\text{Closure}(S) = S$$

$$\text{Int}(S) = \emptyset$$

$$\text{RelInt}(S) = \emptyset$$

$$\text{Bd}(S) = S$$

$$\text{Aff}(S) = \mathbb{R}^2$$



(2.6) $S = \{(1,2), (2,1)\}$

(3) $S \subseteq \mathbb{R}^2$: $S = \{ \underline{x} : x_1 + x_2 = 1 \}$

$$\text{Closure}(S) = S$$

$$\text{Int}(S) = \phi$$

$$\text{RelInt}(S) = S$$

$$\text{Rel Bd}(S) = \phi$$

④ $S \subseteq \mathbb{R}^3$:

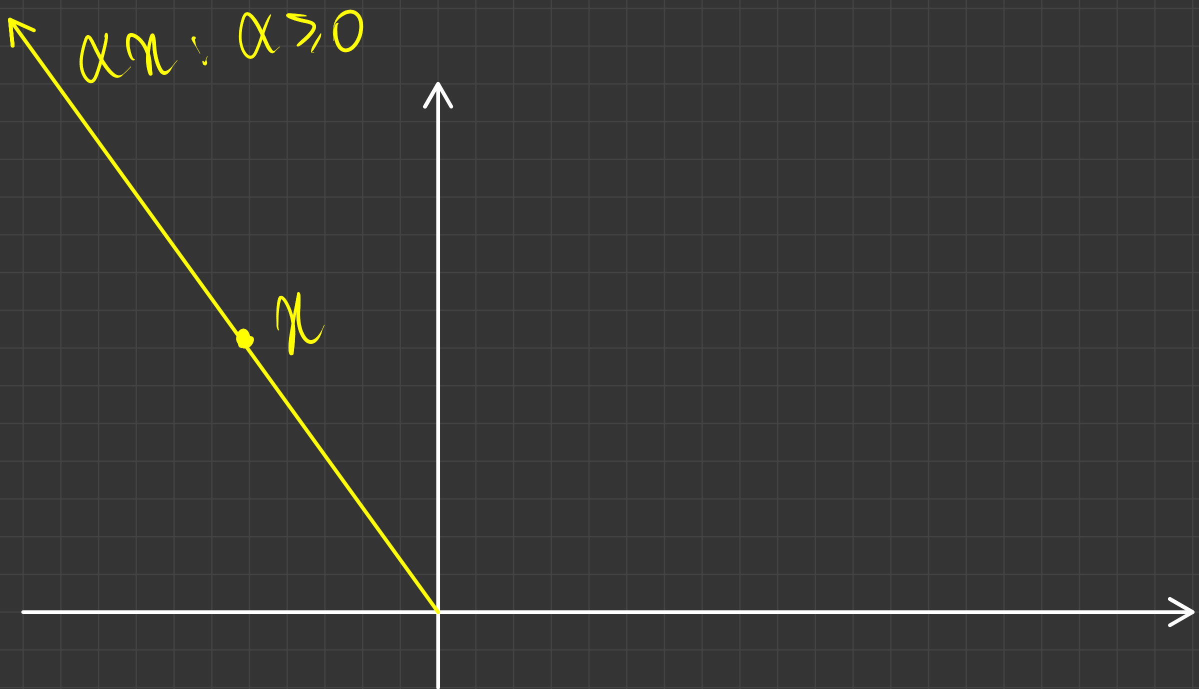
$$S = \left\{ \underline{x} : \underline{x} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad \alpha \in [0, 1] \right\}$$

6 $S \subseteq \mathbb{R}^3$ $\quad S = \left\{ \underline{x} \in \mathbb{R}^3 : \begin{array}{l} x_1^2 + x_2^2 \leq 1 \\ x_3 = 1 \end{array} \right\}$

Cones, and convex cones

S is a cone if $\forall \underline{x} \in S$ & $\alpha \geq 0$,

$\alpha \underline{x} \in S$



Conic combinations : $\sum_{i=1}^{m} \theta_i \underline{x}_i \qquad \theta_i \geq 0 \quad \forall i$

<u>Claim</u> : S is closed under conic combinations iff it is a convex cone.

Suppose S is a convex cone.

$\underline{x}_1 \ldots \underline{x}_m \in S, \qquad \theta_1 \ldots \theta_m \geq 0$

$$\sum_{i=1}^{m} \theta_i \underline{x}_i = (\theta_1 + \theta_2 + \ldots + \theta_m) \underbrace{\sum_{i=1}^{m} \left(\frac{\theta_i}{\theta_1 + \ldots + \theta_m} \right) \underline{x}_i}_{\substack{\text{convex comb} \\ \in S}}$$

$\in S$

Suppose closed under conic combn

$x \in S \Rightarrow \theta x \in S \quad \forall \; \theta \geq 0$

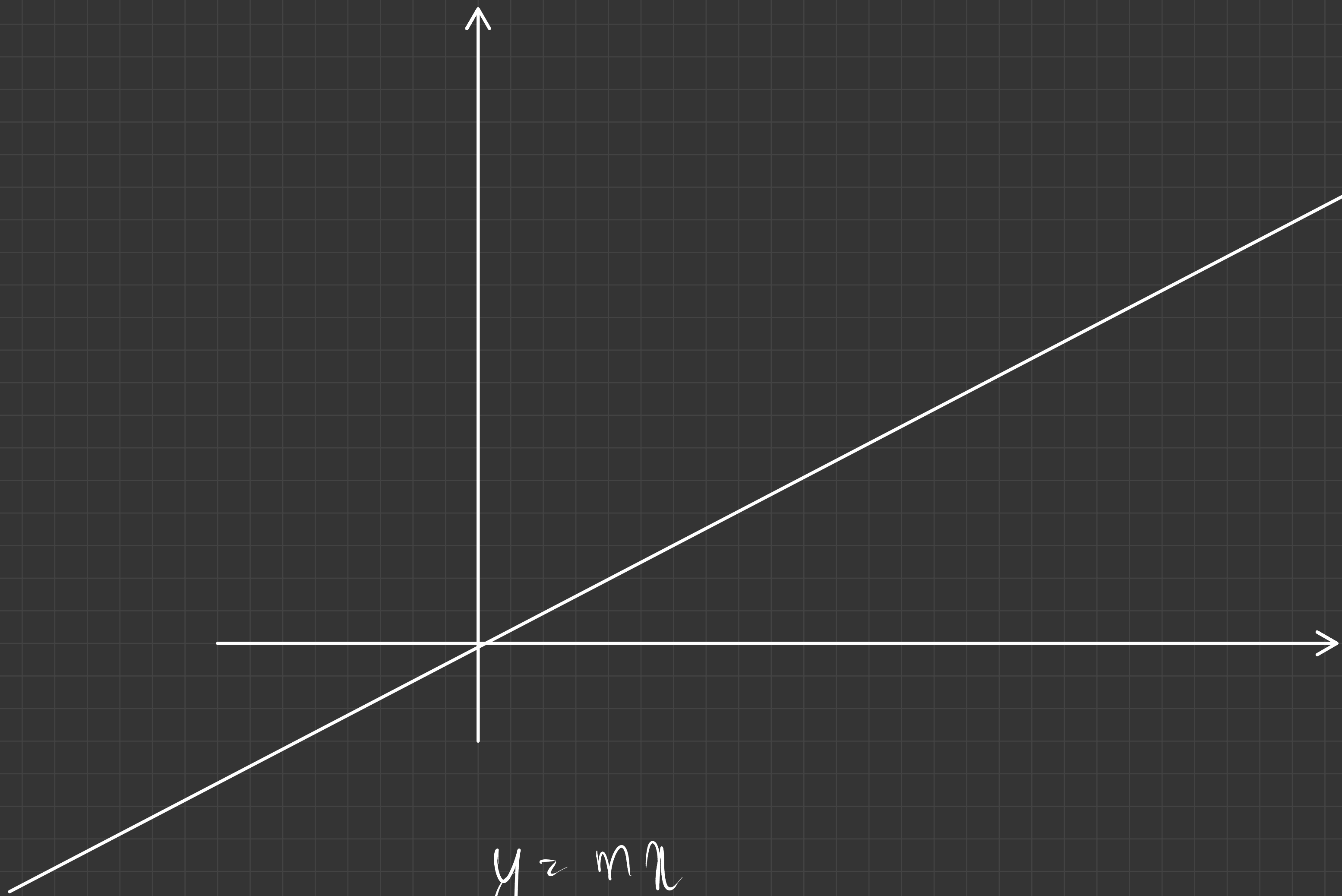
Hence S is a cone

$\Downarrow$

Closed under convex combn.

$x_1 \; - \; x_m$

Eg



$y = mx$

Q: What is the equation of a st line in $\mathbb{R}^n$

$$\{ \underline{x} : W\underline{x} = \underline{b} \}$$

$$\underline{x} \in \mathbb{R}^n$$

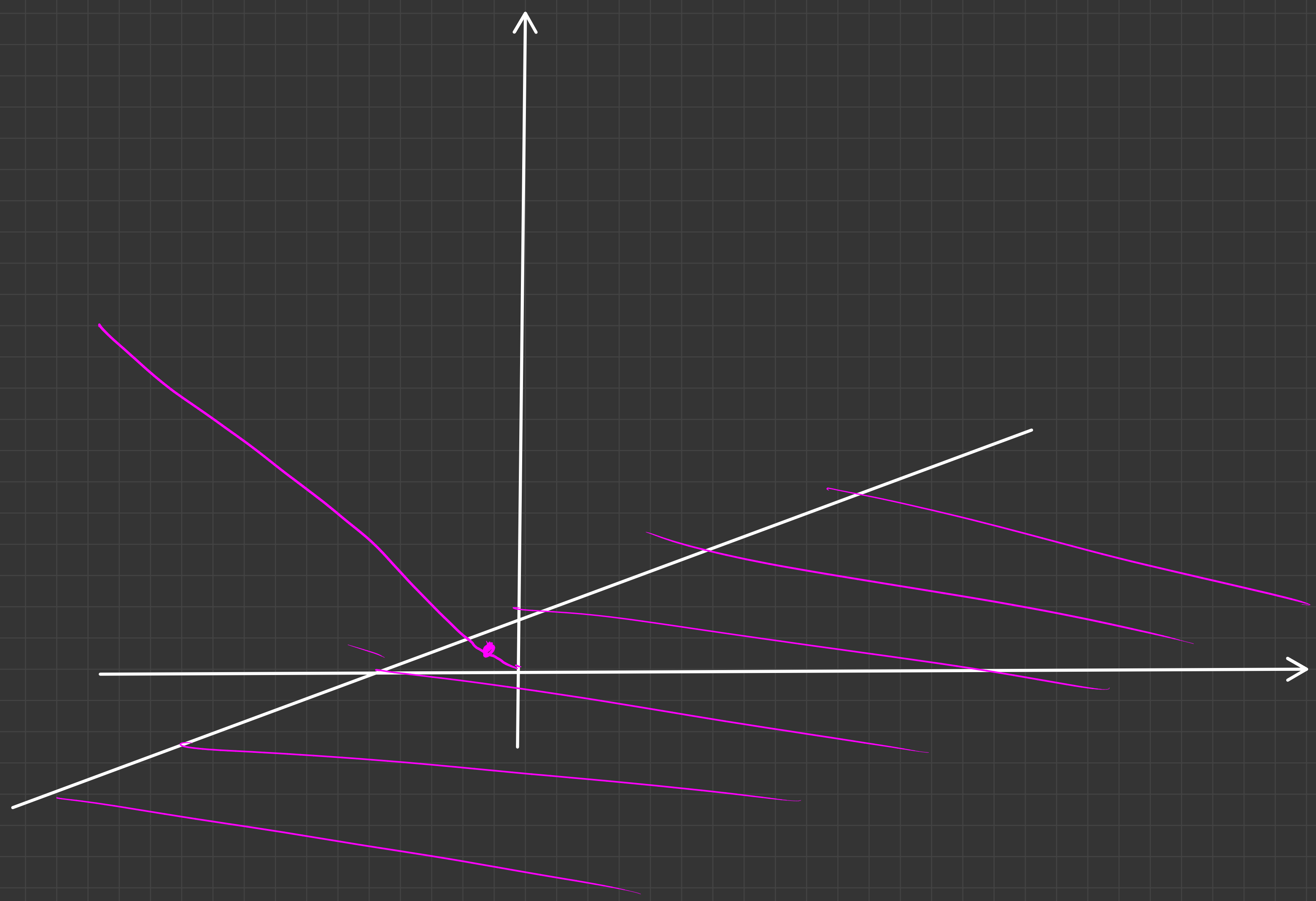
$W \to (n-1) \times n$ full rank matrix

Eq of st line passing through origin

$$= \{ \underline{x} : W\underline{x} = \underline{0} \}$$

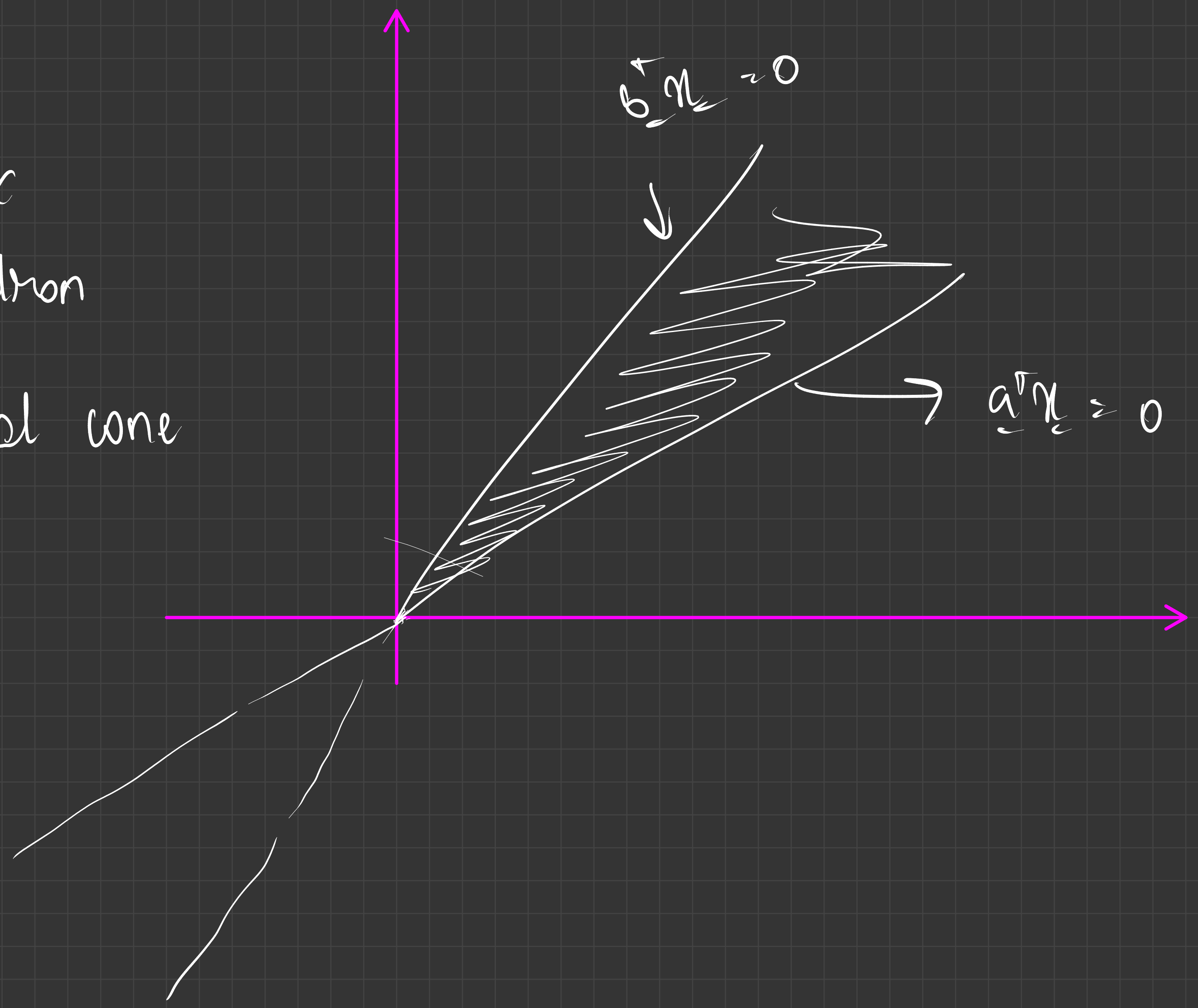
$$\text{rank}(W) = n-1$$

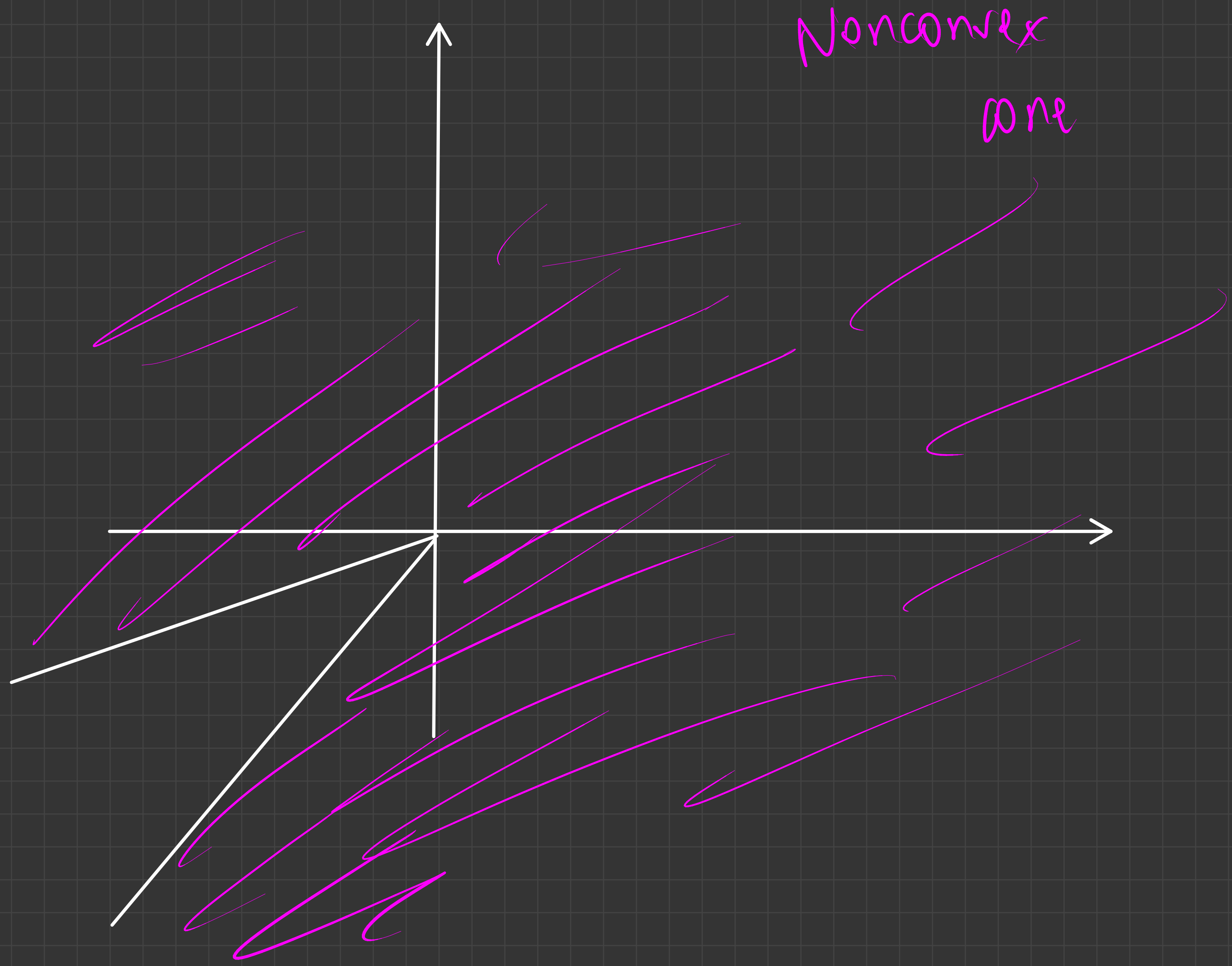
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

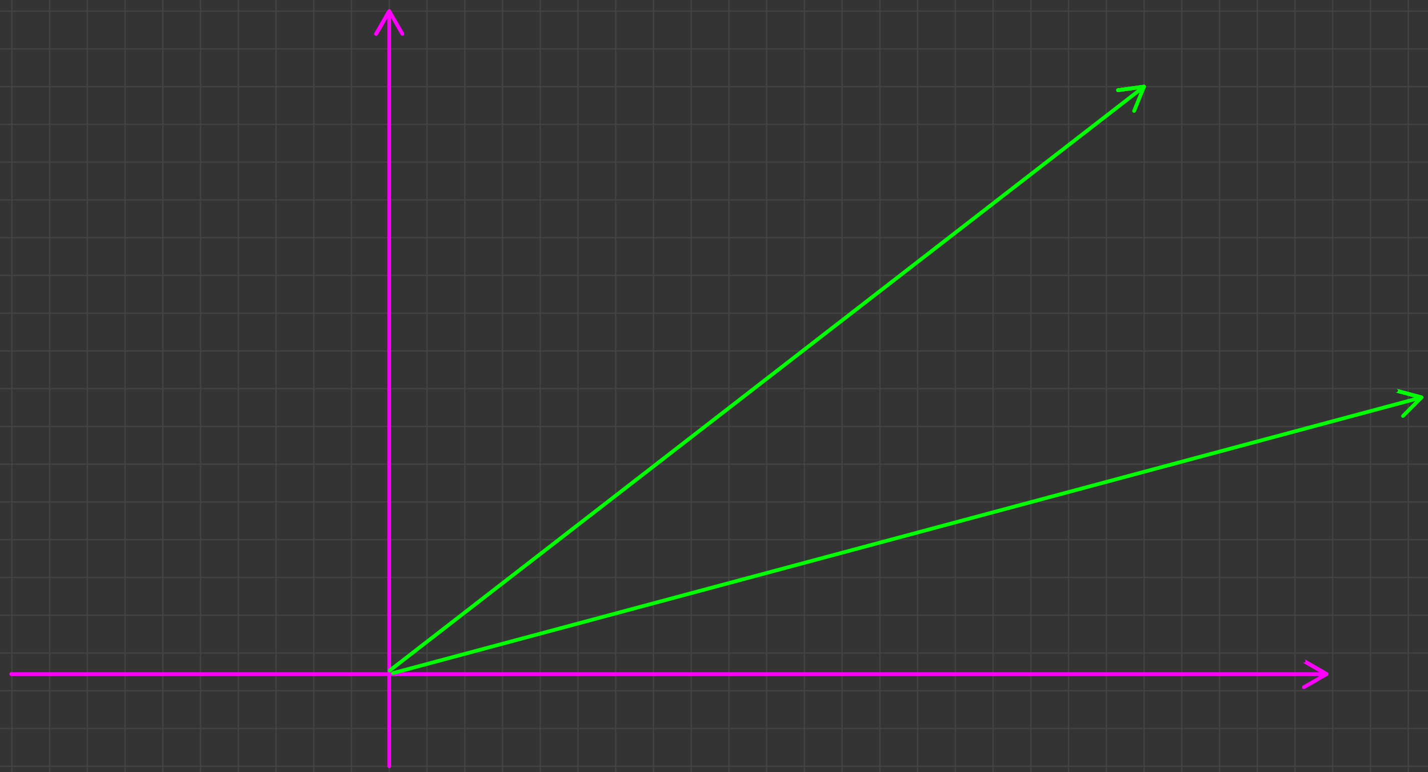


- Cone
- Convex
- Polyhedron

Polyhedral cone




Nonconvex
cone

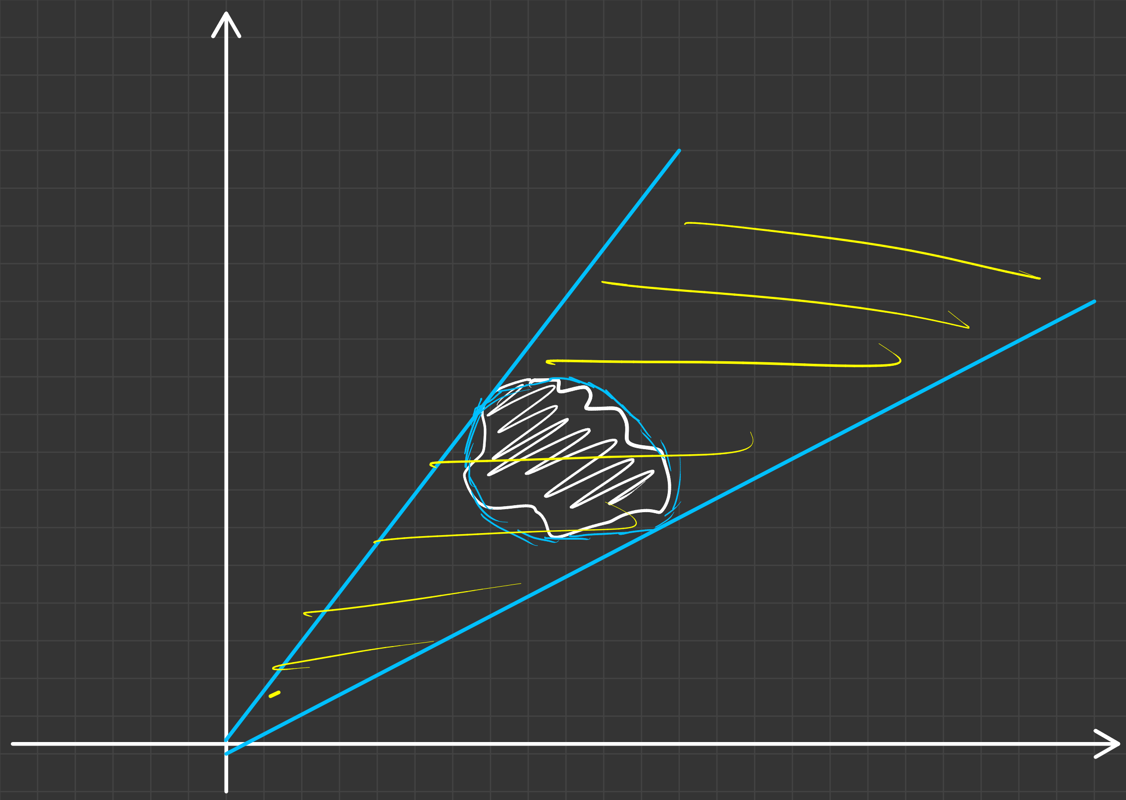


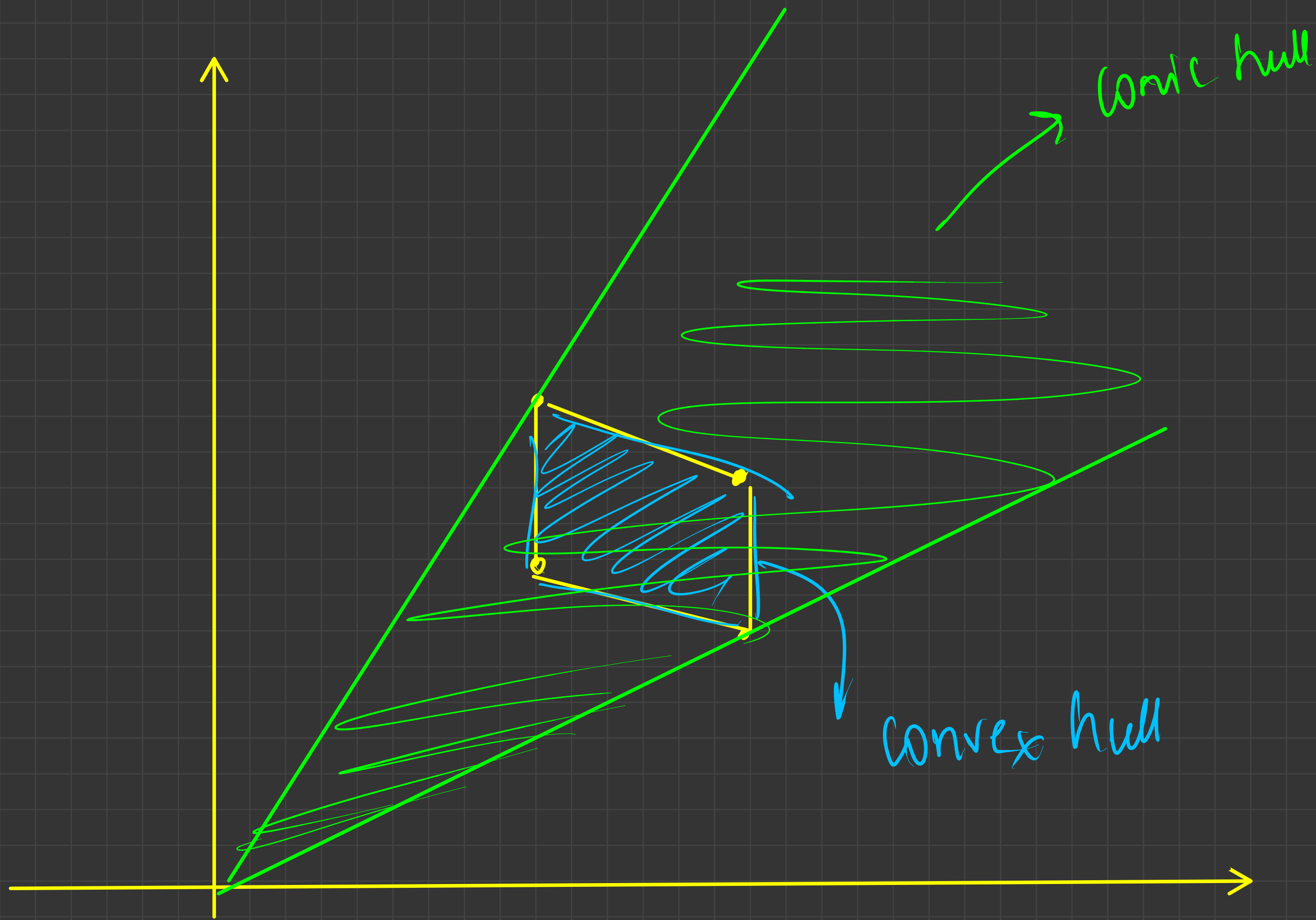
<u>Note</u>: Union of cones is a cone

Conic hull

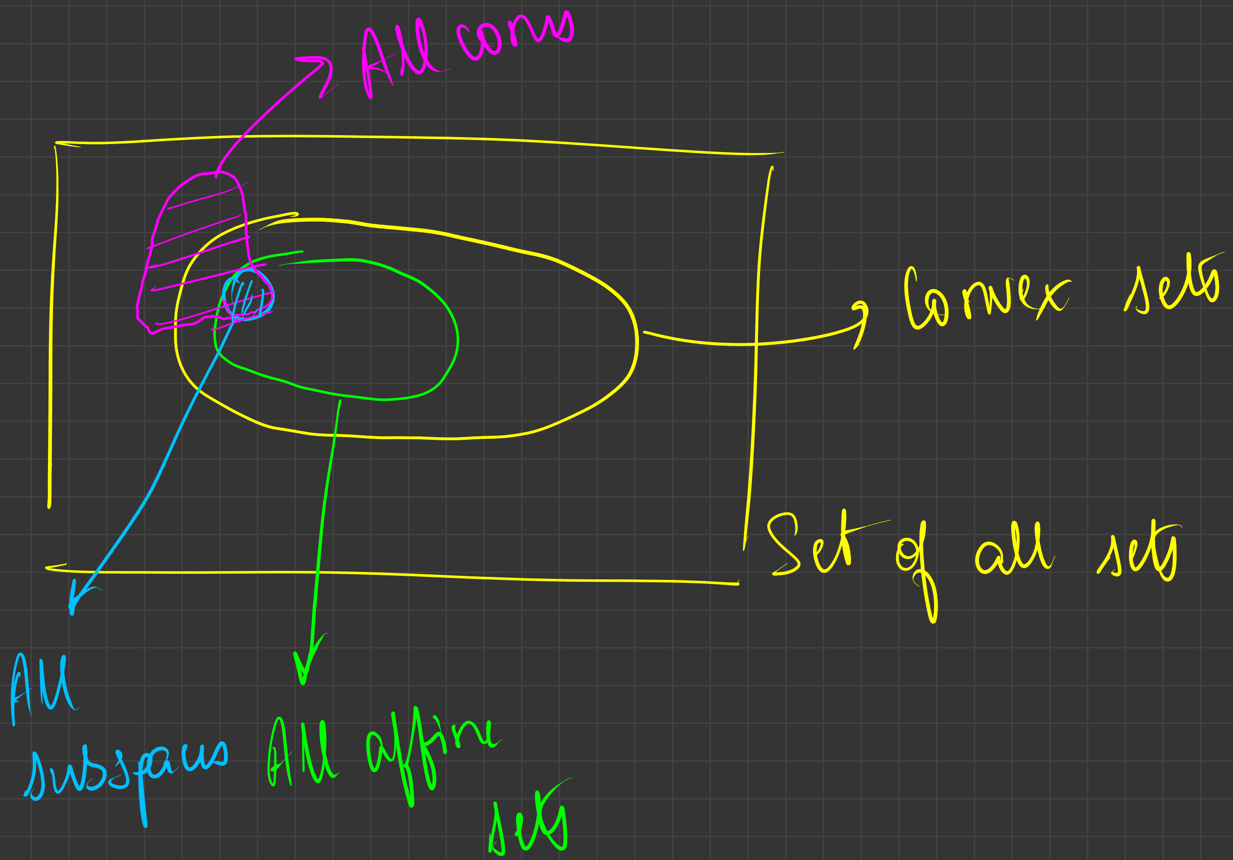
: Smallest convex cone containing S

$$= \left\{ x = \sum_{i=1}^{m} \theta_i x_i \; : \; \begin{array}{l} x_i \in S \\ \theta_i \geq 0 \end{array} \right\}$$




conic hull
convex hull

What does a convex cone in $\mathbb{R}^2$ look like?


All cones
Convex sets
Set of all sets
All subspaces
All affine sets

Any vector subspace

Convex?

Affine?

Cone?

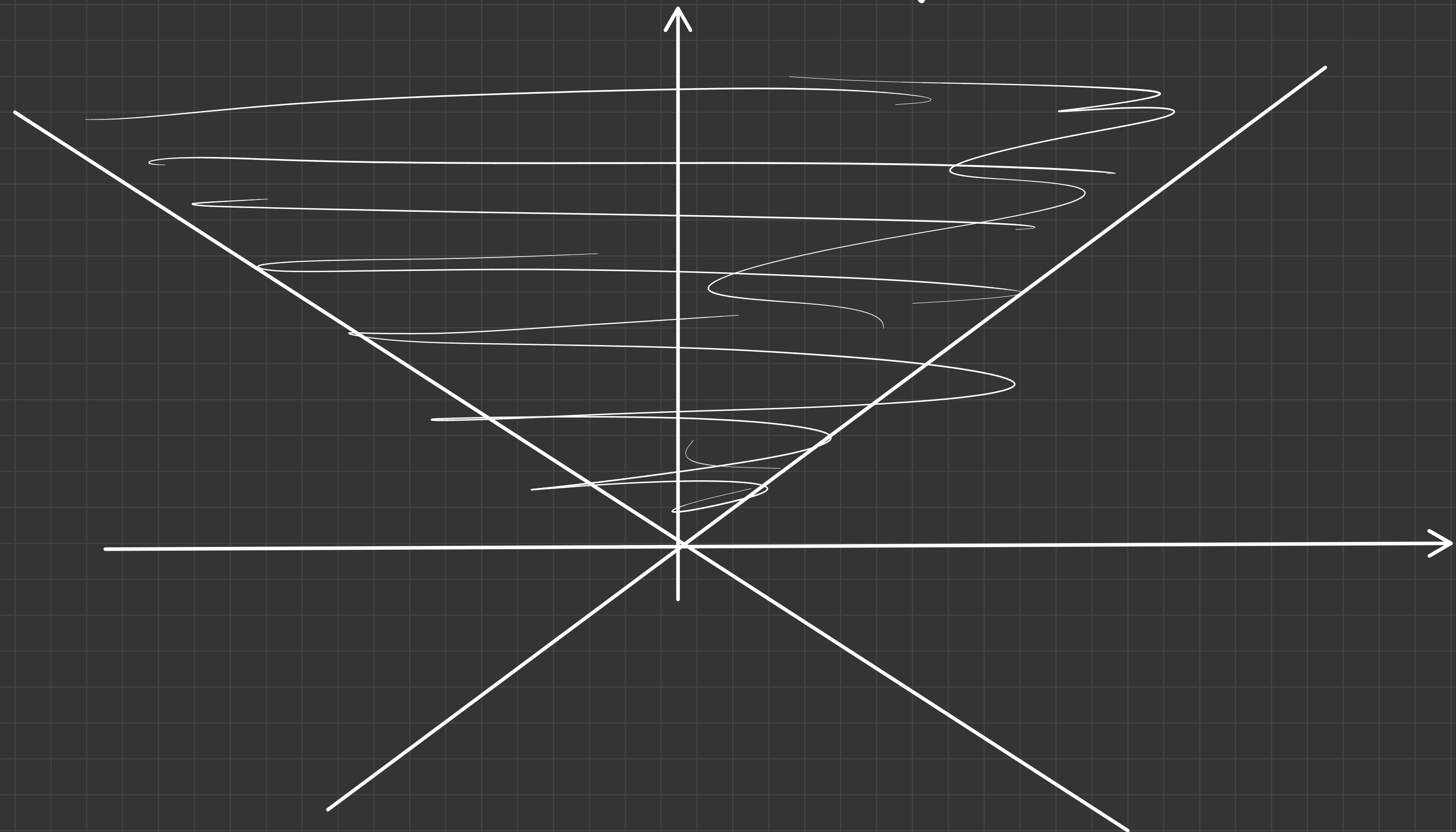
The norm cone

$$\{(\underline{x}, t) : \underline{x} \in \mathbb{R}^{n-1},\ t \in \mathbb{R}_{\geq 0},\ \|\underline{x}\| \leq t\}$$

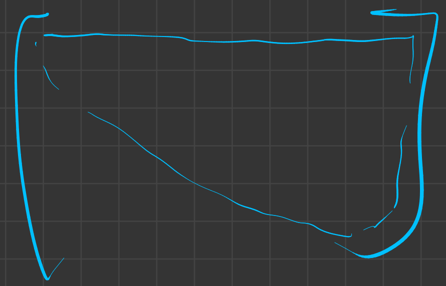
- ✗ Vector space?
- ✓ Convex? Prove
- ✗ Affine?
- ✓ Cone? Prove

In $\mathbb{R}^2$, $\{ (x,y) : |x| \leq y \}$

$y \geq 0$



The set of symmetric matrices



$\dim = 1+2+3+\cdots+n$

Vector space?

Convex?

Affine?

Cone?

The set of positive semidefinite matrices

Vector space? ✗

Convex? ✓

Affine? ✗

Cone? ✓

A, B

$x^T A x \geq 0$

$x^T B x \geq 0$

$\alpha A + (1-\alpha) B$

$x^T (\alpha A + (1-\alpha) B) x$

$= \alpha x^T A x + (1-\alpha) x^T B x \geq 0$

$-1 I + 2 \, 0$ Not PSD

Give examples of

1. Polyhedral cone

2. Non-polyhedral cone

3. Non-convex cone

Operations that preserve convexity

1. Arbitrary intersections

$A_1 \quad A_2$ convex

$A_1 \cap A_2$ is also convex

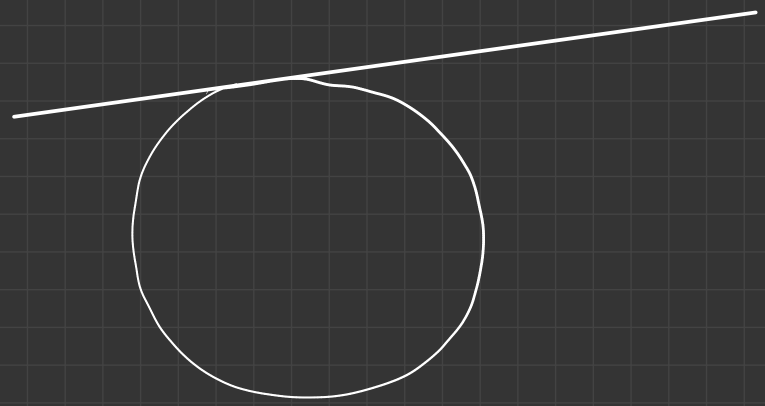
$$\mathcal{A}_{\underline{a}} = \{ X \in S^n : \underline{a}^T X \underline{a} \geq 0 \} \to \text{halfspace}$$

$f(A) \geq 0$

$A \, \mathrm{vec}(X)$ linear operator

$$S^n_+ = \bigcap_{\underline{a} \in \mathbb{R}^n} \mathcal{A}_{\underline{a}}$$

(set of all $n \times n$ sym PSD)



2. Images and inverse images of affine functions

S is a convex set

$A \in \mathbb{R}^{m \times n}$ $\qquad b \in \mathbb{R}^{n}$

$f(\underline{x}) = A\underline{x} + b$

$f(S) = \{ y = A\underline{x} + b : \underline{x} \in S \}$ is convex

$y_1 = A\underline{x}_1 + \underline{b} \qquad y_2 = A\underline{x}_2 + \underline{b} \qquad \underline{x}_1, \underline{x}_2 \in S$

$$\alpha y_1 + (1-\alpha) y_2 = \alpha A x_1 + \alpha \underline{b} + (1-\alpha) A\underline{x}_2 + (1-\alpha)\underline{b}$$

$$= A\underbrace{(\alpha x_1 + (1-\alpha) x_2)}_{\underline{x} \in S} + \underline{b}$$

$$= A\underline{x} + \underline{b} \in f(S)$$

$$f^{-1}(S) = \{ \underline{y} : \underline{x} = A\underline{y} + \underline{b} \quad \text{for some } \underline{x} \in S \}$$

is convex.

Second order conic program (SOCP)

$$\begin{aligned} \text{Minimize} \quad & \underline{c}^T \underline{x} \\ \text{ST} : \quad & F\underline{x} = g \\ & \|A_i \underline{x} + b_i\|_2 \le c_i^T \underline{x} + d_i \quad i = 1, 2, \dots m \end{aligned}$$

$$\{\underline{x} : \|A\underline{x} + b\|_2 \le \underline{c}^T \underline{x} + d\}$$

$$= \{\underline{x} : f(\underline{x}) \in SOC\}$$

affine transformation

Second order cone:

$$\{ (\underline{x}, t) \in \mathbb{R}^{n+1} : \quad t \geq 0, \quad \|\underline{x}\|_2 \leq t \}$$

$$\begin{bmatrix} \underline{y} \\ t \end{bmatrix} = \begin{bmatrix} A\underline{x} + \underline{b} \\ c^T \underline{x} + d \end{bmatrix}$$

$$= \begin{bmatrix} A \\ c^T \end{bmatrix} \underline{x} + \begin{bmatrix} \underline{b} \\ d \end{bmatrix}$$

$$\{ \underline{x} \in \mathbb{R}^n : \quad \begin{bmatrix} \underline{y} \\ t \end{bmatrix} \in SOC \}$$

$$= \{ \underline{x} \in \mathbb{R}^n : \quad \| A\underline{x} + b \|_2 \leq c^T x + d \}$$

$\mathbb{R}^3$

$$\{\underline{x} : \|A\underline{x} + b\|_2 \leq c^T x + d\}$$

$A : 2 \times 3$

$$A \to \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \to \underline{b}$$

$$c^T \to \begin{bmatrix} 7 & 8 & 9 \end{bmatrix}$$

$$\{\underline{x} : \|A\underline{x}+\underline{b}\|_2 \le c^T \underline{x} + d\}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \underline{b} = \underline{0}$$

$$c^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \qquad d = 0$$

$$\{(x_1, x_2, x_3) : \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_2 \le x_3 \} \rightarrow \text{SOC}$$

Example: Robust linear programming

Minimize $\underline{c}^T \underline{x}$

ST

$\underline{a}_i^T \underline{x} \leq b_i \qquad i = 1, 2, 3, \ldots m$

Eg :

$x_i \rightarrow$ # of stocks I purchase for comp i

$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \leq b$

$c_i \rightarrow$ expected returns for comp i

Max $\underline{c}^T \underline{x}$

ST $\underline{a}^T \underline{x} \leq b$

$$\max \quad \underline{c}^T \underline{x}$$

ST

$$(\underline{a} + \underline{e})^T \underline{x} \leq b$$

$$\| \underline{e} \|_2 \leq 1$$

$$\Downarrow$$

$$\underline{a}^T \underline{x} + \underline{e}^T \underline{x} \leq b$$

$$|||$$

$$\underline{a}^T \underline{x} + \| \underline{x} \| \leq b$$

$$\| x \| \leq -a^T x + b$$

Recap

- Cones and convex cones
- Conic combinations and conic hull
- Examples
- Second order conic program (SOCP)
- Example: Robust linear programming

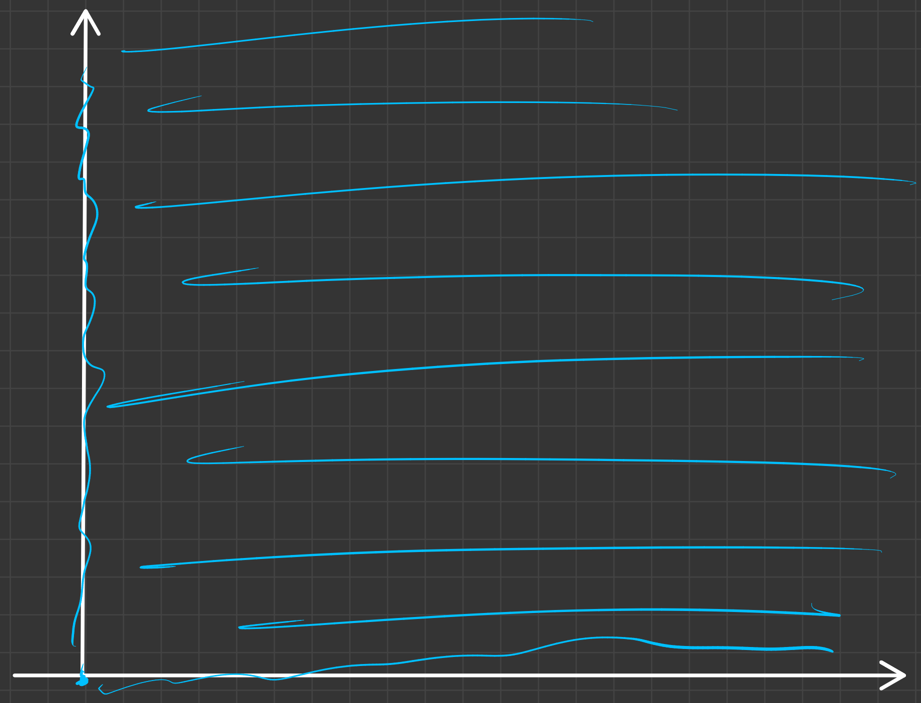
Proper cone

① **Convex** $\forall\, \underline{x}_1, \underline{x}_2 \in K,$ $\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \in K$ $\forall\, \alpha \in [0,1]$

② **Closed** $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0 \;\&\; x_2 > 0$ or $x_1 = x_2 = 0\}$

③ **Solid** K has a nonempty interior
$\exists\ \epsilon > 0 \;\&\; \underline{x} \in K$ ST $B(\underline{x}, \epsilon) \subseteq K$

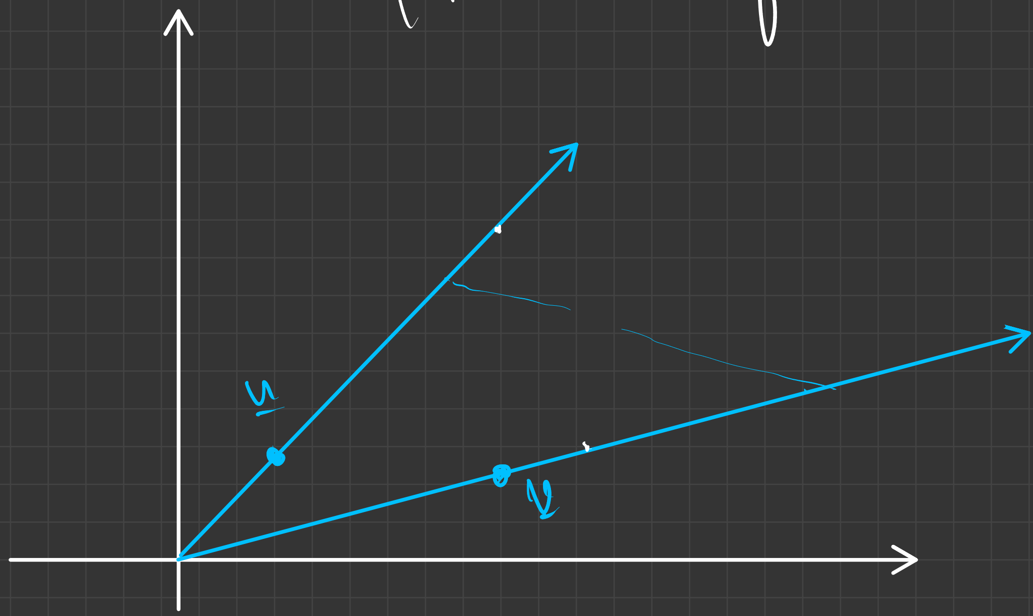
④ **Pointed** $\underline{x} \in K \;\&\; \underline{x} \neq \underline{0} \Rightarrow -\underline{x} \notin K$



Examples:

① Nonconvex: $K = \{\underline{x} = \alpha \underline{v} \text{ for some } \alpha \geq 0\} \cup$

$\{x = \alpha \underline{u} \text{ for some } \alpha \geq 0\}$



- Not solid
- Closed
- Pointed

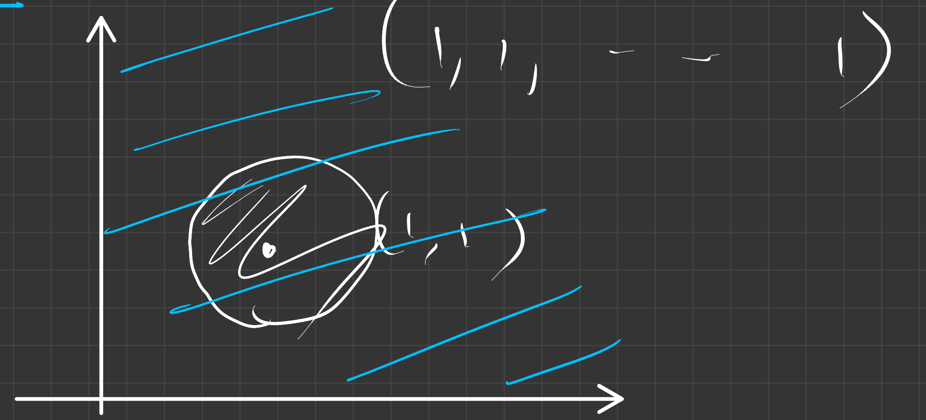
$$\underline{x}_k = \begin{cases} \alpha_k \underline{u}, & k \text{ odd} \\ \beta_k \underline{v}, & k \text{ even} \end{cases}$$

② Nonnegative orthant

$$K = \{ \underline{x} \in \mathbb{R}^n : x_i \geq 0 \quad \forall i \}$$

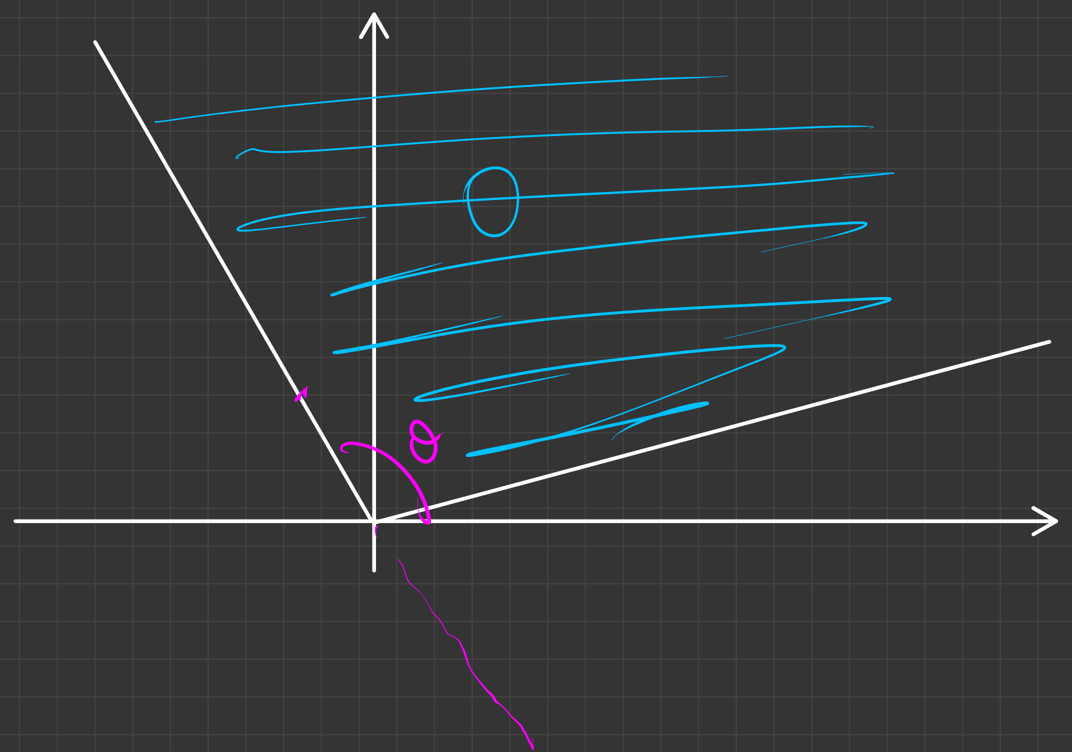
- Cone
- Convex — $\underline{x}_1 \; \underline{x}_2 \in K$, $\theta_1 \underline{x}_1 + \theta_2 \underline{x}_2 \in K$ for $\theta_1 \geq 0, \theta_2 \geq 0$
- Closed
- Pointed — Consider any $\underline{x} \in K$, $-\underline{x} \in K \Rightarrow \underline{x} = \underline{0}$
- Solid

This is a proper cone



③ Consider any subspace of $\mathbb{R}^n$.

- Convex cone
- Closed
- Solid only if dim $= n$
- Pointed? No, unless dim $= 0$



$\theta < 180^\circ \Leftrightarrow$ pointed

① PSD matrices → Proper cone

- Convex cone
- Closed
- Solid ✓ — Pointed ✓

↓

$\exists$ some PSD A, $\epsilon > 0$ st

$\|X - A\| \leq \epsilon \Rightarrow X \in S^n_+$

$A = I$

$\forall X$ st $\|X - I\| \leq \epsilon \Rightarrow X$ is PSD

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} - I = \begin{bmatrix} a-1 & c \\ c & b-1 \end{bmatrix}$$

$$(a-1)^2 + (b-1)^2 + 2c^2 \le 0.1$$

$$\lambda I - A = \begin{bmatrix} \lambda - a & -c \\ -c & \lambda - b \end{bmatrix}$$

$$\det(\lambda I - A) = (\lambda - a)(\lambda - b) - c^2$$

$$= \lambda^2 - \lambda(a+b) + ab - c^2$$

A is PSD iff $a^2 + b^2 + 4c^2 - 2ab \le 0$

$$(a-b)^2 + 4c^2 \le 0$$

$$\left.\begin{array}{l} a+b \geq 0 \\ ab-c^2 \geq 0 \qquad ab \geq c^2 \end{array}\right\} \text{Conditions for } A \text{ to be PSD}$$

Suppose,
$$\underbrace{(a-1)^2}_{\geq 0} + \underbrace{(b-1)^2}_{\geq 0} + 2c^2 \leq 0.1 \quad \text{— (1)}$$

Exercise: ST (1) $\Rightarrow$ A is PSD.

Matrix inner product

$$\langle A, B \rangle = \mathrm{Tr}(A^T B)$$

$$\|A\| = \sqrt{\mathrm{tr}(A^T A)}$$

$$\left\| \begin{bmatrix} a & c \\ c & b \end{bmatrix} \right\| = \sqrt{a^2 + b^2 + 2c^2}$$

$$\|A\| = \sqrt{\sum_i a_{ii}^2 + \sum_i \sum_{j=i+1}^{n} 2a_{ij}^2}$$

$$B(A, \epsilon) = \{ X \in S^n : \|X - A\| \le \epsilon \}$$

Solid cones

Solid v/s open set

Every open set is solid

eg: $\{x \in \mathbb{R}^n ; \ \|\underline{x}\|_2 < 1\}$

$\{x \in \mathbb{R}^n ; \ \|\underline{x}\|_2 \leq 1\}$

Not every solid set is open

But every solid set contains some open set.

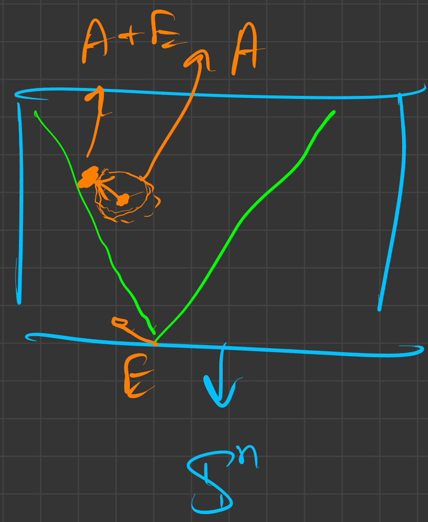
The set of PSD matrices is a solid cone

<u>Claim</u> : S^n_{++} (set of P.D matrices) = $\text{int}(S^n_+)$

If A is P.D. , then we want to ST $\exists\ \epsilon > 0$

so for all symmetric E with
$tr(E^T E) \le \epsilon^2$

$A + E \in S^n_+$



Let $\lambda_1\ \lambda_2 \ldots \lambda_n$ are eigenvalues of A

$\underline{v}_1\ \underline{v}_2 \ldots \underline{v}_n$ are orthonormal e.v.

$$\lambda_{min} = \min_{1 \le i \le n} \lambda_i > 0 \quad \text{as } A \text{ is P.D}$$

Choose $\epsilon < \lambda_{min}$

$$E \in S^n , \qquad tr(E^T E) \le \epsilon^2$$

$$tr(E^2) \le \epsilon^2$$

Assume $e_1\ e_2 \ldots e_n$ are eigenvalues of E

$$\Rightarrow e_1^2\ e_2^2 \ldots e_n^2 \quad \text{———}''\text{———} \quad \epsilon^2$$

$$\sum_{i=1}^{n} e_i^2 \le \epsilon^2$$

$$|e_i| \le \epsilon$$

$$-\epsilon \le e_i \le \epsilon$$

$$\lambda_{min}(A+E) > 0$$

Suppose $\underline{u}_1\ \underline{u}_2 \cdots \underline{u}_n$ are eigenvectors of E

$e_1\ e_2 \cdots e_n$

$$C = A + E$$

$\downarrow$

$\underline{w}_1 \cdots \underline{w}_n$ are eigenvectors of C

$\mu_1 \cdots \mu_n$ are eigenvalues

$$\underline{x} = \sum_{i=1}^{n} \alpha_i \underline{v}_i = \sum_{i=1}^{n} \beta_i \underline{u}_i = \sum_{i=1}^{n} \gamma_i \underline{w}_i$$

$$\|x\|^2 = \sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n} \beta_i^2 = \sum_{i=1}^{n} \gamma_i^2$$

$$\underline{x}^T C \underline{x} = \left(\sum_{i=1}^{n} r_i \underline{w}_i\right)^T C \left(\sum_{j=1}^{n} r_j \underline{w}_j\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} r_i r_j \; \underline{w}_i^T C \underline{w}_j$$

$$= \sum_{i=1}^{n} r_i r_j \; \mu_j \; \underline{w}_i^T \underline{w}_j$$

$$= \sum_{i=1}^{n} r_i^2 \; \mu_j$$

$$\underline{x}^T A \underline{x} = \sum_{i=1}^{n} \alpha_i^2 \lambda_i$$

$$\underline{x}^T E \underline{x} = \sum_{i=1}^{n} \beta_i^2 e_i$$

$$\underline{x}^T C \underline{x}$$

$$x^T C x = \sum_{i=1}^{n} r_i^2 \mu_i = \sum_{i=1}^{n} \left(\alpha_i^2 \lambda_i + \beta_i^2 e_i \right)$$

convex combn of μ_j

eigenvalues of $A+E$

e.v of A

e.v of E

$$\|x\| = 1 \Rightarrow \underbrace{\sum_{i=1}^{n} r_i^2 = 1}_{\text{convex comb.}}$$

$$\sum_{i=1}^{n} r_i^2 \mu_i \geq \sum_{i=1}^{n} \left(\alpha_i^2 \lambda_{min} + \beta_i^2 e_{min} \right)$$

$$= \left(\sum_{i=1}^{n} \alpha_i^2 \right) \lambda_{min} + \left(\sum_{i=1}^{n} \beta_i^2 \right) e_{min}$$

$$= \lambda_{min} + e_{min}$$

μ_{min} can be obtained by choosing some $(r_1 \dots r_n)$

$$\mu_{min} \geq \lambda_{min} + \ell_{min}$$

$$\mu_{max} \leq \lambda_{max} + \ell_{max}$$

But from our choice of E, ϵ,

min E.V of $E = \ell_{min} \geq -\epsilon$

min e.v of $A = \lambda_{min} > \epsilon$

$$\mu_{min} > \epsilon - \epsilon = 0$$

$\therefore$ $A + E$ is P.S.D.

$$\sum_{i=1}^{n} r_i^2 \mu_i \geq \lambda_{min} + \ell_{min} \qquad \forall (r_1 \cdots r_n)$$

$$\text{ST } \sum_i r_i^2 = 1$$

$$\mu_1 \geq \mu_2 \cdots \geq \mu_n$$

Choose $r_1 = 0 = r_2 \cdots = r_{n-1}$ & $r_n = 1$

$$\mu_{min} = \mu_n \geq \lambda_{min} + \ell_{min}$$

<u>EX</u> : ST if A is PSD NOT P.D, then A is not an interior pt of S_+^n

Generalized inequalities

Consider $\underline{x}, \underline{y} \in \mathbb{R}^n$

$\underline{x} \preceq \underline{y}$ if $x_i \leq y_i \ \forall i$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 2 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

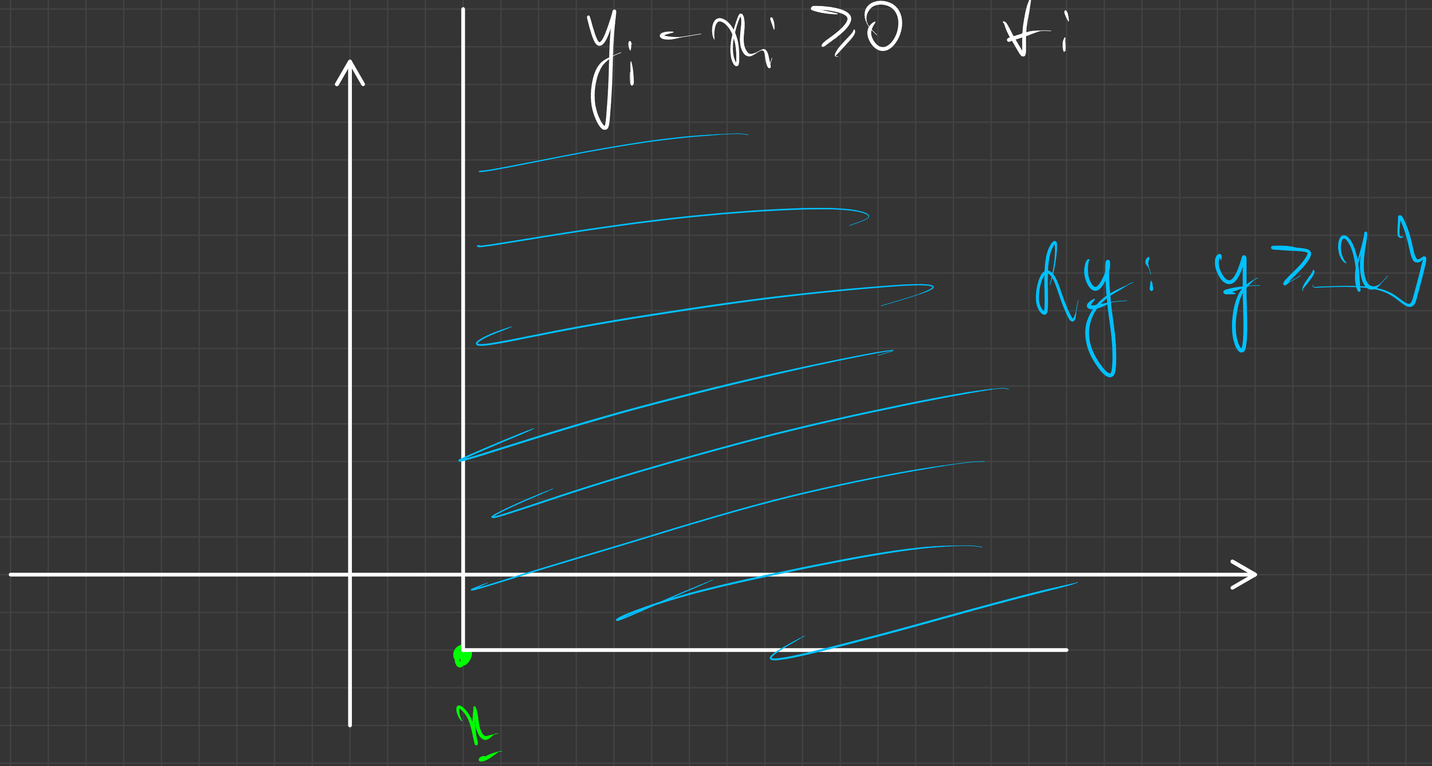
$$\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

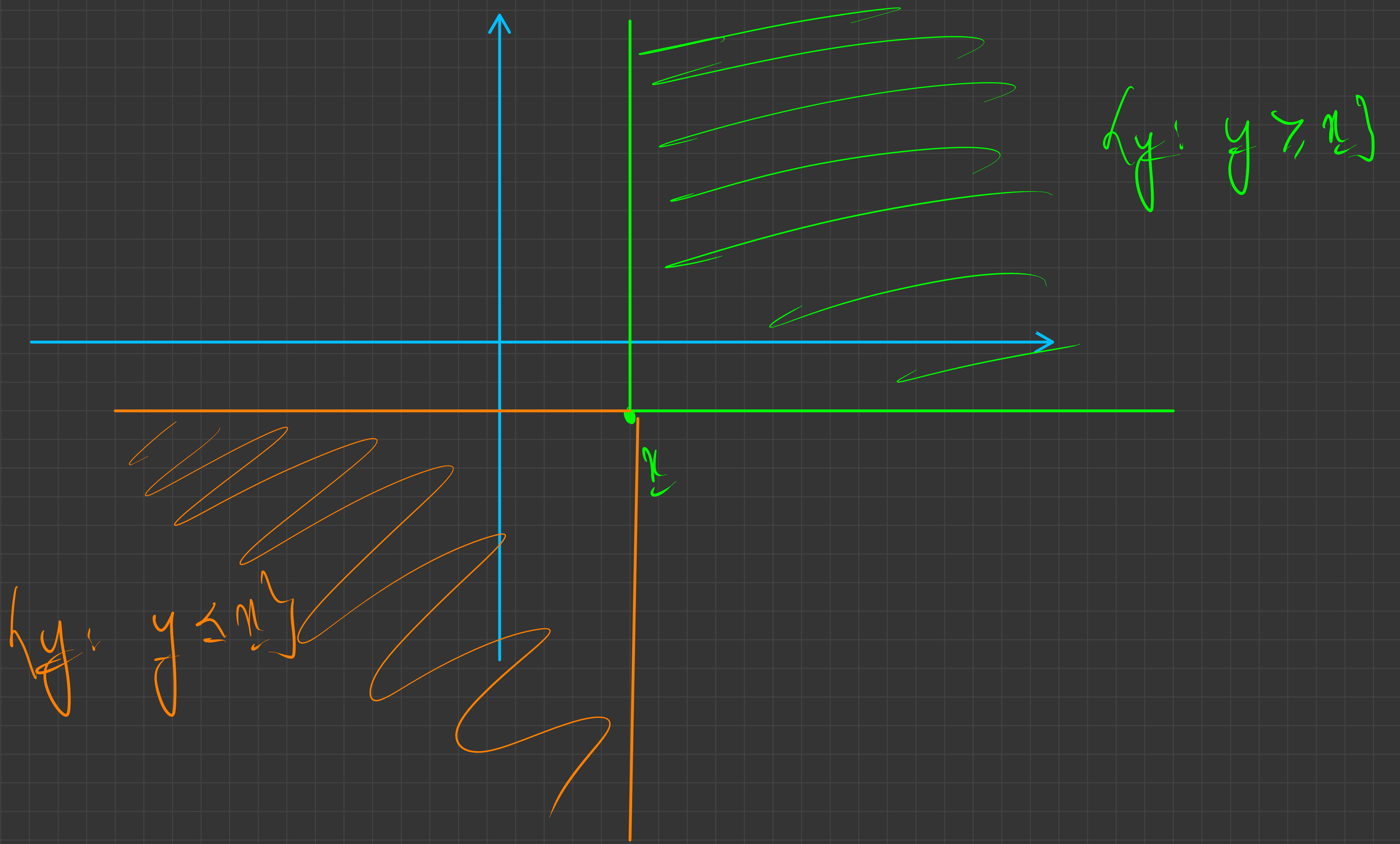
$f_1(\underline{x}) \qquad f_2(\underline{x})$

Defn: If K is a proper cone, we say $\underline{x} \preceq_K \underline{y}$ if $\underline{y} - \underline{x} \in K$

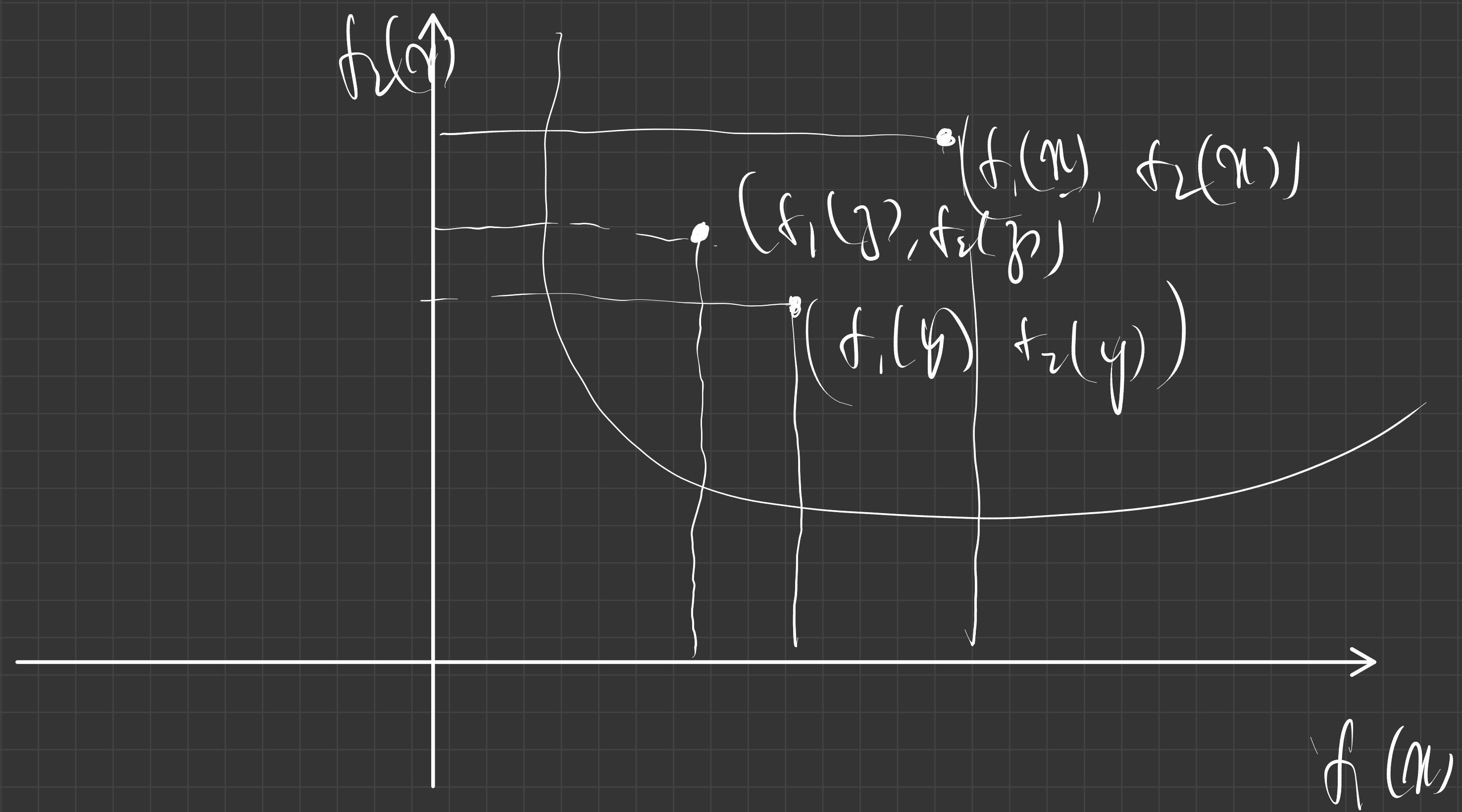
Eg: $\underline{x} \leq \underline{y}$ $\quad x_i \leq y_i \quad \forall i$

$y_i - x_i \geq 0 \quad \forall i$





$\{y : y \geq x\}$
x
$\{y : y \leq x\}$


$f_2(x)$
$(f_1(x), f_2(x))$
$(f_1(y), f_2(y))$
$(f_1(y), f_2(y))$
$f_1(x)$

Generalized inequality induces a partial ordering

We say that $\preceq$ is an inequality that induces a partial order if

① Reflexive : $x \preceq x \quad \forall x$

② Antisymmetric : $x \preceq y \;\&\; y \preceq x \Rightarrow y = x$

③ Transitive : $x \preceq y \;\&\; y \preceq z \Rightarrow x \preceq z$.

Total order → ④ $\forall x, y \quad x \preceq y$ or $y \preceq x$

Eg: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \preceq \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ if $x_1 \leq y_1$ $\quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Not antisymmetric

① Reflexive $\quad \underline{x} \in \mathbb{V}$

$\underline{x} \preceq_K \underline{x}$ because $\underline{x} - \underline{x} = \underline{0} \in K$

$\Downarrow$

since K is a cone

② Antisymmetric $\quad \underline{x} \preceq_K \underline{y}$ & $\underline{y} \preceq_K \underline{x}$

$\Downarrow \qquad \Downarrow$

$\underline{y} - \underline{x} \in K \qquad \underline{x} - \underline{y} \in K$

$-(\underline{y} - \underline{x}) \in K$

This can happen only if $\underline{y} - \underline{x} = 0$ (as K is pointed)

(3) Transitive

$x \preceq_K y$ & $y \preceq_K z$

$(y-x) \in K$ $(z-y) \in K$

Since K is a convex cone,

$(y-x) + (z-y) \in K$

$z-x \in K$

$\Rightarrow$ $x \preceq_K z$

Other properties

① $\underline{x} \preceq_K \underline{y}$ & $\underline{u} \preceq_K \underline{v}$ $\Rightarrow$ $\underline{x}+\underline{u} \preceq_K \underline{y}+\underline{v}$

② $\underline{x} \preceq_K \underline{y}$ & $\alpha \geq 0$ $\Rightarrow$ $\alpha\underline{x} \preceq_K \alpha\underline{y}$

③ $\underline{x}_i \preceq_K \underline{y}_i$ $\Rightarrow$ $\lim_{i\to\infty} \underline{x}_i \preceq_K \lim_{i\to\infty} \underline{y}_i$

Minimum and minimal elements

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

① Given S & a proper cone K,

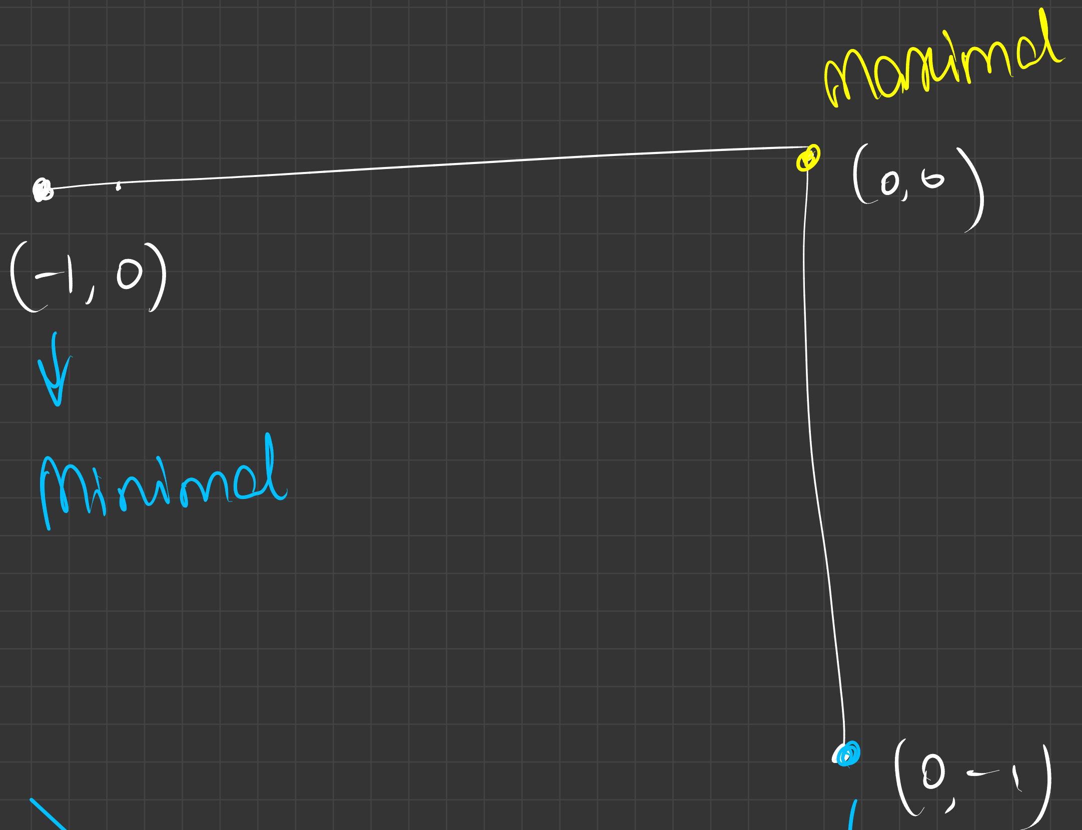
We say that $\underline{x}^*$ is the minimum of S under $\preceq_K$

if $\underline{x}^* \preceq_K \underline{x} \quad \forall \underline{x} \in S$

② We say that y is a <u>minimal</u> element of S if

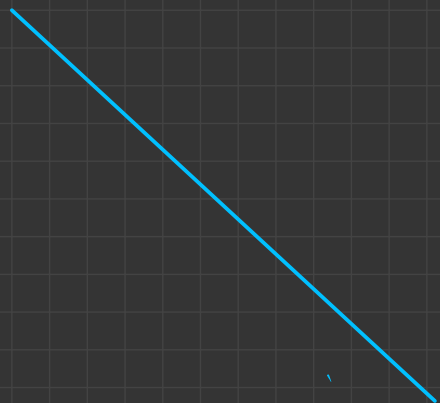
if $\underline{x} \preceq_K y \Rightarrow y = \underline{x}$

1



$$\left\{ \underline{x} = \begin{bmatrix}\alpha\\0\end{bmatrix} \text{ or } \begin{bmatrix}0\\\alpha\end{bmatrix} \quad \alpha \in [-1, 0] \right\}$$

2



In 2, all points are minimal

(3) $K = \{ \underline{x} \in \mathbb{R}^n : x_i \leq 0 \ \forall i \}$

$\underline{x} \preceq_K \underline{y} \qquad \underline{y} - \underline{x} \in K$

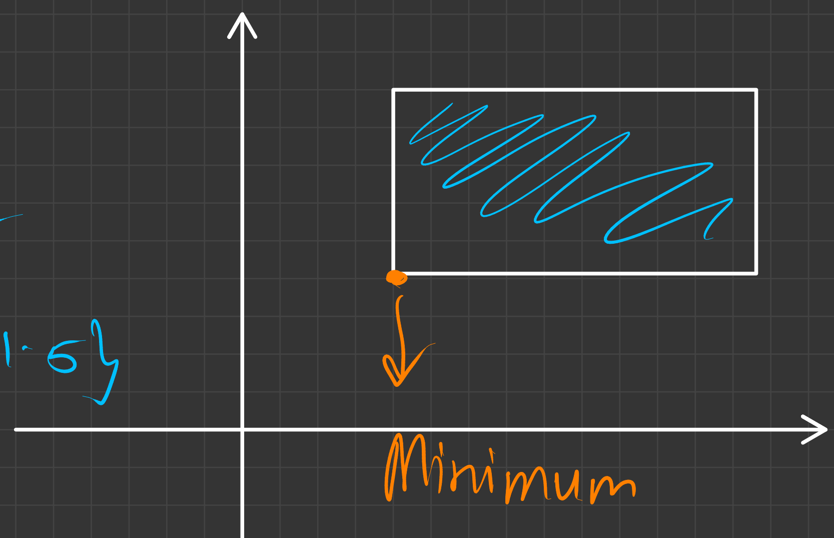
$y_i - x_i \leq 0$

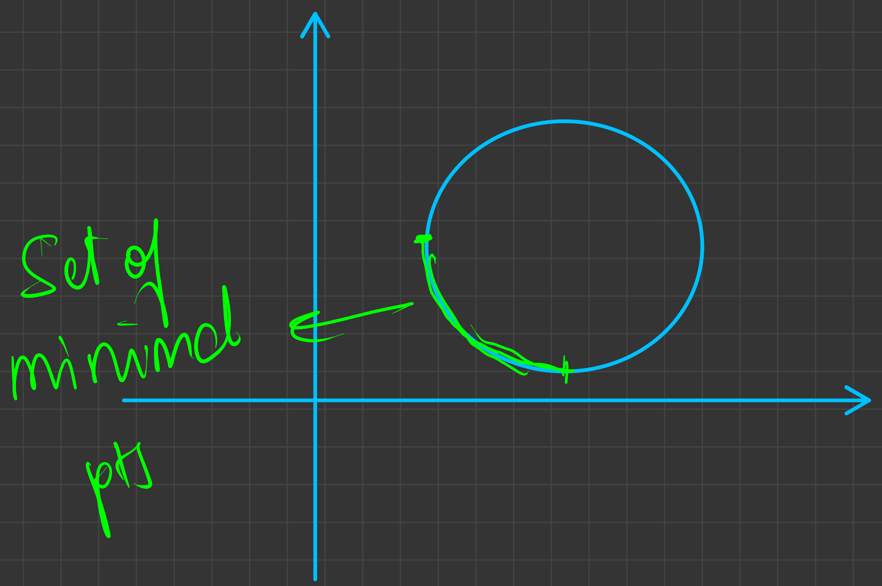
$y_i \leq x_i$

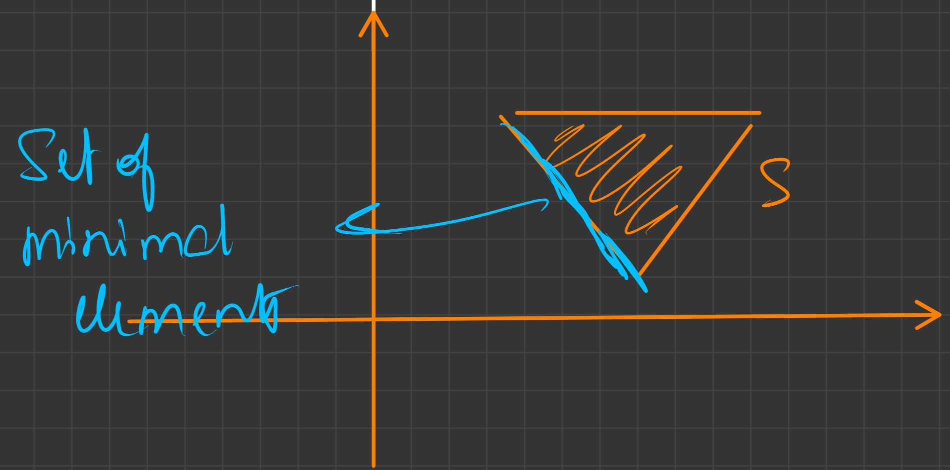
Examples

① Consider the componentwise inequality

$$S = \{ (x_1, x_2) : 1 \le x_1 \le 2, \quad 1 \le x_2 \le 1.5 \}$$





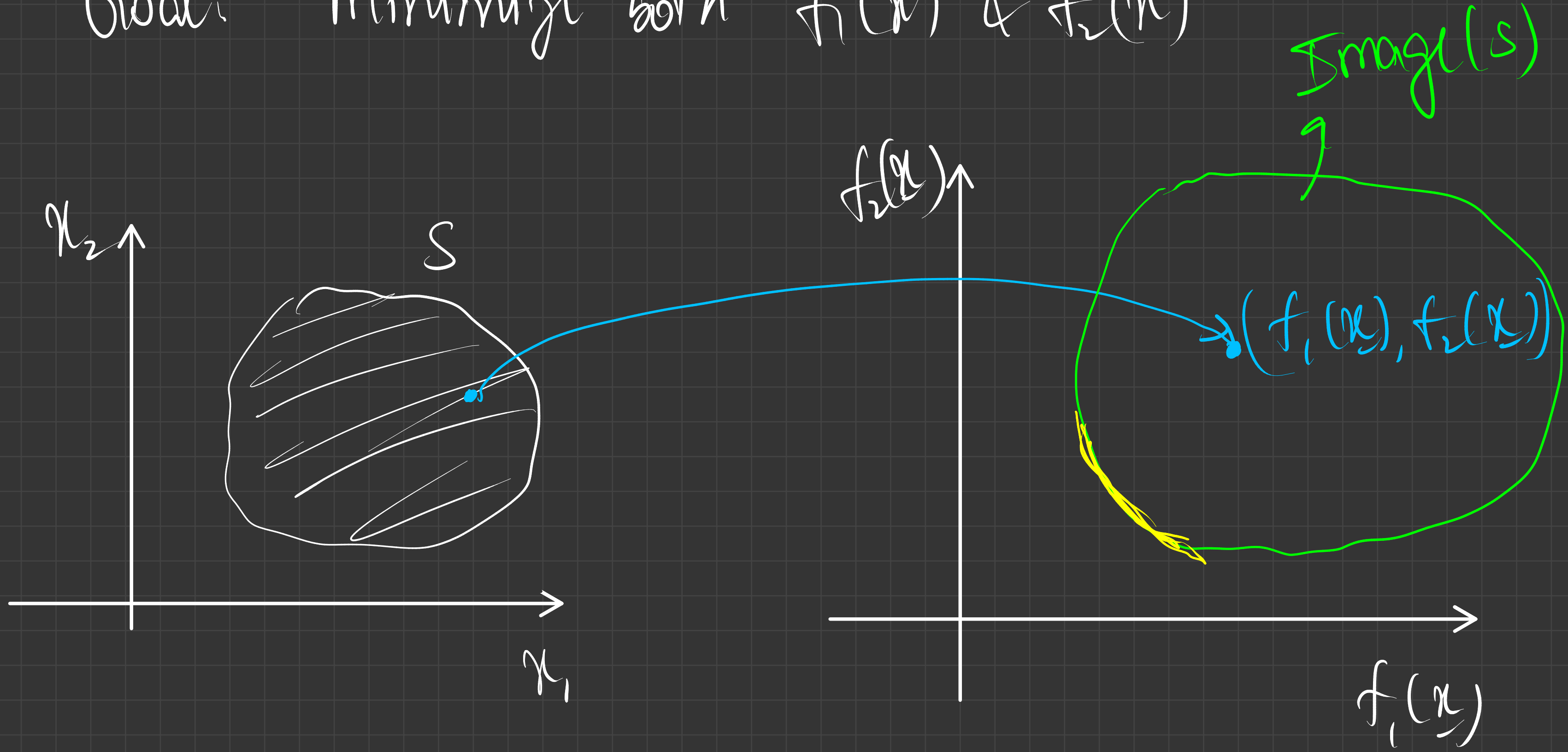


Examples

$f_1(\underline{x}) \quad f_2(\underline{x})$

$\underline{x} \in S$

Goal: Minimize both $f_1(\underline{x})$ & $f_2(\underline{x})$



Goal: Find set of all minimal pts of Im(S)
(under componentwise inequality)

= Pareto optimal points

$f : V \to \mathbb{R}$

$$f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) \leq \alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)$$

$f : V \to \mathbb{R}^m$ $\longrightarrow$ WRT some generalized inequality $\preceq_K$

we say f is convex if

$$f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) \preceq_K \alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)$$

$$\underline{f}(\underline{x}) = \begin{bmatrix} f_1(\underline{x}) \\ f_2(\underline{x}) \\ \vdots \\ f_m(\underline{x}) \end{bmatrix}$$

Dual Cone

Given a cone K, the dual cone

$$K^* = \{ \underline{x} : \langle \underline{x}, y \rangle \geq 0 \quad \forall\, y \in K \}$$

Q1: Is K^* a cone? YES

If $\underline{x} \in K^*$, $\langle \underline{x}, y \rangle \geq 0 \quad \forall y \in K$

$\Rightarrow \langle \alpha \underline{x}, y \rangle \geq 0 \; \forall\, y \in K$
$\forall\, \alpha \geq 0$

$\Rightarrow \alpha \underline{x} \in K^* \quad \forall\, \alpha \geq 0$

② K^* is convex

$$x_1, x_2 \in K^* \quad \& \quad \theta_1\ \theta_2 \geq 0$$

$$\langle \theta_1 x_1 + \theta_2 x_2, y \rangle = \theta_1 \langle x_1, y \rangle + \theta_2 \langle x_2, y \rangle$$

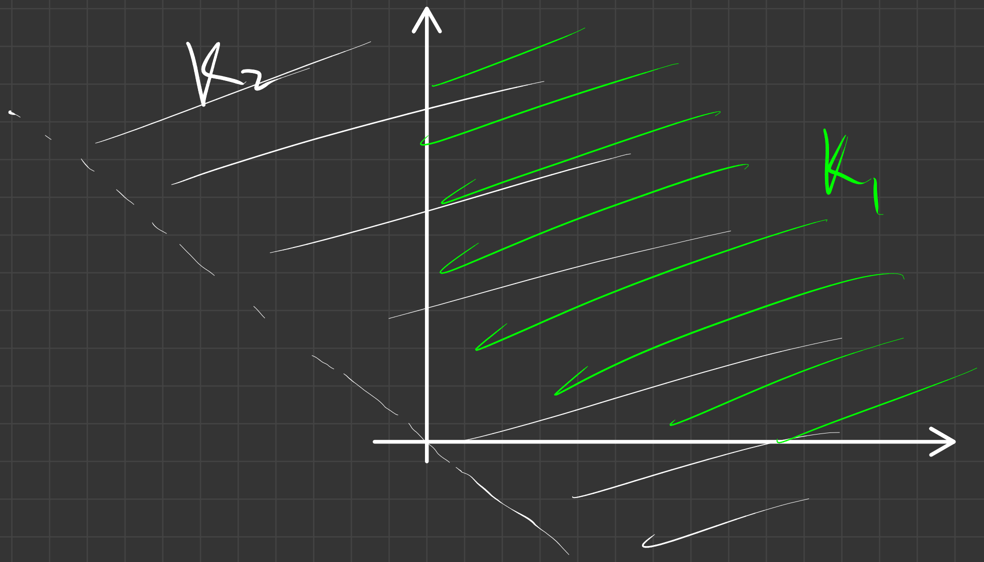
$$\geq 0 \quad \text{as } x_1, x_2 \in K^*$$

$$\Rightarrow \theta_1 x_1 + \theta_2 x_2 \in K^*$$

③ $K_1 \subseteq K_2 \qquad K_1^* \supseteq K_2^*$

Consider any $x \in K_2^* \qquad \langle x, y \rangle \geq 0 \quad \forall y \in K_2$

$$\Rightarrow x \in K_1^*$$



$$K_1 = \{ \underline{x} : x_i \geq 0 \ \forall i \}$$

$$K_2 = \{ \underline{x} : x_1 + x_2 \geq 0 \}$$

$$K_3 = \{ \underline{x} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \alpha \geq 0 \}$$

$$K_2 \supseteq K_1 \supseteq K_3$$

$$K_2^* \subseteq K_1^* \subseteq K_3^*$$

$$K_1^* = K_1, \quad K_2^* = K_3 \ \& \ K_3^* = K_2$$

$K_1 = \{ \underline{x} : x_i \geq 0 \ \forall i \}$

Suppose $\underline{y}$ $\quad y_j < 0$ $\qquad \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ \vdots \end{bmatrix} \rightarrow j$ th comp

$\underline{y} \notin K^*$

Any $\underline{y} \in K_1$ also lies in K_1^*

$K_1 = K_1^*$ (Self dual)

$K_2 = \{ \underline{x} : x_1 + x_2 \geq 0 \}$

Every $\begin{bmatrix} \alpha \\ \alpha \end{bmatrix} : \alpha \in \mathbb{R}_{\geq 0}$ lies in K_2^*

Consider $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} : \alpha \neq \beta$

If $\alpha \neq \beta$ $\qquad$ $\alpha - \beta < 0$ $\quad$ OR $\quad$ $\beta - \alpha < 0$

$< -\epsilon$ $\qquad\qquad$ $< -\epsilon$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} 1 \\ -1+\delta \end{bmatrix} \qquad\qquad \begin{bmatrix} -1+\delta \\ 1 \end{bmatrix}$$

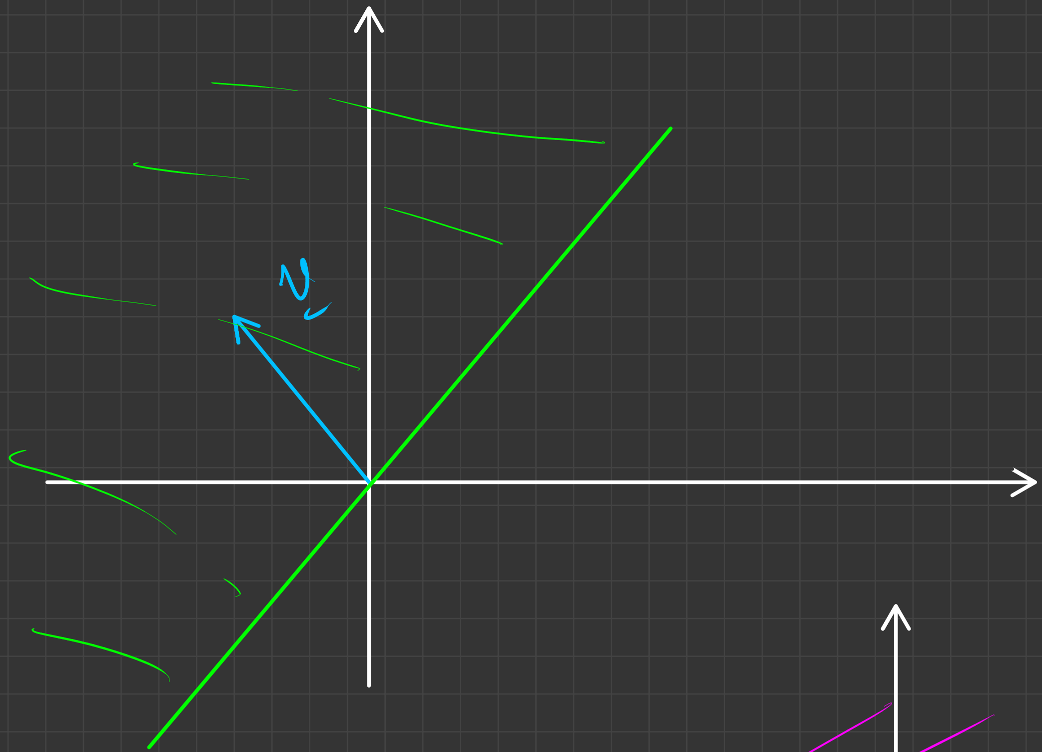
$$\underbrace{\alpha - \beta}_{< -\epsilon} + \underbrace{\beta\delta}_{< \epsilon/2}$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \notin K_2^*$$

$$\Rightarrow \quad K_2^* = K_3$$

$$\bar{K}_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \ x_1 + x_2 > 0 \ \text{ OR } \ x_1 = x_2 = 0 \right\}$$

$$\bar{K}_2^* = K_3$$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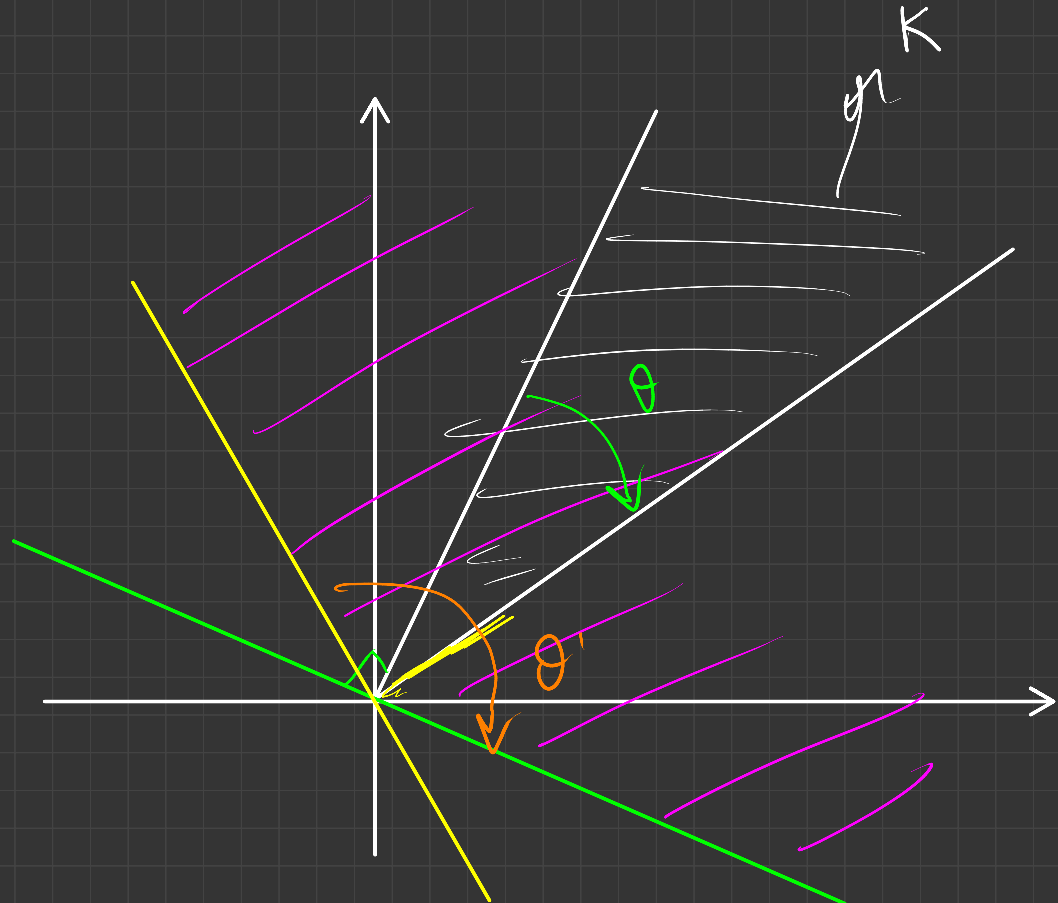
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K

Properties

① K^{**} = Closure of Convex hull of K

$K^{**} = K$ if K is closed & convex

② If K is proper, then K^{*} is also proper

Examples

1. $K =$ a subspace

2. $K =$ ray

3. Halfspace

4. Second order cone

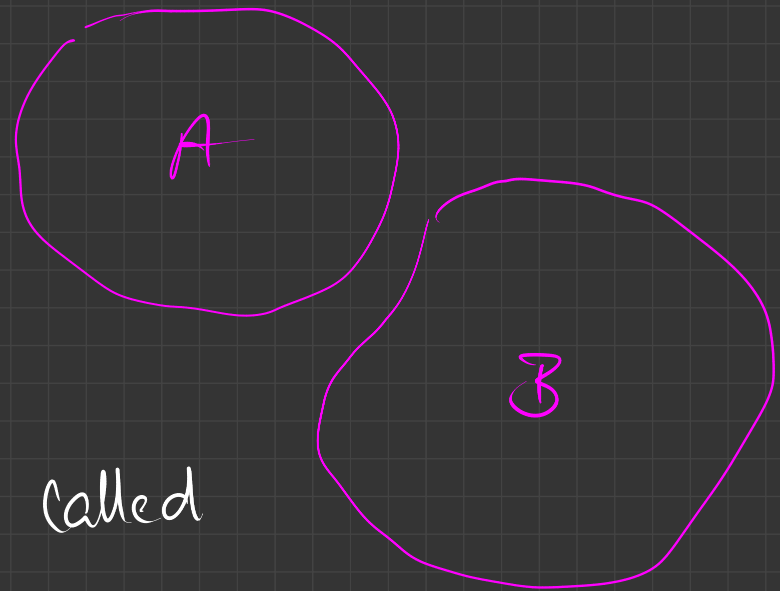
5. PSD matrices

Hyperplane:

$$\{\underline{x} : \langle \underline{a}, \underline{x} \rangle = b\}$$

$\underline{x} \in A \Rightarrow \langle \underline{a}, \underline{x} \rangle \geq b$

$\underline{x} \in B \Rightarrow \langle \underline{a}, \underline{x} \rangle < b$



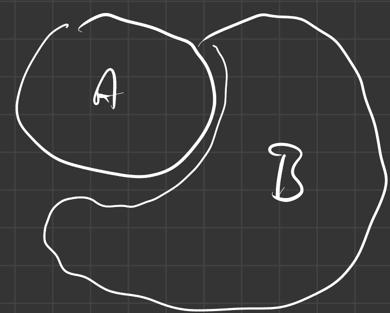
If $\exists$ $(\underline{a}, b)$, then

$\{\underline{x} : \langle \underline{a}, \underline{x} \rangle = b\}$ is called

a separating hyperplane

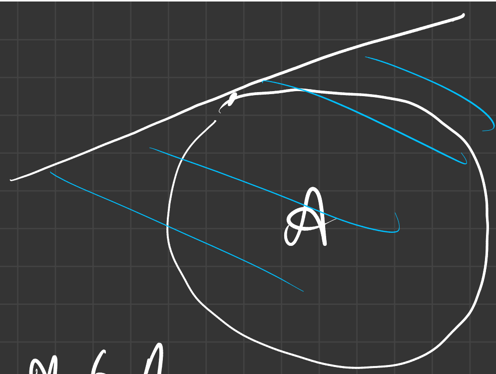
Theorem: If A & B are convex, & $A \cap B = \phi$,

then $\exists$ a separating hyperplane.



Supporting hyperplane:

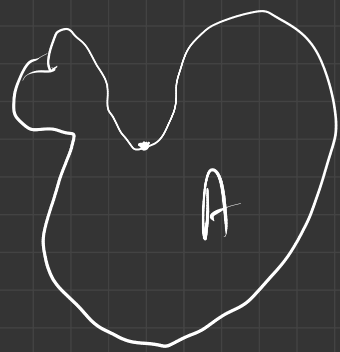
Consider $x_0 \in bd(A)$.



If $\langle \underline{a}, \underline{x} \rangle \geq \langle \underline{a}, \underline{x}_0 \rangle \ \forall\, \underline{x} \in A$,

then $\{ y : \langle \underline{a}, y \rangle = \langle \underline{a}, \underline{x}_0 \rangle \} \to$ supporting hyperplane

Theorem: If A is convex, Every point on $bd(A)$ has a supporting hyperplane



Note: Any closed & convex set A can be expressed as an intersection of halfspaces.

$A' = \mathrm{relint}(A)$

$B' = \{x_0\}$

polytope

x_0 is a vertex of A iff

$\exists$ a supporting hyperplane through

x_0 that contains exactly one pt of A.