Convex Sets

Reference: Chapter 2, Boyd & Vandenberghe

Why convex?

$$x^* = \underset{g(x) \le 0}{\arg\min} f(x)$$

- Linear programming: f & g are both linear
- Quadratic programming
- Semidefinite programming
- Convex optimization f is convex

  constraints form a convex set

Lines, and line segments

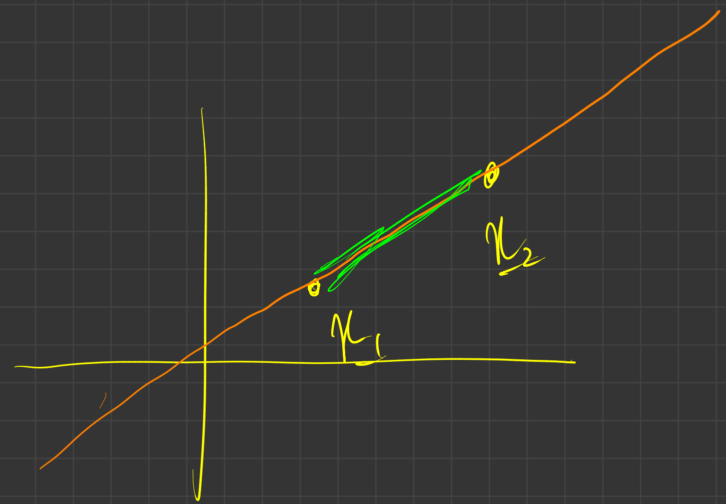
$x_1, x_2 \in \mathbb{R}^n$

$\{\alpha x_1 + (1-\alpha) x_2 : \alpha \in \mathbb{R}\}$ is the line passing through x_1, x_2

↓

$x_1 + (1-\alpha)(x_2 - x_1)$

base — direction



$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\begin{bmatrix} x - x_1 \\ y - y_1 \end{bmatrix} = \begin{bmatrix} \alpha(x_2 - x_1) \\ \alpha(y_2 - y_1) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \alpha \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \alpha \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \alpha \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + (1-\alpha) \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$= \alpha\, \underline{x}_1 + (1-\alpha)\, \underline{x}_2$$

$$\{ \underline{x} = \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 : 0 \leq \alpha \leq 1 \}$$

is the line segment joining $\underline{x}_1$ & $\underline{x}_2$

Affine sets

A set S is affine if

$\underline{x}_1, \underline{x}_2 \in S$, then $\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2 \in S$

$\forall \alpha \in \mathbb{R}$.

Consider $\mathbb{R}^2$.

- Any straight line is affine
- $\{\underline{x}\}$ is affine
- $\mathbb{R}^2$
- $A\underline{x} = \underline{b}$

x_1, x_2 are solutions of $Ax = b$

$$A(\alpha x_1 + (1-\alpha) x_2) = \alpha A x_1 + (1-\alpha) A x_2$$

$$= \alpha b + (1-\alpha) b$$

$$= b$$

Affine combination of $x_1, x_2 \ldots x_k$

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$$

$$\sum_{i=1}^{k} \alpha_i = 1$$

Every affine set is a shift of a vector subspace

Consider S affine, $\underline{c} \in S$

Claim: $S - \underline{c} = \{ \underline{x} - \underline{c} : \underline{x} \in S \}$ is a vector subspace

$$\alpha(\underline{x}_1 - \underline{c}) + \beta(\underline{x}_2 - \underline{c}) = \alpha \underline{x}_1 + \beta \underline{x}_2 - (\alpha + \beta)\underline{c}$$

$$= \underbrace{\alpha \underline{x}_1 + \beta \underline{x}_2 + (1 - \alpha - \beta)\underline{c}} - \underline{c}$$

Affine comb of $\underline{x}_1, \underline{x}_2, \underline{c}$ & hence in S

Dimension of affine space

$$\dim(S) = \dim(S - \underline{c})$$

↳ vector space dimension

Every affine set is the solution space of a system of linear equations

Every affine set can be written as

$$S = V + b$$

$$\downarrow$$

$$\{x : Ax = 0\}$$

$$S = \{x + b : Ax = 0\}$$

$$= \{y : A(y - b) = 0\}$$

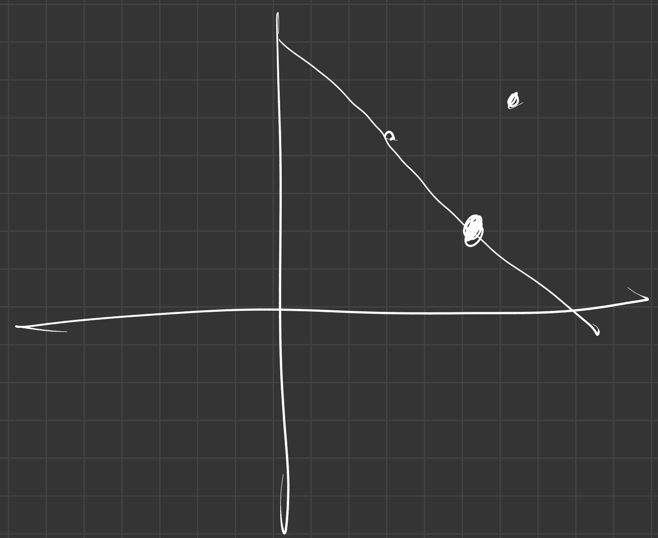
$$= \{y : Ay = Ab\}$$

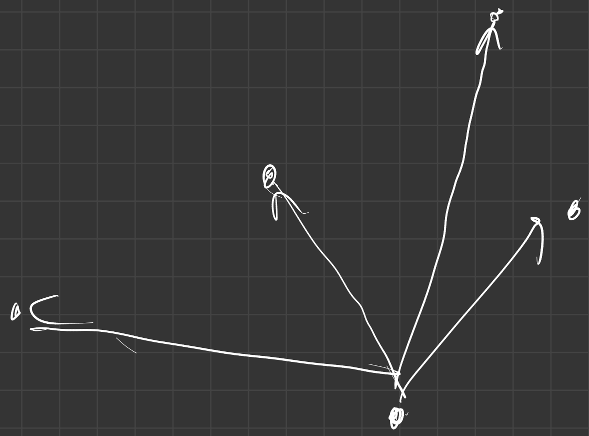
Affine hull, examples

Given a set $C \subseteq \mathbb{R}^n$, the affine hull

$$\mathrm{aff}(C) = \{ x = \sum_{i=1}^{k} \alpha_i x_i : x_1 \ldots x_k \in \mathbb{R}^n, \; \sum_{i=1}^{k} \alpha_i = 1 \}$$

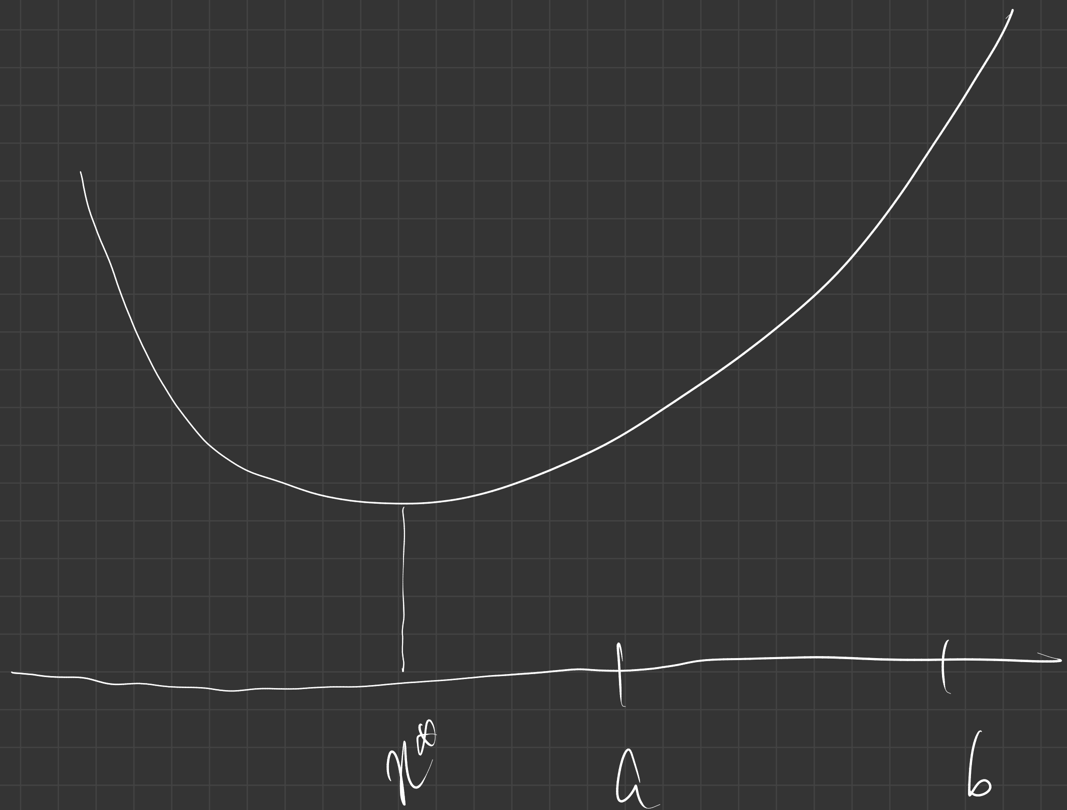
$\rightarrow$ Smallest affine set that contains C



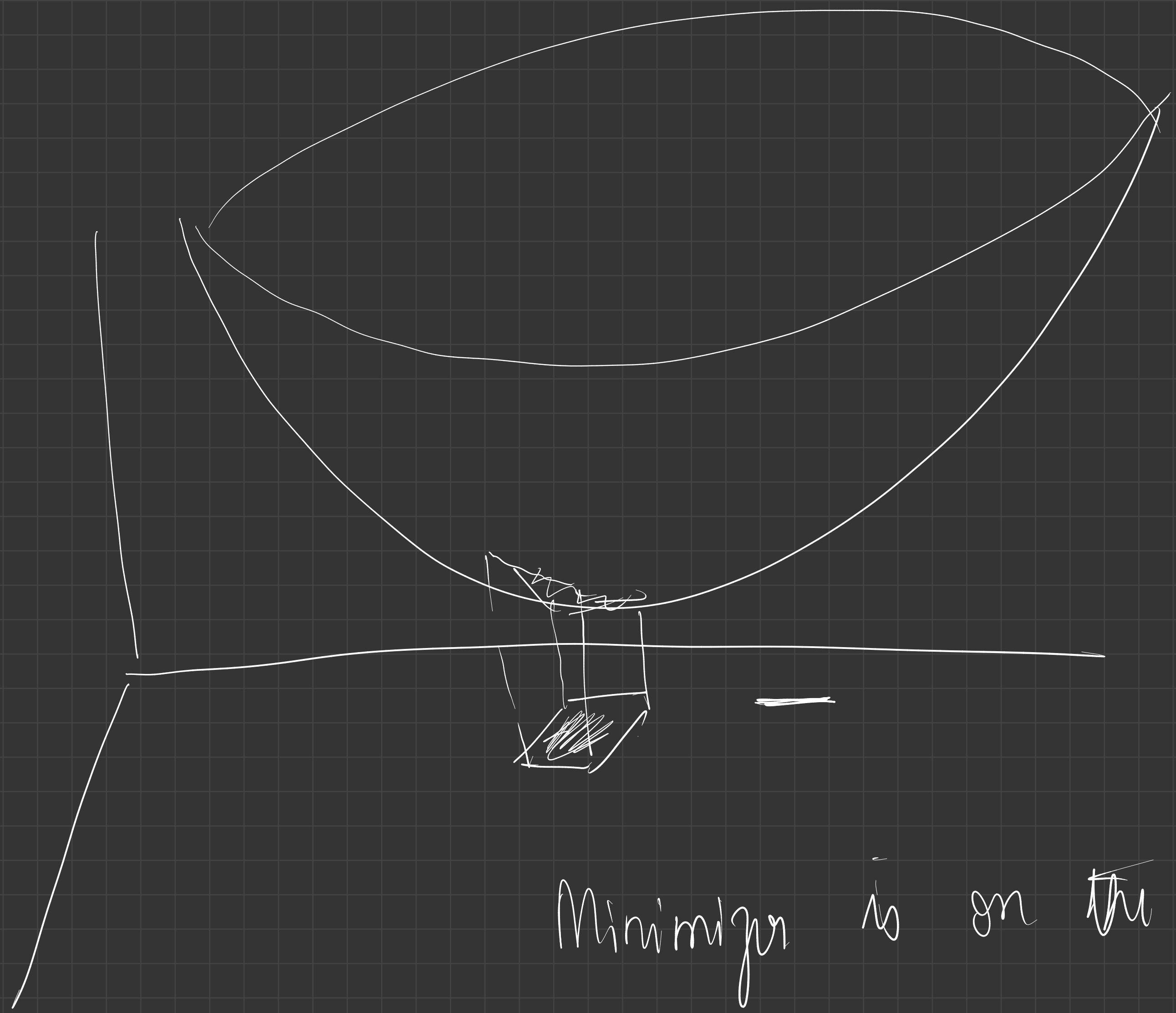


$\underline{x_0}$

P I $\quad \underset{x \in [a,b]}{\operatorname{argmin}} f(x)$



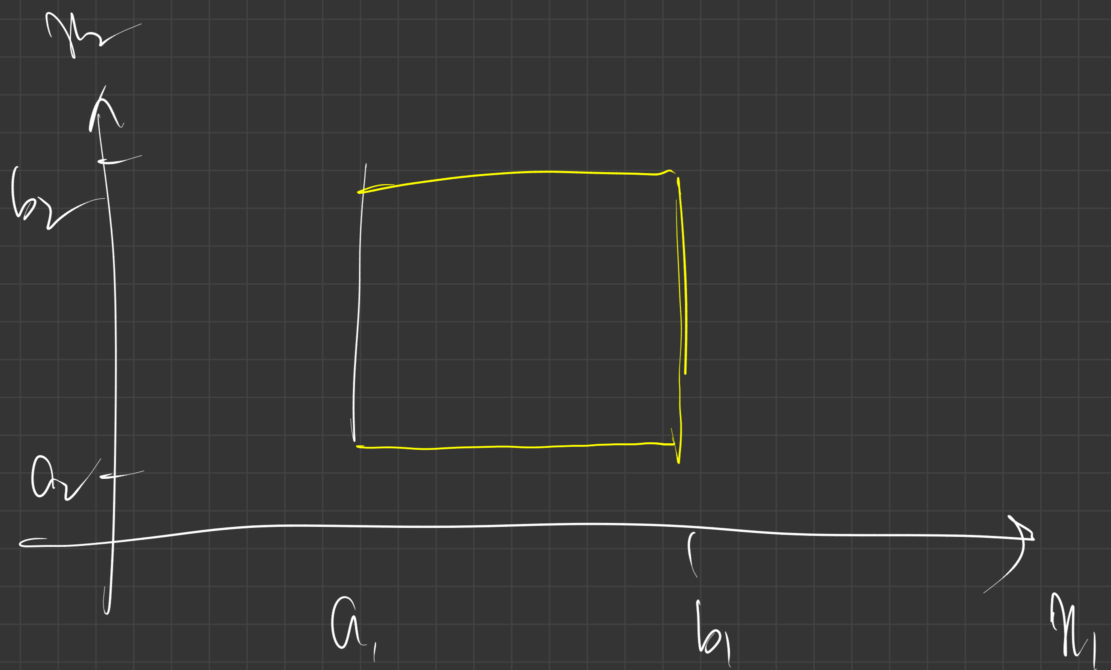
P II $\quad \operatorname{argmin} f(x)$



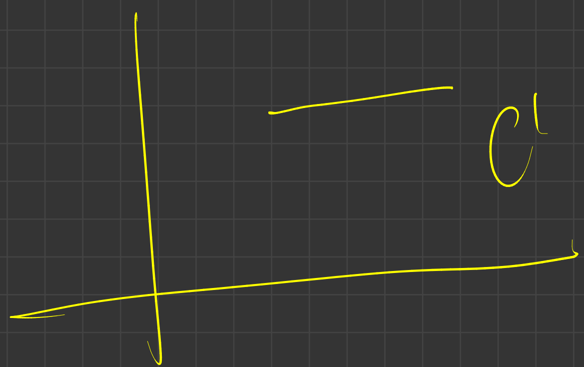
Minimizer is on the boundary

$$C = \left\{ \begin{array}{l} a_1 \le x_1 \le b_1 \\ a_2 \le x_2 \le b_2 \end{array} \right\}$$

$$\text{int}(C) = \left\{ \begin{array}{l} a_1 < x_1 < b_1 \\ a_2 < x_2 < b_2 \end{array} \right\}$$



$\text{bd}(C)$

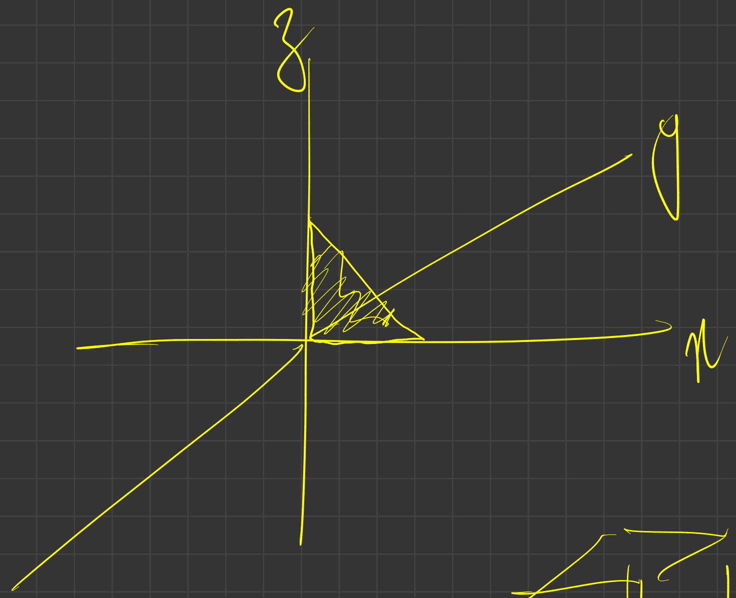


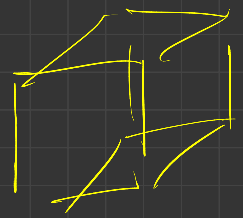
$\text{int}(C') = \phi$

Relative interior and boundary of a set

$$\text{Rel int}(C) = \{ \underline{x} : B_n(\underline{x}, \varepsilon) \cap \text{aff}(C) \subseteq C \text{ for some } \varepsilon > 0 \}$$

$$\text{bd}(C) = \text{closure}(C) \setminus \text{rel int}(C)$$


z
y
x



Convex combinations, convex sets and convex hulls

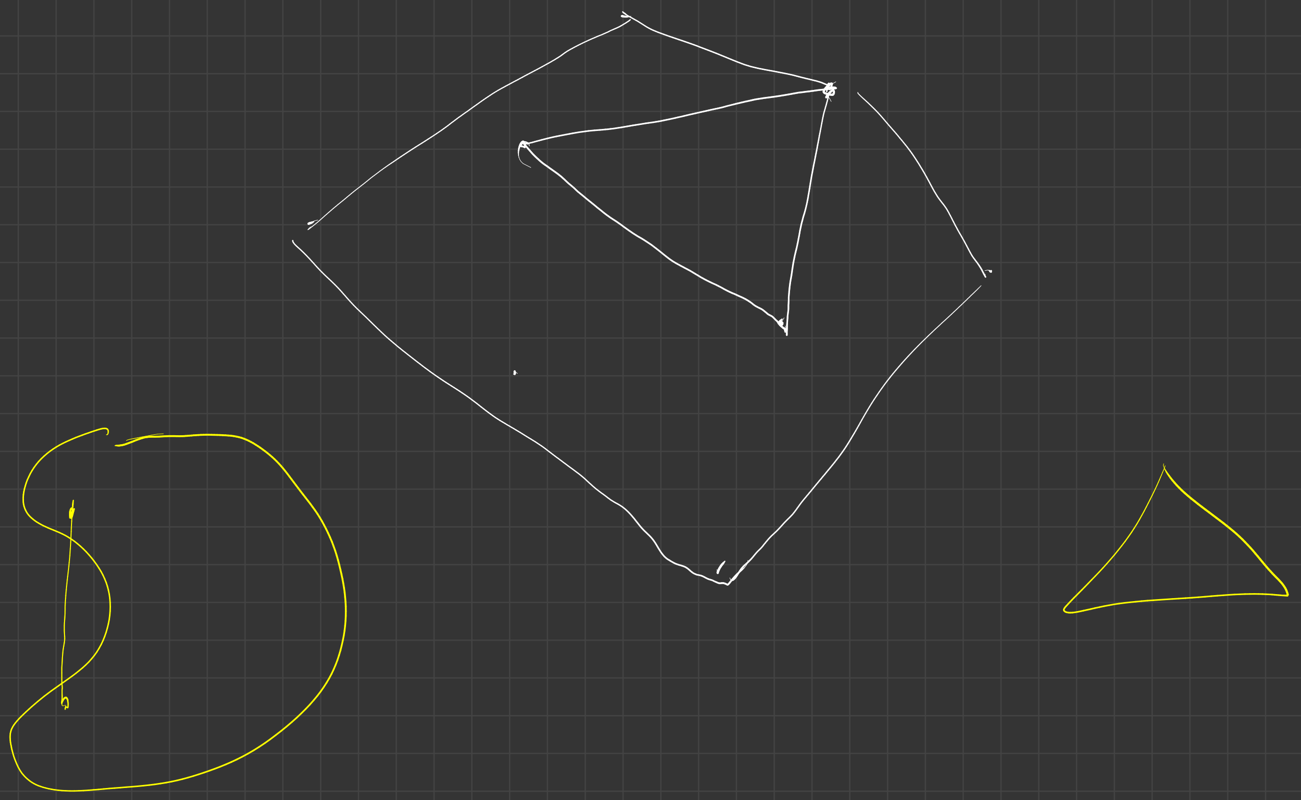
$$\alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 + \cdots + \alpha_k \underline{x}_k \;:\; \sum_{i=1}^{k} \alpha_i = 1$$

Convex combination of $\underline{x}_1 \ldots \underline{x}_k$ $\qquad \alpha_i \geq 0 \;\; \forall i$

C is convex if $\underline{x}_1, \underline{x}_2 \in C \Rightarrow \alpha_1 \underline{x}_1 + (1-\alpha) \underline{x}_2 \in C$

$\forall \alpha \in [0, 1]$

Convex hull : $\text{conv}(C)$ = set of all convex comb. of pts in C

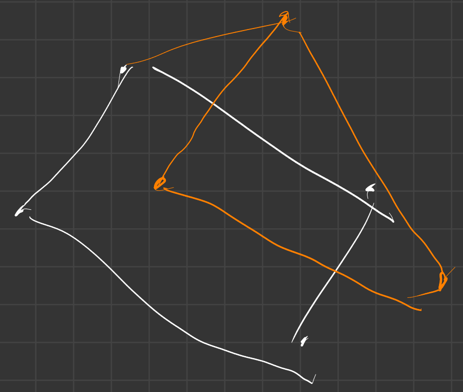


A, B $\quad A \cap B = \emptyset$

$\text{conv}(A) \cup \text{conv}(B)$

$\neq$

$\text{conv}(A \cup B)$



Infinite convex combinations

C $x_1, x_2, \ldots, x_n, \ldots$

$$\sum_{i=1}^{\infty} \alpha_i x_i \quad , \quad \alpha_i \geq 0 \quad \forall i$$

$$\sum_{i=1}^{\infty} \alpha_i = 1$$

$\underbrace{}$?? exists

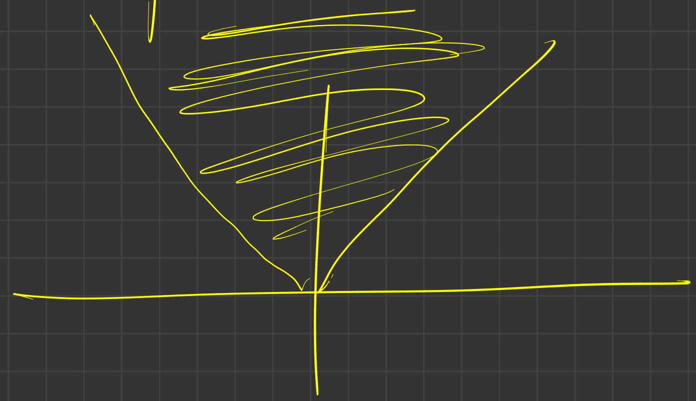
C $\quad f \rightarrow$ pdf on C $\quad \int_{x \in C} f(x)\, dx = 1$

$$\int_{x \in C} x\, f(x)\, dx$$

Cones and conic combinations

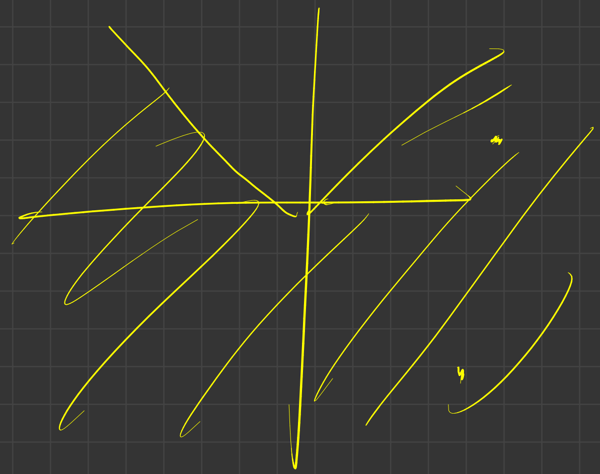
C is a cone if $\underline{x} \in C$, then

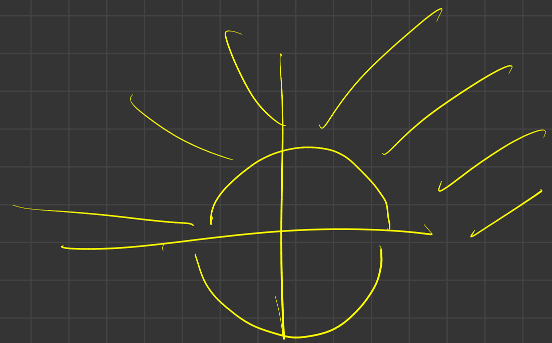
$\alpha \underline{x} \in C \quad \forall \alpha \geq 0$

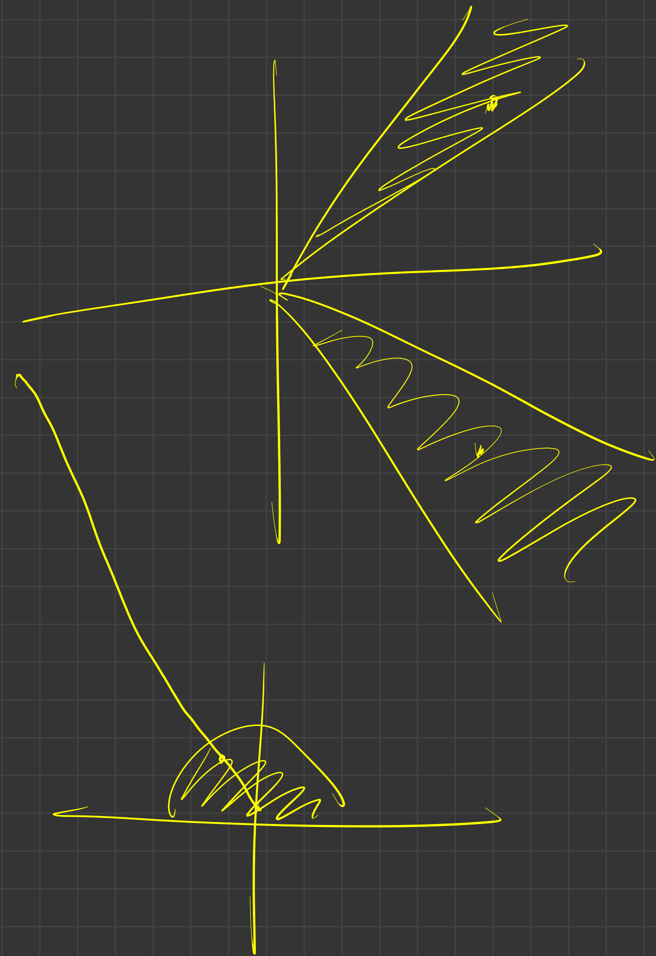


$\{ (x, |x|) : x \in \mathbb{R}$

$\{ (x, y) : y \geq |x| \}$





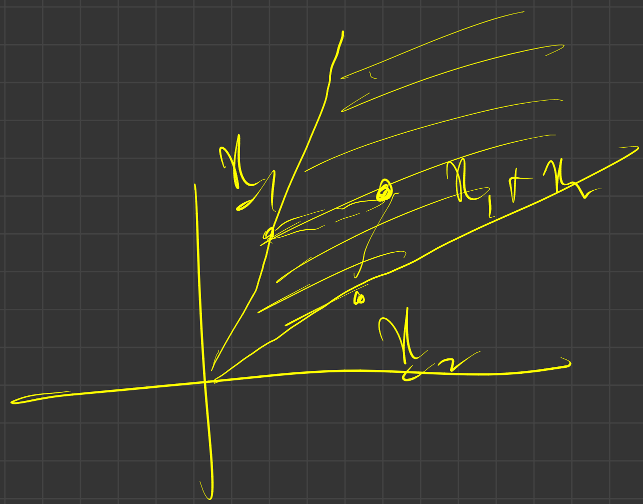


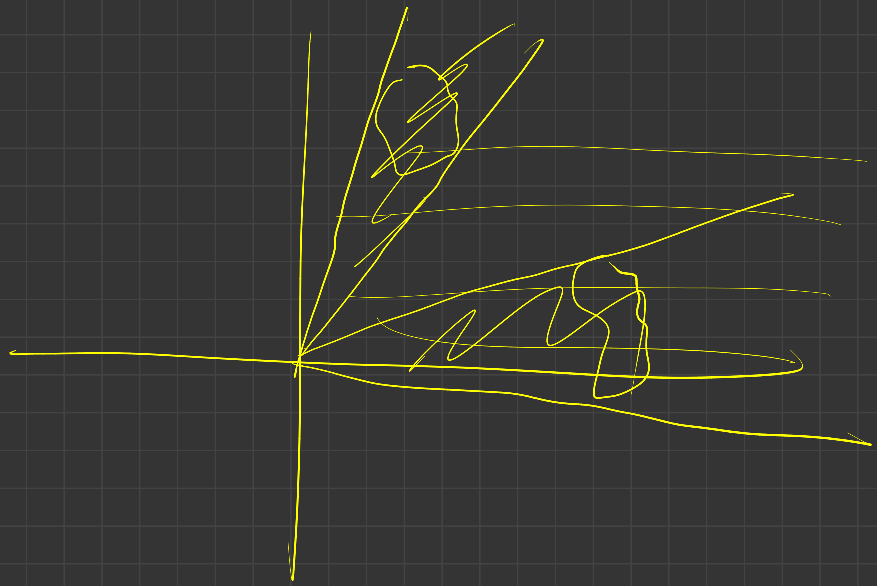
Conic hull

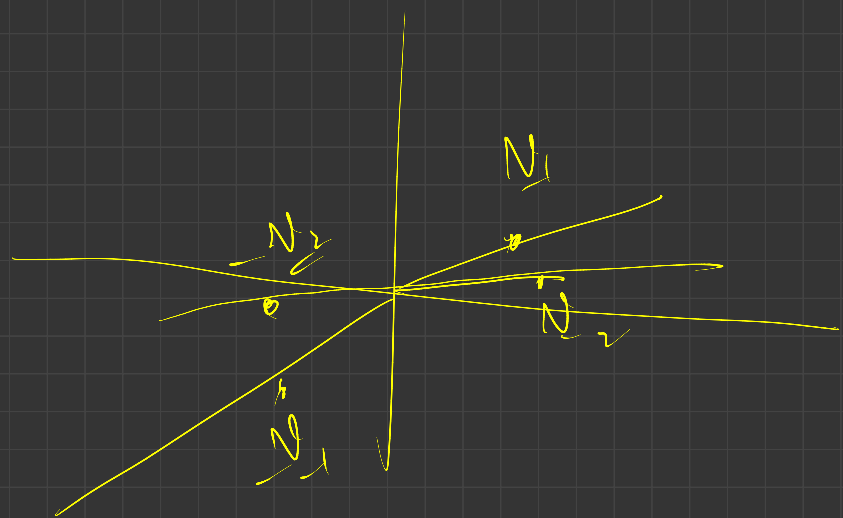
Conic combination:

$x_1, x_2, \ldots, x_k$

$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k$: $\alpha_i \geq 0$ $\forall i$







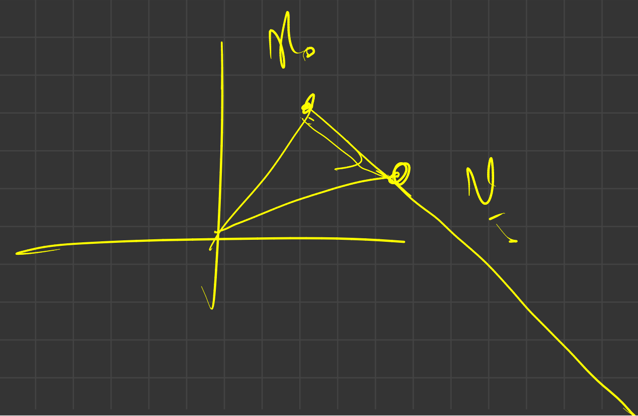
$$\alpha_1 v_1 + \alpha_2 v_2$$

$$-\alpha_3 v_1 - \alpha_4 v_2$$

$$(\alpha_1 - \alpha_3) v_1 + (\alpha_2 - \alpha_4) v_2$$

Examples

		convex	affine	cone
①	$\emptyset$	✓	✓	✓
②	Subspace of $\mathbb{R}^n$	✓	✓	✓
③	line segment	✓	✗	✗
④	$\{\underline{x}_0 + \alpha(\underline{v} - \underline{x}_0) : \alpha \geq 0\}$, $\underline{x}_0, \underline{v} \in \mathbb{R}^n$	✓	✗ Yes if $\underline{x}_0 = \underline{v}$	No general. Yes if $\underline{x}_0 = \underline{0}$



Hyperplanes and halfspaces

$$H = \{ \underline{x} : \underline{a}^T \underline{x} = b \}$$

$$\underline{a} \in \mathbb{R}^n \setminus \{\underline{0}\} \quad b \in \mathbb{R}$$

↓

Affine set of dim $n-1$

$$\left.\begin{array}{l} \{ \underline{x} : \underline{a}^T \underline{x} \geq b \} \\ \{ \underline{x} : \underline{a}^T \underline{x} \leq b . \} \end{array}\right\} \text{Halfspace}$$

$a_1^T x \le b_1$

$a_2^T x \le b_2$

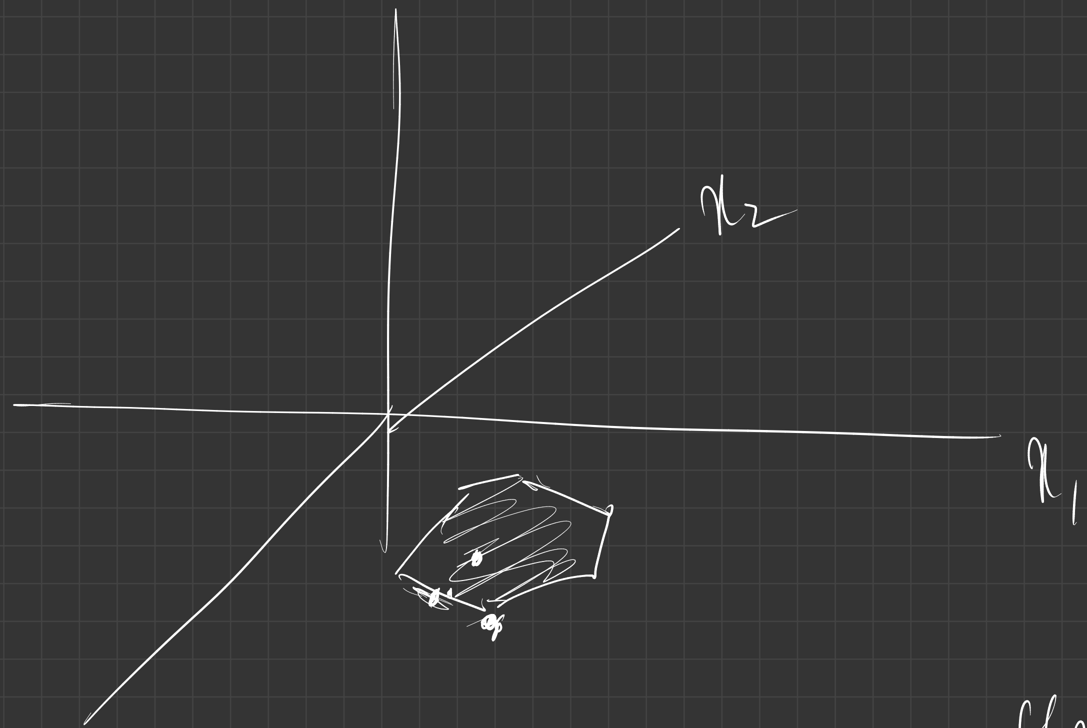
$\vdots$

$a_k^T x \le b_k$

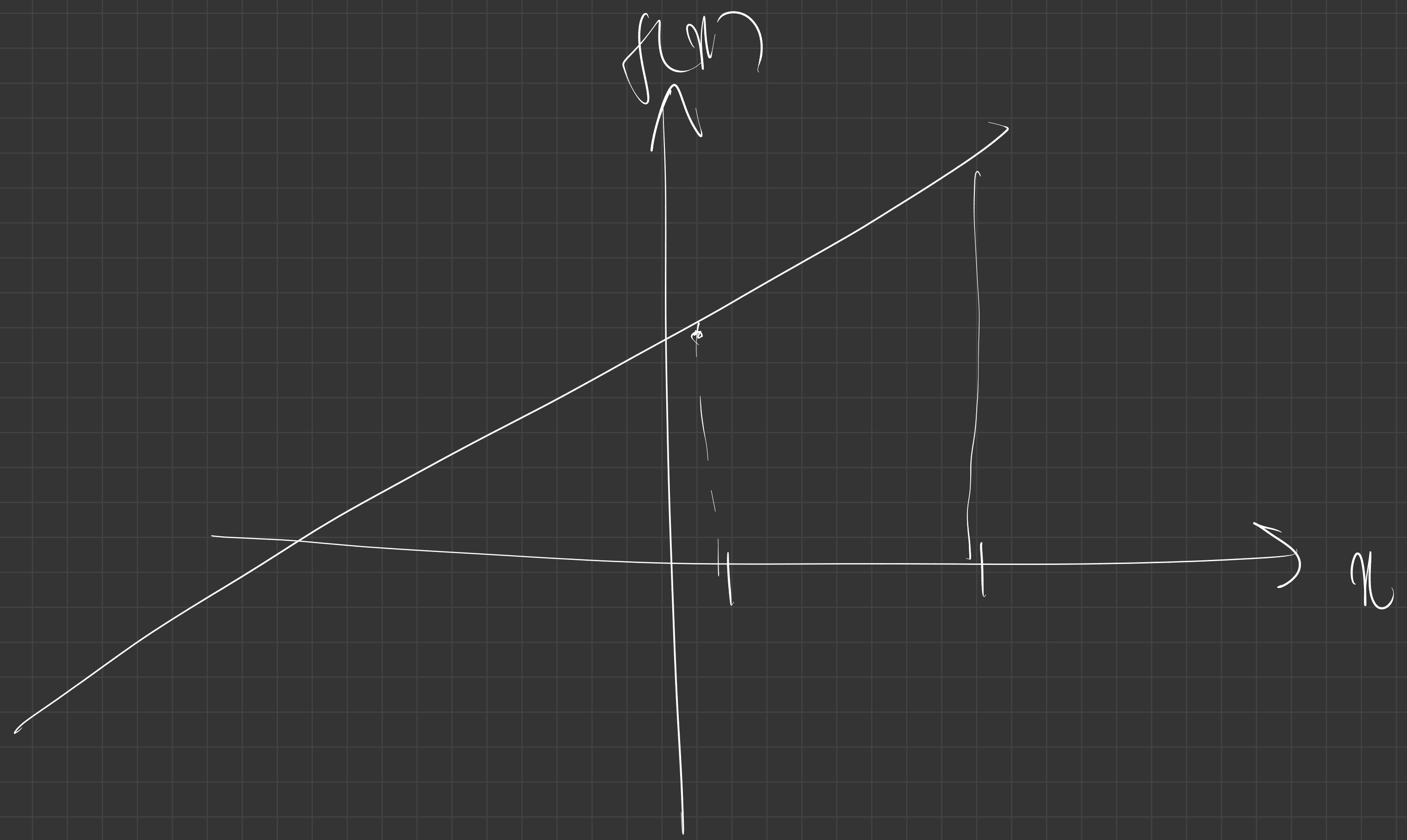
$a_i \in \mathbb{R}^n$

$b_i \in \mathbb{R}$

Polyhedron/polytope : Intersection of halfspace



$$f(x_1, x_2) = \alpha x_1 + \beta x_2$$


f(x)
x

Linear program

$$f(\underline{x}) = \underline{a}^T \underline{x}$$

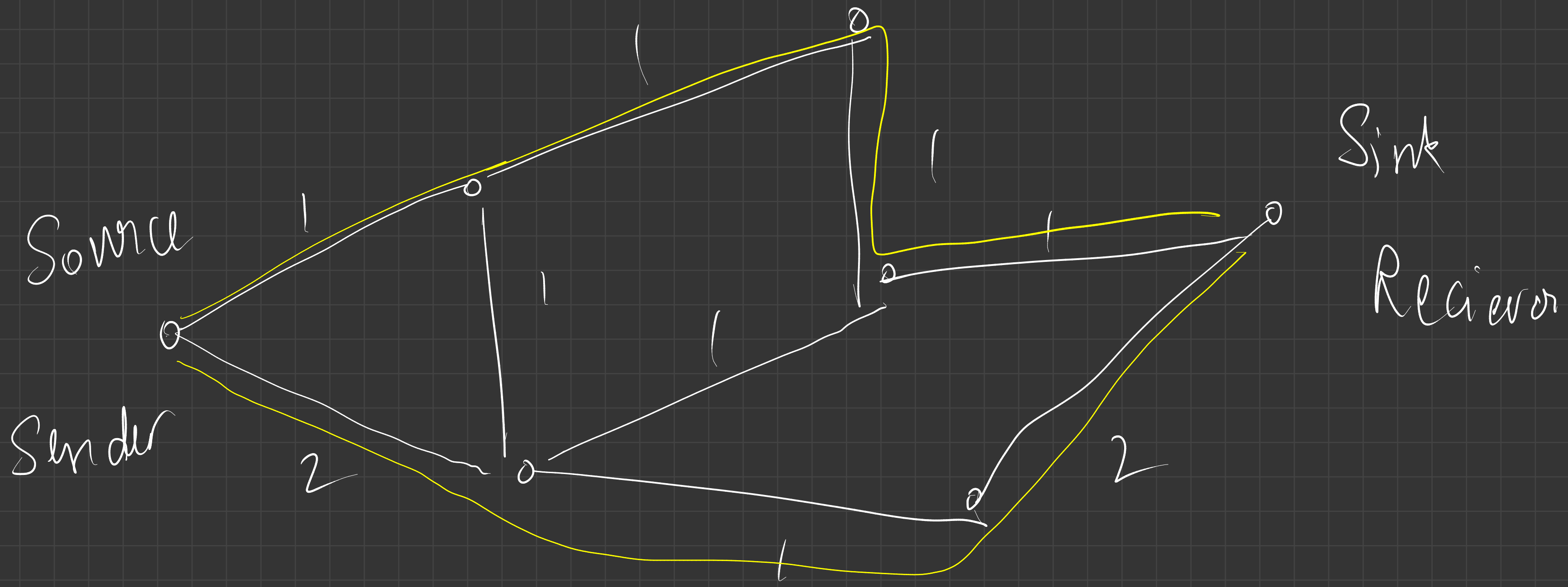
ST

$$\underline{a}_1^T \underline{x} \leq b_1 \qquad \underline{c}_1^T \underline{x} = d_1$$

$$\underline{a}_2^T \underline{x} \leq b_2 \qquad \underline{c}_2^T \underline{x} = d_2$$

$$\vdots \qquad \qquad \vdots$$

Max flow problem



Variables: Flows along each edge

(i,j) : x_{ij} (# of packets sent from i to j per sec)

X — $n \times n$ matrix

Capacities: For each pair of vertices i, j

$c_{ij} \geq 0$ $\quad$ $c_{ij} = c_{ji}$

$c_{ii} = 0$

C — $n \times n$ matrix

Objective

$f(\underline{x})$ = Total flow leaving source / total flow into sink

$$f(\underline{x}) = \sum_{j=1}^{n-1} x_{0j} = \sum_{j=0}^{n-2} x_{j,n-1}$$

Constraints :

① Capacity constraint : $x_{ji} + x_{ij} \leq C_{ij}$

② $x_{ij} \geq 0 \quad \forall i, j$

③ $\forall i, \quad \sum_{\substack{j=0 \\ j \neq i}}^{n-1} x_{ji} - \sum_{\substack{j=0 \\ j \neq i}}^{n-1} x_{ij} = 0$

$1 \leq i \leq n-2$

④ $\sum_{j=1}^{n-1} x_{j0} = 0$

$\sum_{j=0}^{n-2} x_{n-1,j} = 0$

Variable: X : $n \times n$ matrix

Objective: $\max \{ \text{sum}(X[0, :]) \}$

Constraints:

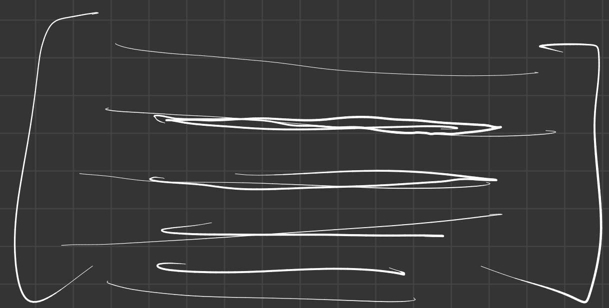
$X \geq 0$ (elementwise)

$X + X^T \leq C$

total inflow

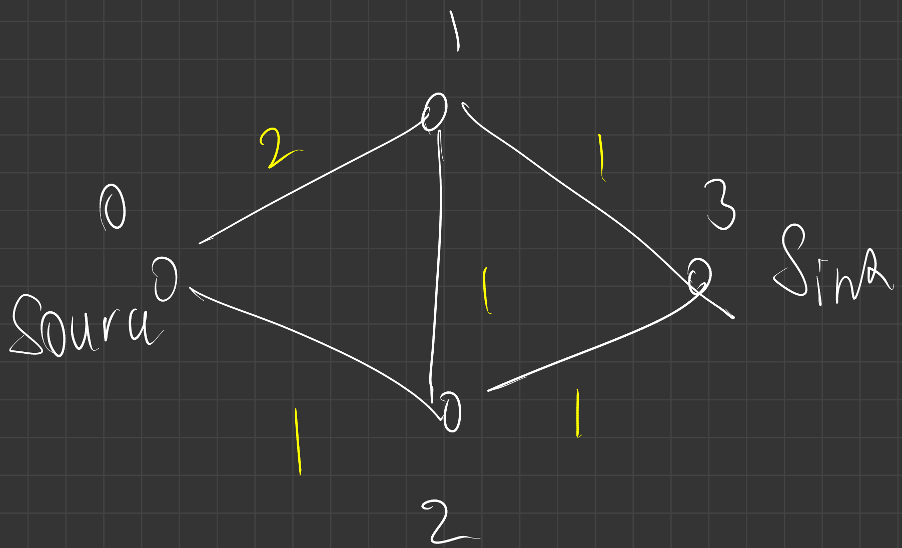
$$\text{sum}(X[1:n-2, :], \text{axis}=1) - \text{sum}(X^T[1:n-2, :], \text{axis}=1) = 0$$

total outflow from a node

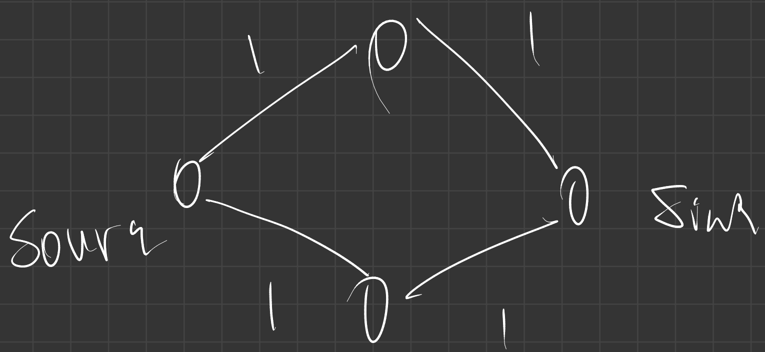


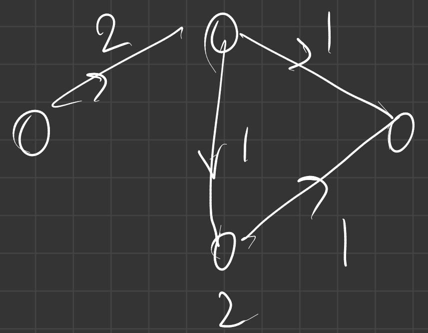
$\text{sum}(X[:, 0]) = 0$

$\text{sum}(X[n-1, :]) = 0$

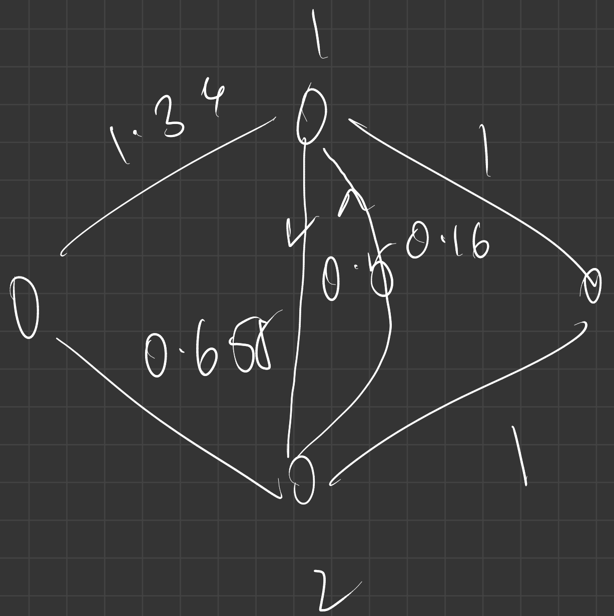


$$C = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$





$$X^* = \begin{bmatrix} 0 & 1.34 & 0.658 & 0 \\ 0 & 0 & 0.5 & 1 \\ 0 & 0.16 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



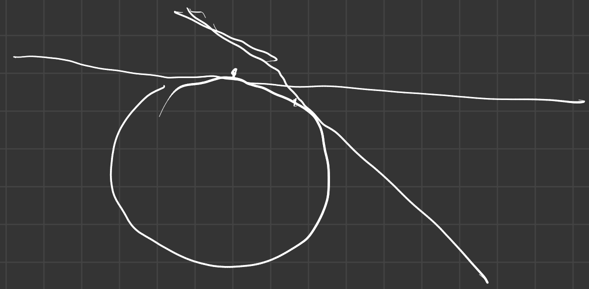
$f(x,y) = x^2+y^2$, Minimize.

$-1 \leq x \leq 1$

$-1 \leq y \leq 1$

Two different ways of looking at a closed convex set

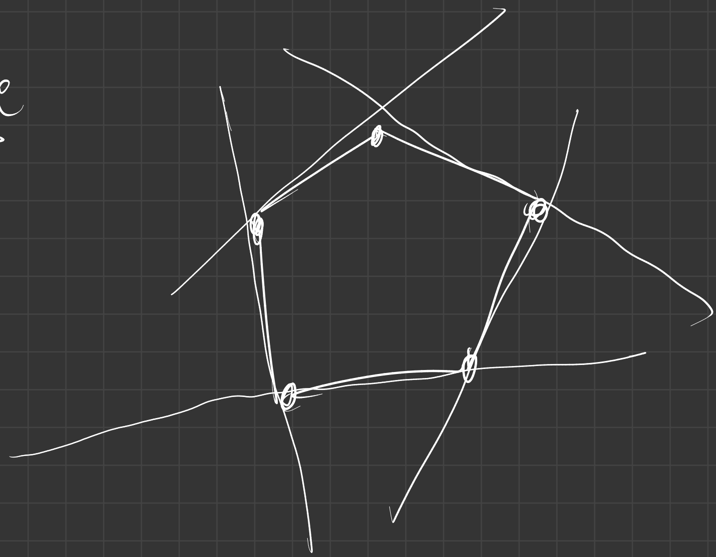
$C = \bigcap \{ H \text{ a halfspace that contains } C \}$



→ Halfspace description of a closed convex set

$C = \text{conv}(C')$ → convex hull description

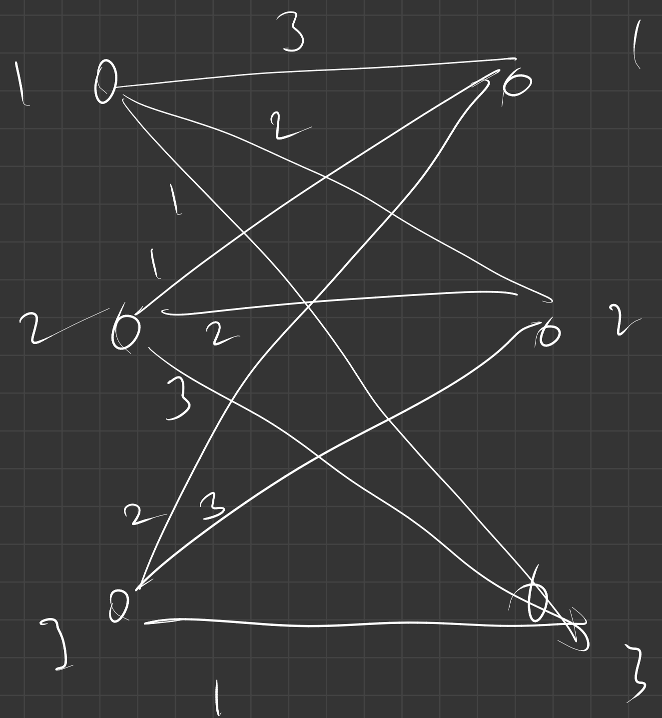
Polytope

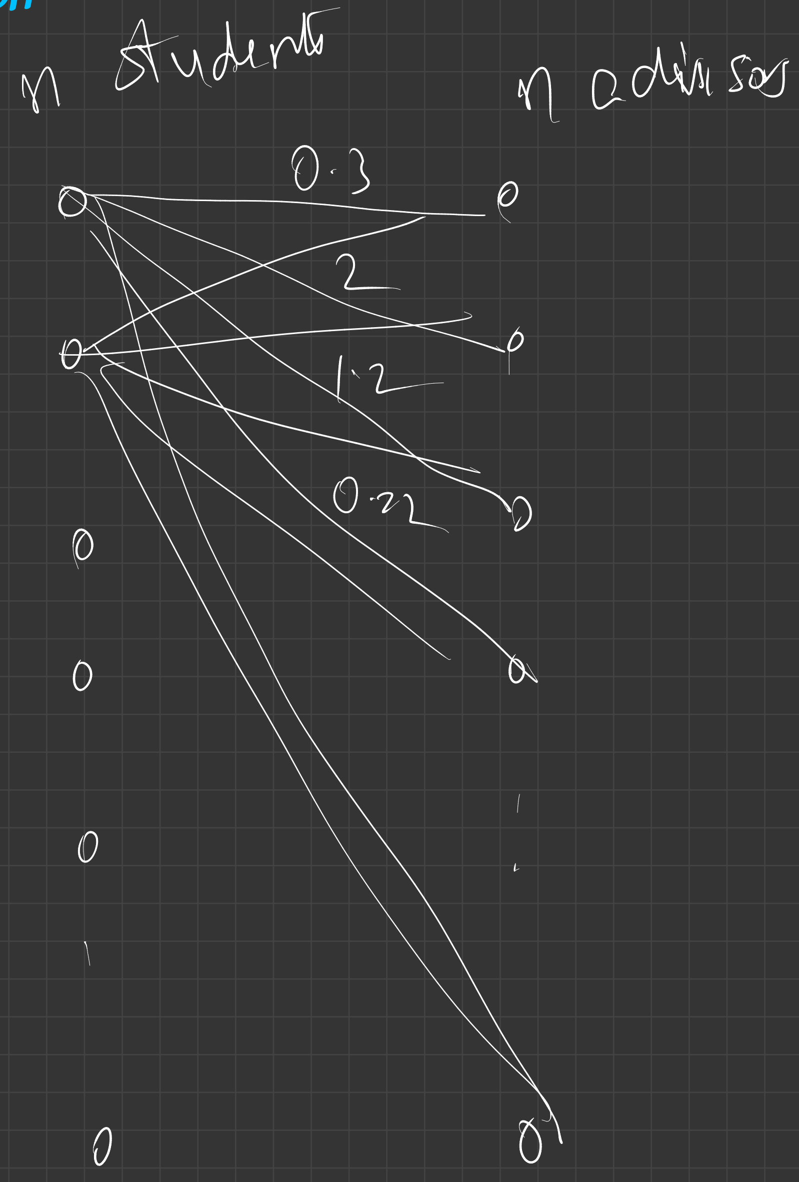


Convex hull of finitely many pts

Two different ways of looking at a polytope

Maximum weight matching on a complete bipartite graph





1 – 1
2 – 2
3 – 3

1 – 1
2 – 3
3 – 2

Problem:

$W : n \times n$

w_{ij} is the score given by i about j

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

Variables:

$X : n \times n$

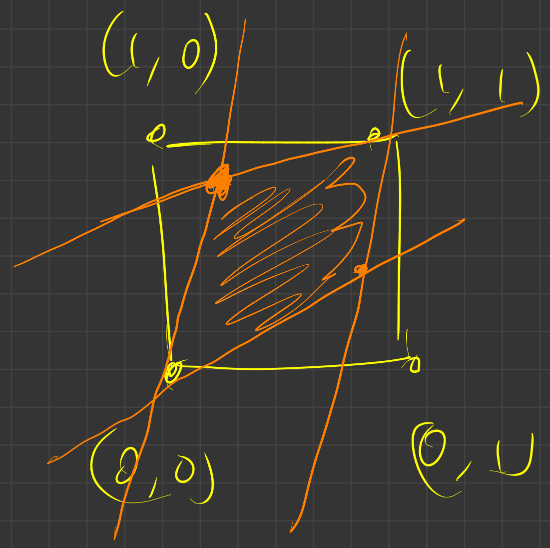
$X_{ij} \in \{0, 1\}$ → relax this to $x_{ij} \in [0, 1]$

$\sum_j X_{ij} = 1 \quad \forall i$

$\sum_i X_{ij} = 1 \quad \forall j$

} Constraints

Objective : $\max_{X} \sum_{i,j} w_{ij} x_{ij}$



Doubly stochastic matrices and the Birkhoff polytope

X is a doubly stochastic matrix if:

$x_{ij} \in [0,1]$

$\sum_j x_{ij} = 1 \; \forall i, \quad \sum_i x_{ij} = 1 \; \forall j$

If $x_{ij} \in \{0,1\}$ in addition, X is a permutation matrix.

* Any convex combination of permutation matrices is doubly stochastic

* Birkhoff-von Neumann theorem: Birkhoff polytope = conv(perm. matrices)

Norm balls and ellipsoids

$$B_r(\underline{x}_c, r) = \{ \underline{x} \in \mathbb{R}^n : \|\underline{x} - \underline{x}_c\| \leq r \}$$

$$\underline{x}_1, \underline{x}_2 \in B_r(\underline{x}_c, r)$$

$$\| \alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 - \underline{x}_c \|$$

$$= \| \alpha (\underline{x}_1 - \underline{x}_c) + (1-\alpha)(\underline{x}_2 - \underline{x}_c) \|$$

$$\leq \| \alpha (\underline{x}_1 - \underline{x}_c) \| + \| (1-\alpha)(\underline{x}_2 - \underline{x}_c) \|$$

$$= \alpha \| \underline{x}_1 - \underline{x}_c \| + (1-\alpha) \| \underline{x}_2 - \underline{x}_c \|$$

$$\leq \alpha r + (1-\alpha) r = r$$

Ellipsoid :

$$\{ \underline{x} : \| A(x - x_c) \| \leq r \}$$

$$\{ \underline{x} : (x - x_c)^T (A^T A) (x - x_c) \leq 1 \}$$

$$\{ \underline{x} : (x - x_c)^T P (x - x_c) \leq 1 \}$$

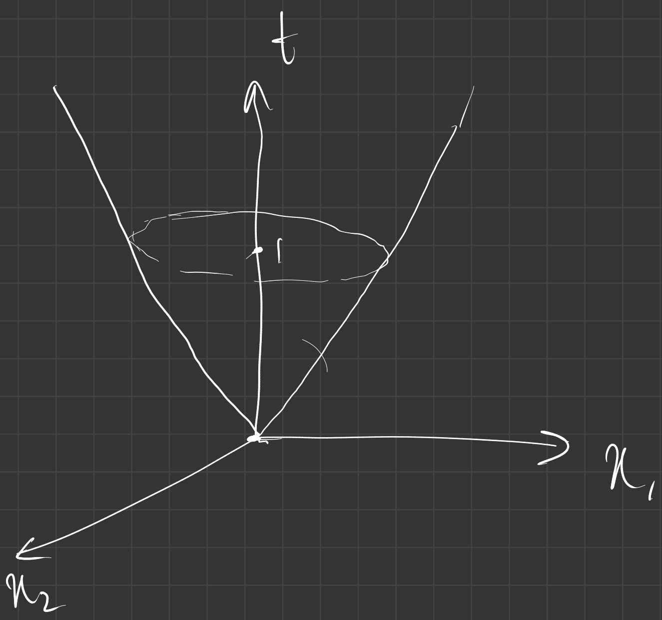
$\downarrow$

symmetric positive definite

$$x^T (A^T A) x = \| A x \|^2$$

Norm cones and the positive semidefinite cone

Norm cone : $\{ (\underline{x}, t) : \underline{x} \in \mathbb{R}^n, t \geq 0, \|\underline{x}\|_2 \leq t \}$



$\alpha_1 \begin{bmatrix} \underline{x}_1 \\ t_1 \end{bmatrix} + \alpha_2 \begin{bmatrix} \underline{x}_2 \\ t_2 \end{bmatrix}$ (each $\in C$) $\quad \alpha_1, \alpha_2 \geq 0$

$\begin{bmatrix} \alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 \\ \alpha_1 t_1 + \alpha_2 t_2 \end{bmatrix}$

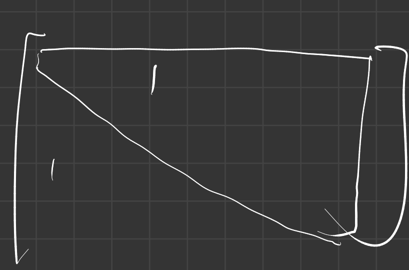
$\|\alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2\| \leq \|\alpha_1 \underline{x}_1\| + \|\alpha_2 \underline{x}_2\|$

$= \alpha_1 \|\underline{x}_1\| + \alpha_2 \|\underline{x}_2\|$

$\leq \alpha_1 t_1 + \alpha_2 t_2$

$\Rightarrow \alpha_1 \begin{bmatrix} \underline{x}_1 \\ t_1 \end{bmatrix} + \alpha_2 \begin{bmatrix} \underline{x}_2 \\ t_2 \end{bmatrix} \in C$

$S^n = \{ A \in \mathbb{R}^{n \times n} : A^T = A \}$



$\dim(S^n) = n(n+1)/2$

$S^n_+ = \{ A \in S^n , \ A \text{ is PSD} \}$

Convex ? A_1, A_2 PSD

$$x^T(\alpha_1 A_1 + \alpha_2 A_2)x = \alpha_1 x^T A_1 x + \alpha_2 x^T A_2 x \geq 0$$

S^n_+ is a convex cone

Transformations of Convex Sets

1. Intersection of convex sets

A_1 A_2 are convex

$A_1 \cap A_2$ is also convex

$\underline{x}_1, \underline{x}_2 \in A_1 \cap A_2$ $\quad$ $y = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$

$\underline{x}_1, \underline{x}_2 \in A_1 \Rightarrow y \in A_1$

$\underline{x}_1, \underline{x}_2 \in A_2 \Rightarrow y \in A_2$

Consider a family of sets $A_t : t \in R$

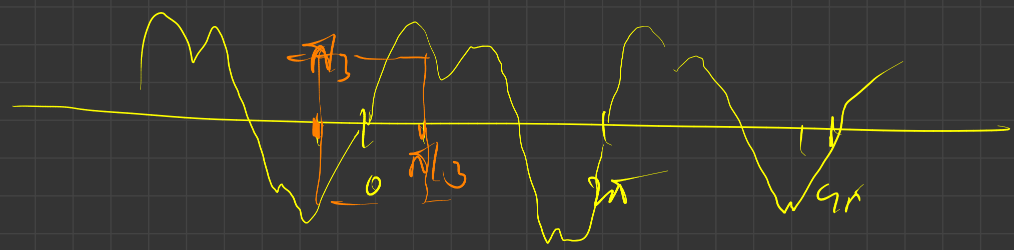
$\underline{x}_1, \underline{x}_2 \in A_t \quad \forall t$

$y \in A_t \Rightarrow y \in \bigcap_t A_t$

Example:

$$A = \left\{ \underline{x} \in \mathbb{R}^n \ \text{ st } \ \Big| \overbrace{\sum_{k=1}^{n} x_k \cos(kt)}^{f_x(t)} \Big| \le 1 \right\}$$

$$\forall \ -\frac{\pi}{3} \le t \le \frac{\pi}{3}$$



$$A_t = \left\{ \underline{x} \in \mathbb{R}^n : \underbrace{\Big| \sum_{k=1}^{n} x_k \cos kt \Big| \le 1}_{} \right\}$$

$$\langle \underline{x}, \underline{\cos kt} \rangle \le 1$$

$$\langle \underline{x}, \underline{\cos kt} \rangle \ge -1$$

$$A = \bigcap_{t \in [-\pi/3, \pi/3]} A_t$$

$$A_t = \left\{ \underline{x} : \sum_{k=1}^{n} x_k e^{kt} \leq 1 \right\}$$

$$A_t = \left\{ (x_1, x_2) : \begin{array}{l} x_1 \cos t + x_2 \cos 2t \leq 1 \\ x_1 \cos t + x_2 \cos 2t \geq -1 \end{array} \right\}$$

$$x_1 \leq \frac{1 - x_2 \cos 2t}{\cos t}$$

$$x_1 \geq \frac{-1 - x_2 \cos(2t)}{\cos t}$$

2. Minkowski sum of convex sets

A_1, A_2 convex

$$A_1 + A_2 = \{ \underline{x}_1 + \underline{x}_2 : \underline{x}_1 \in A_1, \underline{x}_2 \in A_2 \}$$

$$\underline{x}_1, \underline{x}_2 \in A_1 + A_2$$

$$\underline{x}_1 = \underline{x}_{11} + \underline{x}_{12}, \quad \underline{x}_{11} \in A_1, \ \underline{x}_{12} \in A_2$$

$$\underline{x}_2 = \underline{x}_{21} + \underline{x}_{22}, \quad \underline{x}_{21} \in A_1, \ \underline{x}_{22} \in A_2$$

$$\alpha \underline{x}_1 + (1-\alpha) \underline{x}_2 = \alpha x_{11} + \alpha x_{12} + (1-\alpha) x_{21} + (1-\alpha) x_{22}$$

$$\alpha x_{11} + (1-\alpha) x_{21} \in A_1, \quad \alpha x_{12} + (1-\alpha) x_{22} \in A_2$$

$$= \quad \in A_1 + A_2$$

3. Cartesian product of convex sets

A_1 A_2 are convex

$A_1 \subseteq \mathbb{R}^n$ $A_2 \subseteq \mathbb{R}^m$

$A_1 \times A_2 = \{ (x, y) : x \in A_1 \quad y \in A_2 \} \subseteq \mathbb{R}^{n+m}$

4. Affine transform of a convex set

If $\mathcal{A}$ is convex then (for $A \in \mathbb{R}^{m \times n}$, $\underline{b} \in \mathbb{R}^m$)

$\mathcal{A}' = A\mathcal{A} + \underline{b} = \{ A\underline{x} + \underline{b} : x \in \mathcal{A} \}$ is convex

Proper Cone

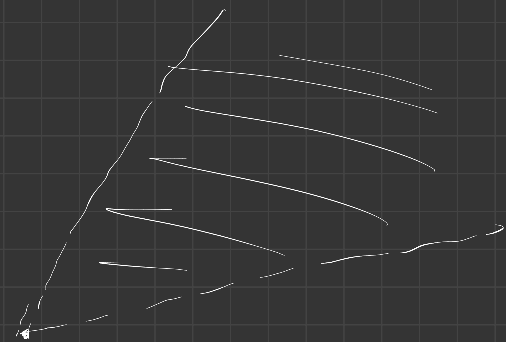
A cone K is proper if

1. It is closed
2. It is convex
3. It is solid: has a nonempty interior
4. It is pointed: $\underline{x} \in K,\ \underline{x} \neq \underline{0} \Rightarrow -\underline{x} \notin K$.

- Cone that is not closed: $\{(x_1, x_2):$

$x_1 > 0$
$x_2 > 0$

or $x_1 = x_2 = 0\}$



$(x_1, a x_1)$

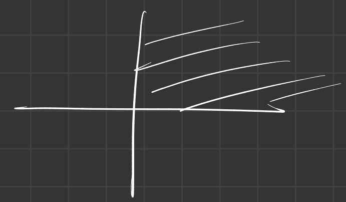
-

- Cone not solid: ray in $\mathbb{R}^2$
  - Not solid
  - Convex
  - closed
  - pointed.

- Cone - not pointed: ① line passing through origin:
  - Not solid
  - Convex
  - closed
  - Not pointed

  ② $\mathbb{R}^2$:
  solid
  convex
  closed
  not pointed

Example :

① $\{ \underline{x} : x_i \geq 0 \quad \forall i \} \rightarrow$ 

② The set of all PSD matrices $\subseteq S^n$

(i) If A is PSD then αA is PSD for $\alpha \geq 0$

(ii) If $A_1, A_2 \in S_+^n$ then $\alpha A_1 + (1-\alpha) A_2$ is PSD

$$\underline{x}^T (\alpha A_1 + (1-\alpha) A_2) \underline{x} \quad \text{for any } \underline{x} \in \mathbb{R}^n$$

$$0 \leq \alpha \leq 1$$

$$= \alpha \underline{x}^T A_1 \underline{x} + (1-\alpha) \underline{x}^T A_2 \underline{x}$$

$$\geq 0$$

Hence convex

(iii) $A \in S_+^n \quad A \neq \underline{0} \quad -A$ is not PSD

$-A \notin S_+^n$

(iv) In $\mathbb{R}^n$ $\quad B_n(r) = \{ \underline{x} \in \mathbb{R}^n : \|x\|_2^2 \le r^2 \}$

$$\|x\|_2^2 = \sum_{i=1}^{n} x_i^2$$

For S^n, (or even $\mathbb{R}^{n \times n}$), we use the Frobenius norm

$$\|A\| = \sqrt{\sum_{i,j} a_{ij}^2}$$

Consider $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ $\quad \|A\|^2 = a_{11}^2 + a_{22}^2 + 2a_{12}^2$

$\downarrow$

3 free variables

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{22} \end{bmatrix}$$

Claim: $\text{int}(S_+^n) = S_{++}^n$

$\supseteq$ If $A \in S_{++}^n$ then $\exists$ a ball of radius ε around A

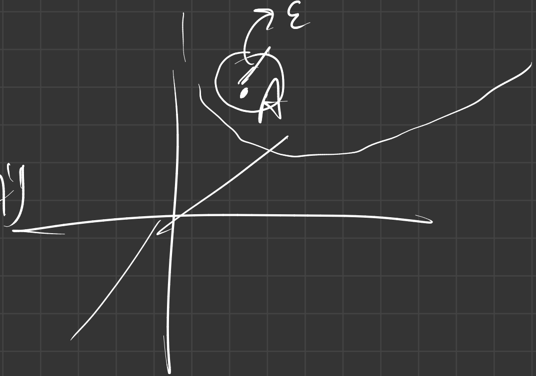
$\subseteq$

Fix any $A \in S_{++}^n$. $\exists\, \varepsilon > 0$ st $A + A'$; $A' \in S^n$

$A \in S_{++}^n$

$A' \in S^n$

$A + A' \in S_{++}^n$ as long as $\|A'\| < \varepsilon$



$\lambda_{min}(A) + \lambda_{min}(A') \leq \lambda_{min}(A+A') \leq \lambda_{max}(A+A') \leq \lambda_{max}(A) + \lambda_{max}(A')$

As long as $\lambda_{min}(A') > -\lambda_{min}(A)$

$\lambda_{min}(A+A') > 0 \Rightarrow A+A'$ is P.D.

Generalized Inequalities

Given any proper cone K.

$$x \preceq_K y \quad \text{iff} \quad y - x \in K$$

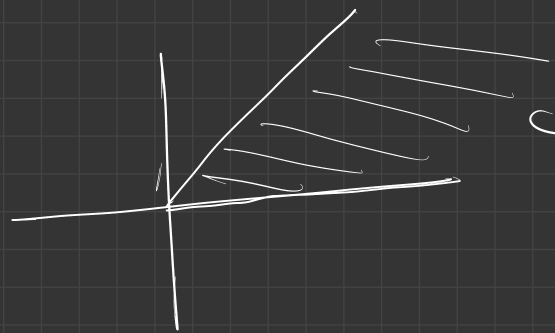
① K the nonnegative orthant (in $\mathbb{R}^n$)

$$K = \{ x \in \mathbb{R}^n : x_i \geq 0 \ \forall i \}$$

$$x \preceq_K y \Rightarrow y - x \in K$$

$$\Rightarrow y_i - x_i \geq 0 \quad \forall i$$

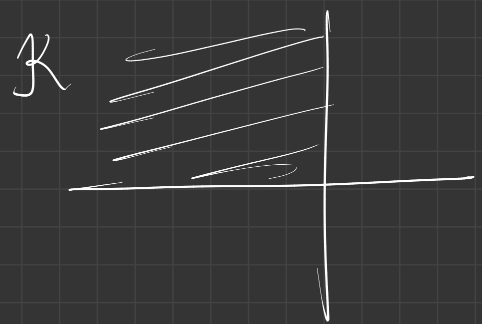
$$y_i \geq x_i \quad \forall i$$

② $K =$  $\hookrightarrow K = \{ (x,y) : x \geq 0, \ y \leq x$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \preceq_K \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \iff \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \in K$$

$$\iff x_2 \geq x_1, \quad y_2 - y_1 \leq x_2 - x_1$$

(3) $\mathbb{R}^2$



$$K = \left\{ (x, y) : \begin{array}{l} x \leq 0 \\ y \geq 0 \end{array} \right\}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \preceq_K \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \iff \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \in K$$

$$\iff x_2 - x_1 \leq 0 \ \& \ y_2 - y_1 \geq 0$$

$$\iff x_2 \leq x_1 \ \& \ y_2 \geq y_1$$

(4) $K = S^n_+$

$A \preceq_K B$ iff $B - A$ is PSD

Properties:

① $x \preceq_K y$ then $x + z \preceq_K y + z \quad \forall z \in \mathbb{R}^n$

② $x \preceq_K y$ & $y \preceq_K z$ then $x \preceq z$

$\Downarrow \qquad\qquad \Downarrow$

$y - x \in K \qquad z - y \in K$

Since K is a convex cone, $(y - x) + (z - y) \in K$

$\Rightarrow z - x \in K$

$\Rightarrow x \preceq_K z$

③ $\underline{x} \preceq_K y \Rightarrow \alpha x \preceq_K \alpha y \quad \forall \alpha \geq 0$

$y - x \in K \Rightarrow \alpha(y - x) \in K$

④ $\underline{x} \preceq_K \underline{x}$ since $\underline{0} \in K$

⑤ $\underline{x} \preceq_K y$ & $y \preceq \underline{x} \Rightarrow \underline{x} = y$

$y - \underline{x} \in K$ & $(\underline{x} - y) = -(y - x) \in K$

$\Rightarrow y = \underline{x}$

⑥ If $\underline{x}_n \preceq_K y_n$ for $n = 1, 2, 3, 4, \ldots$

$\lim_{n\to\infty} \underline{x}_n \preceq_K \lim_{n\to\infty} y_n$

If $x_n \leq y_n \quad \forall n$

then is $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$?

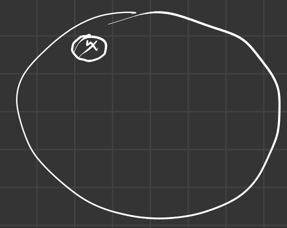
$x_n = \frac{1}{n} \qquad y_n = \frac{2}{n}$

Want to ST S^n_{++} is open in S^n

ST for any $A \in S^n_{++}$, a ball of radius $\varepsilon << 1$ centered at A & contained in S^n_{++}

B

$A+B \in S^n_{++}$ for any B $\|B\| < \varepsilon$



Example : If $\underline{X}$ is a random vector

Correlation matrix : $\mathbb{E}[\underline{X}\,\underline{X}^T] = C_X$

If X, Y rv's

Correlation matrix $= \begin{bmatrix} \mathbb{E}X^2 & \mathbb{E}XY \\ \mathbb{E}XY & \mathbb{E}Y^2 \end{bmatrix}$

$$\underline{u}^T C_X \underline{u} = u^T \mathbb{E}[\underline{X}\,\underline{X}^T]\,\underline{u}$$

$$= \mathbb{E}[\underline{u}^T X\, X^T \underline{u}]$$

$$= \mathbb{E}[(u^T X)^2] \geq 0$$

X, Y, Z rv's

Given:

$$C_X = \begin{bmatrix} 1 & \rho_{XY} & \rho_{XZ} \\ \rho_{XY} & 1 & \rho_{YZ} \\ \rho_{XZ} & \rho_{YZ} & 1 \end{bmatrix}$$

$-0.2 \leq \rho_{XY} \leq 0.3$

$\rho_{XZ} \geq 0$

What is max ρ_{YZ} ?

<u>Problem</u>: max ρ_{YZ} st: $-0.2 \leq \rho_{XY} \leq 0.3$

$\rho_{XZ} \geq 0$

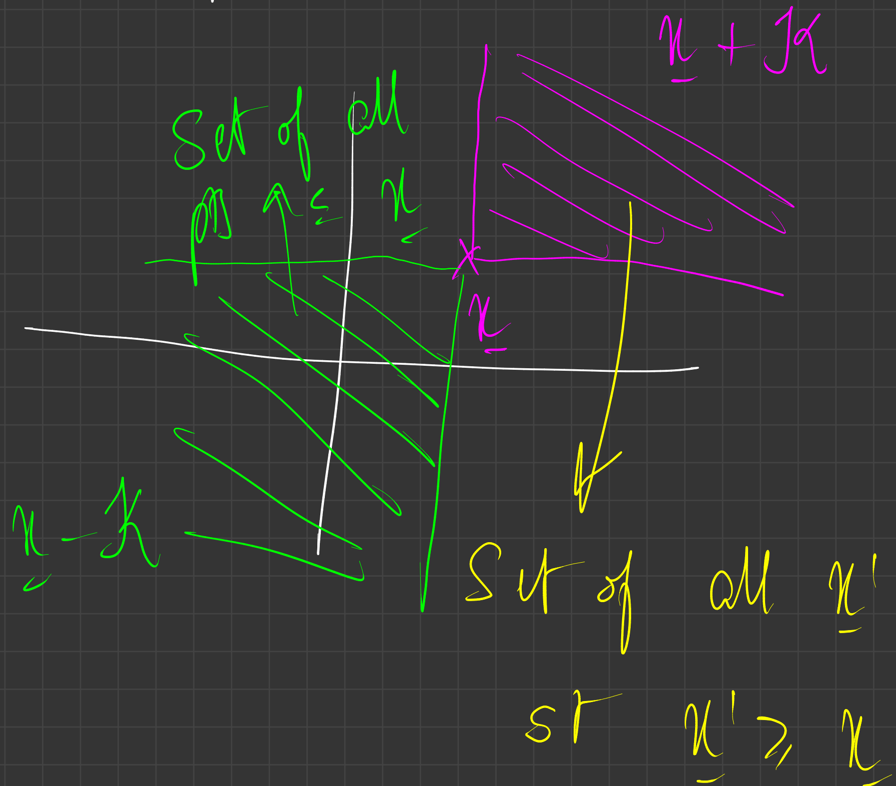
& C_X is PSD.

Generalized inequalities only form a partial order

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \not\leq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

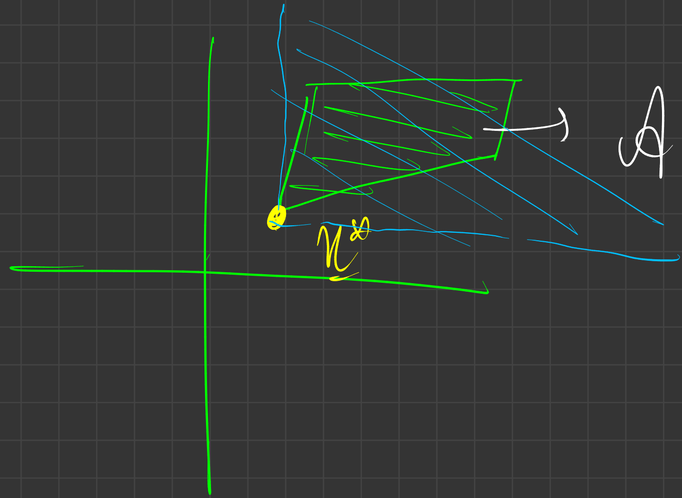
Not all pairs of elements are comparable.

Componentwise inequality



Componentwise inequality:

Does A have a minimum?



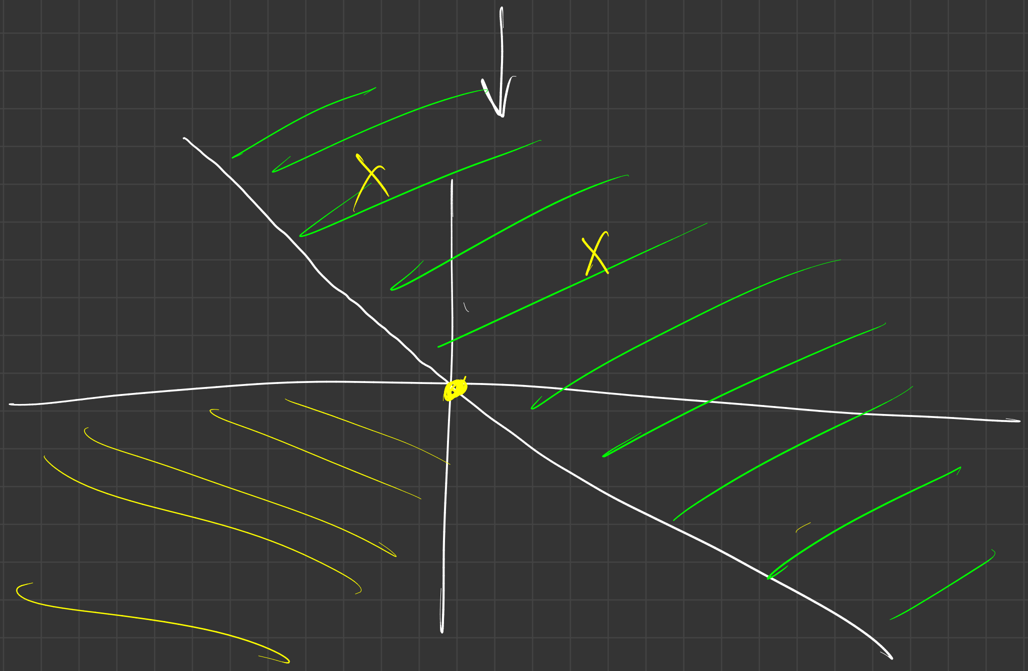
$\underline{x}^* = \min A$ iff $\underline{x}^* \leq \underline{x} \ \forall x \in A$

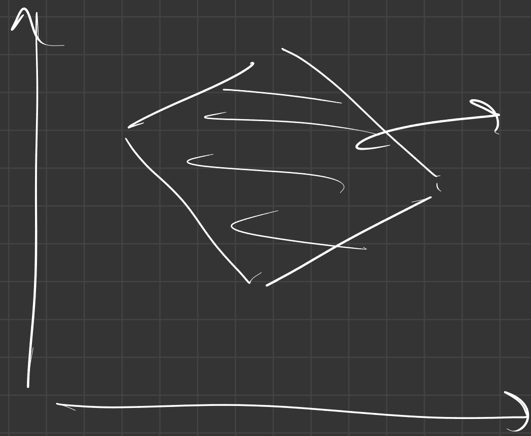
$\underline{x}^* \in A$

<u>Observation</u>: $\underline{x}^* = \min A$ iff $(\underline{x}^* + K) \supseteq A$

& $x^* \in A$

(2) $\{(x_1, x_2) : x_1 + x_2 \geq 0\}$





Minimal points We say that $\underline{x} \in A$ is a minimal pt of A if $\underline{y} \leq \underline{x}$ & $\underline{y} \in A \Rightarrow \underline{y} = \underline{x}$

Similarly $x \in A$ is a maximal pt of A iff

$y \geq x$ & $y \in A \Rightarrow y = x$

Separating hyperplane

Given A, B. We say that $\{a^T x = b\}$ is a separating hyperplane for A, B if

$$a^T x \geq b \quad \forall x \in A$$

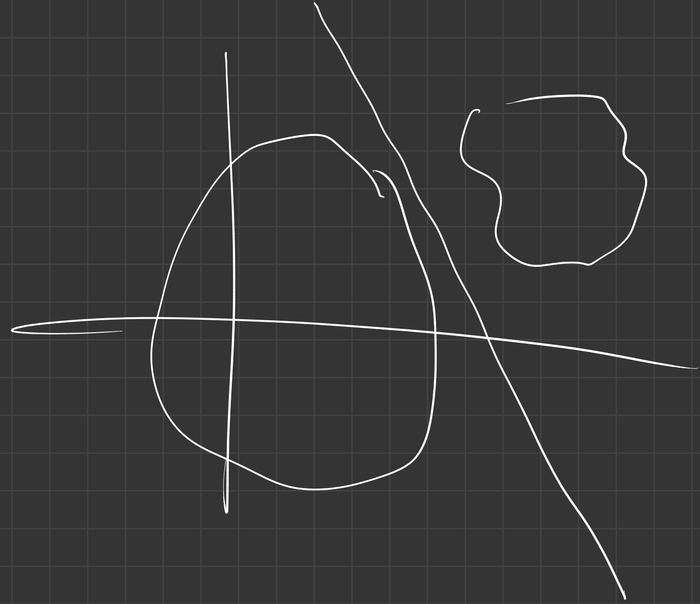
$$a^T x \leq b \quad \forall x \in B$$

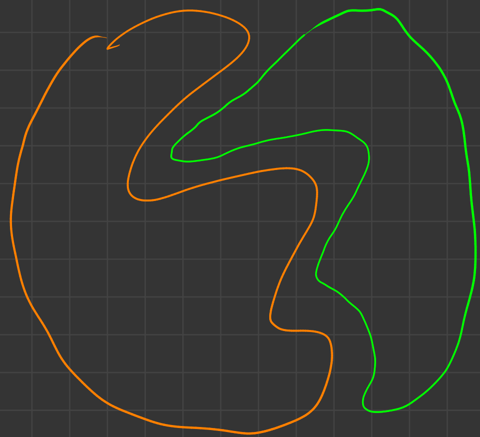
The hyperplane strictly separates A, B if

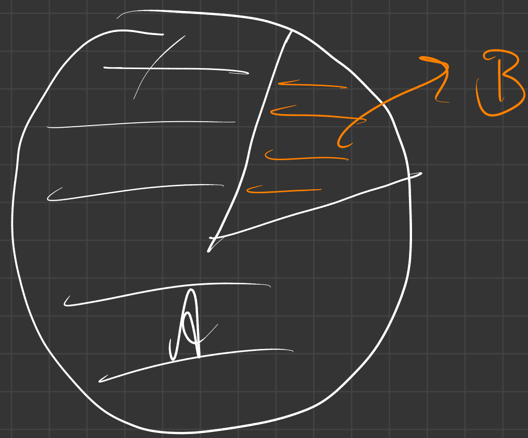
$$a^T x > b \quad \forall x \in A$$

$$a^T x < b \quad \forall x \in B.$$

Theorem If A, B convex & $A \cap B = \emptyset$, then there is a hyperplane that separates A & B.

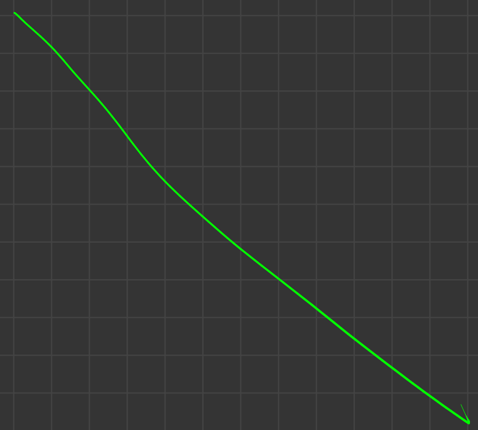






$A = \{x : a^T x > b\}$

$B = \{x : a^T x \leq b\}$



* Converse not true in general
* Converse true with additional constraints
  - Strict separation, or
  - One of A, B is open

Supporting hyperplane

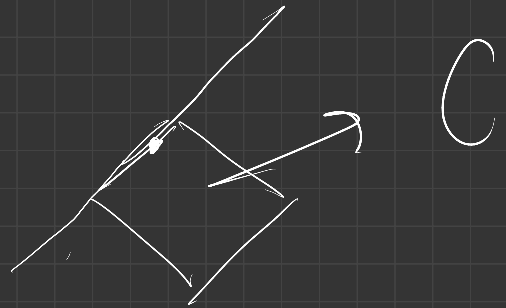
$x_0 \in bd(A) \rightarrow$ boundary

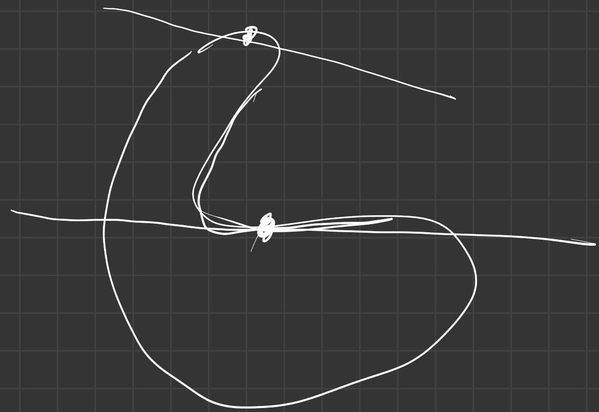
$= cl(A) \setminus int(A)$

We say that $\{x : a^T x = a^T x_0\}$ is a supporting hyperplane for A if all of A lies on one side

$$a^T x \geq a^T x_0 \quad \forall x \in A$$

Claim: Every pt on the boundary of a convex set has a supporting hyperplane





Take $\{\underline{x}_0\} = A$

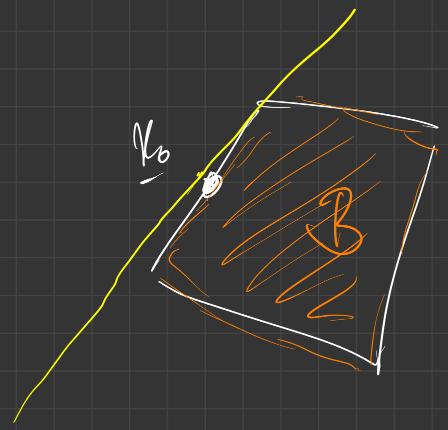
$B = \text{int}(C)$

$A \cap B = \phi$

A, B convex

$\exists\ \underline{a} \in \mathbb{R}^n,\ b \in \mathbb{R}$ st $\underline{a}^T \underline{x} \geq b \quad \forall\ \underline{x} \in B$

$\underline{a}^T \underline{x} \leq b \quad \forall\ \underline{x} \in A$

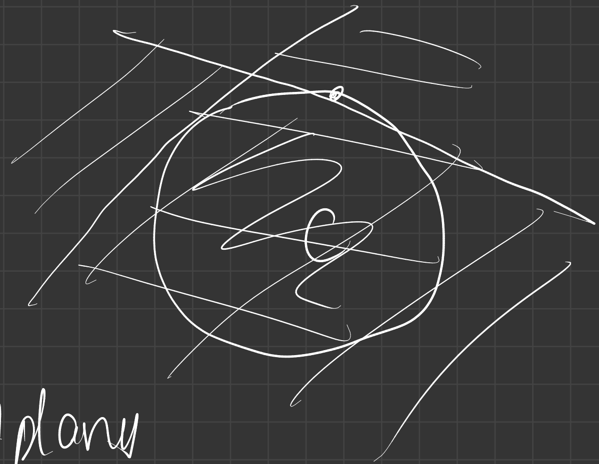


x_0 is on the separating hyperplane

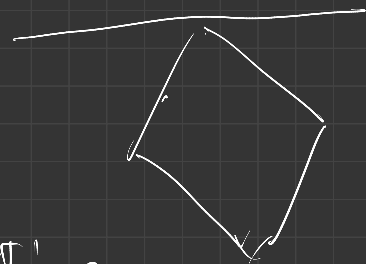
$\Rightarrow \quad \underline{a}^T \underline{x} \geq b \quad \forall \, \underline{x} \in C$

$\therefore$ this is a supporting hyperplane.

Every closed convex set is the intersection of half spaces defined by the supporting hyperplanes

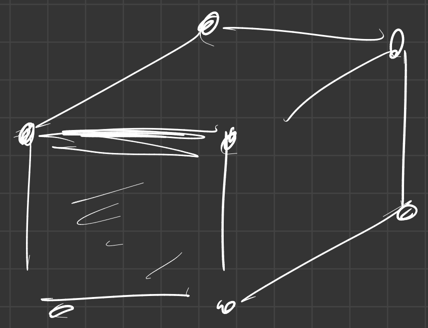


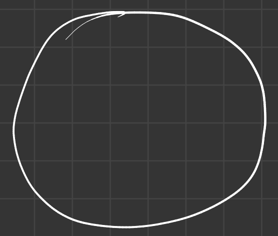
Extreme point: $\underline{x} \in A$ is an extreme pt if supporting hyperplane has only one pt from A ($\underline{x}$ itself)



Property:

A closed convex set is the convex hull of its extreme points





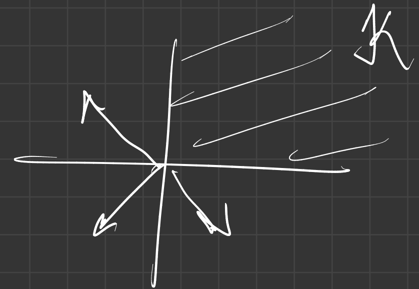
Dual cone

If K is a cone. The dual cone

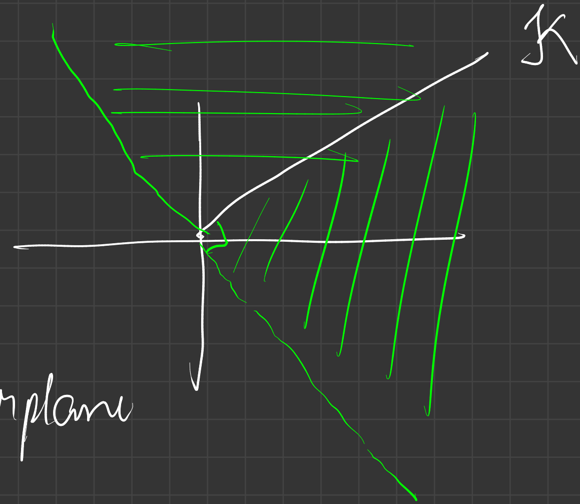
$$K^* = \{ \underline{x} \in \mathbb{R}^n : \underline{x}^T y \geq 0 \;\; \forall y \in K \}$$

① Nonnegative orthant

$$K^* = K$$



K_1



K_2

$K_2 \subseteq K_1$

$K_1^* \subseteq K_2^*$

② $K = \{ \alpha \underline{x} : \alpha \geq 0 \}$

K^* = halfspace defined by hyperplane normal to $\underline{x}$

$$= \{ y : \underline{x}^T y \geq 0 \}$$

③ $\mathcal{K}$ = any k-dim subspace of $\mathbb{R}^n$

$$\underline{x} \in \mathcal{K} \qquad \underline{y} \in \mathcal{K}^*$$

Suppose $\underline{x}^T \underline{y} > 0 \qquad (-\underline{x})^T \underline{y} < 0$

$$\Rightarrow \underline{x}^T \underline{y} = 0 \quad \forall\, \underline{y} \in \mathcal{K}^*$$

$\mathcal{K}^* = \mathcal{K}^\perp$, the dual space of $\mathcal{K}$

④ $\mathcal{K} = \mathbb{S}^n_+ \qquad \langle A, B \rangle = \sum_{i,j} a_{ij}\, b_{ij} = \mathrm{tr}(A^T B)$

Claim: $\mathcal{K}^* = \mathcal{K}$

Take $A \in \mathcal{K}^*$ $\qquad$ (want to show $\underline{x}^T A \underline{x} \geq 0$)

$$\underline{y}^T (\underline{x}\,\underline{x}^T)\, \underline{y} = (\underline{y}^T \underline{x})(\underline{x}^T \underline{y}) = (\underline{x}^T \underline{y})^2 \geq 0$$

$\Rightarrow \underline{x}\,\underline{x}^T$ is PSD

$A \in K^* \Rightarrow x^T A x = \underbrace{tr(x x^T A) \geq 0}_{\text{since } A \in K^* \;\&\; xx^T \in K}$

Consider any PSD matrix A, B

$$A = \sum_{i=1}^{n} \lambda_i \, q_i q_i^T$$

$$tr(A^T B) = tr\left(\sum_{i=1}^{n} \lambda_i \, q_i q_i^T B \right) = \sum_{i=1}^{n} tr\left(\lambda_i \, q_i q_i^T B\right)$$

$$= \sum_{i=1}^{n} \lambda_i \, tr\left(q_i q_i^T B\right)$$

$$= \sum_{i=1}^{n} \underbrace{\lambda_i}_{\geq 0} \; \underbrace{(q_i^T B q)}_{\geq 0}$$

$\Rightarrow A \in K^*$

$\Rightarrow K^* = K$

Properties of dual cone

① K^* is closed & convex

$x_1, x_2 \in K^*$, $\quad x_1^T y \geq 0 \quad \forall y \in K$

$x_2^T y \geq 0 \quad \forall y \in K$

$(\alpha x_1 + (1-\alpha) x_2)^T y \geq 0 \quad \forall y \in K$

$\Rightarrow \quad \alpha x_1 + (1-\alpha) x_2 \in K^*$

② $K_1 \subseteq K_2 \Rightarrow K_1^* \supseteq K_2^*$

Take any $x \in K_2^*$ $\quad x^T y \geq 0 \quad \forall y \in K_2$

$\Rightarrow x^T z \geq 0 \quad \forall z \in K_1$

$\Rightarrow x \in K_1^*$

③ If K has nonempty interior, then K^* is pointed.

Proof: Suppose K^* is not pointed.

$\exists$ nonzero $x \in K^*$ s.t. $-x \in K^*$

Take any $y \in K$ $\quad y^T x = 0$

$\dim(K) \leq n-1$

$\Rightarrow$ K does not have a nonempty interior

④ If closure of K is pointed, K^* has nonempty interior

Proof: Suppose K^* does not have a nonempty interior

$\Leftrightarrow$ $\dim(K^*) \leq n-1$

$\Rightarrow$ $\exists$ $y \in \mathbb{R}^n$ s.t. $y^T x = 0$ $\forall x \in K^*$

$\Rightarrow$ $y \in K$ & K is not pointed,

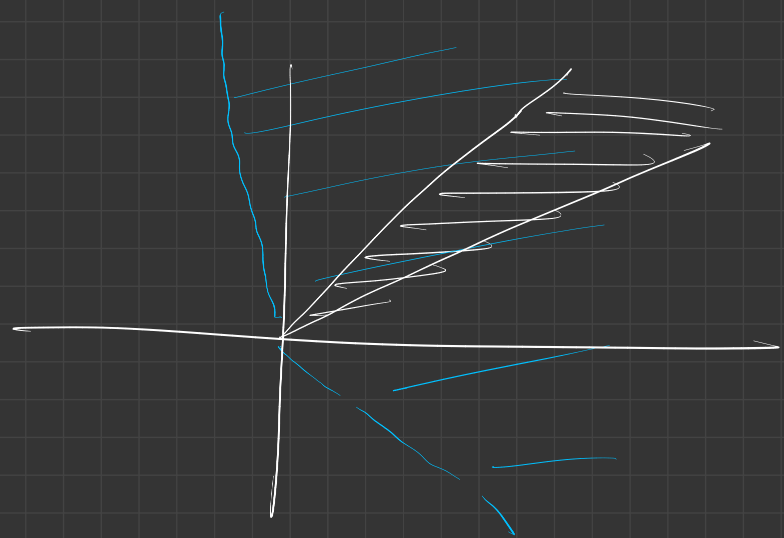
K^{**} is not pointed

But K^{**} is not pointed $\Longrightarrow$ $cl(K)$ is not pointed

(If closure of K is pointed, then the convex hull of the closure of K cannot be pointed)

However, $K = \{0\} \cup \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 > 0\}$ is a pointed cone, but closure is not pointed. The closure is $\{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \geq 0\}$ & $(-1, 0)$ & $(1, 0)$ lie in this set.

The dual of K is $\{(0, x_2) : x_2 \geq 0\}$. This has empty interior.



Property: If K is a proper cone, then K^* is a proper cone

$$\underline{x} \preceq_K y \quad \text{iff} \quad y - \underline{x} \in K$$

$$y \succeq_K \underline{0} \Rightarrow y \in K$$

PROP: $\underline{x} \preceq_K y$ & $\left(\underline{\lambda} \in K^* \Rightarrow \underline{\lambda} \succeq_{K^*} \underline{0} \right)$

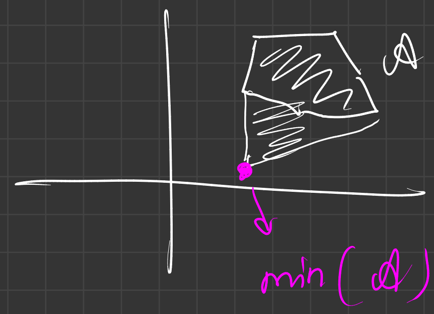
$$\Rightarrow \underline{\lambda}^T \underline{x} \leq \underline{\lambda}^T y$$

$$y - \underline{x} \in K \Rightarrow \lambda^T (y - x) \geq 0$$

$$\Rightarrow \lambda^T y \geq \lambda^T x$$

Consider $A \subseteq \mathbb{R}^n$ K a proper cone

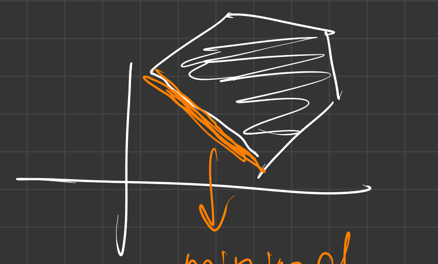
$x^* = \min(A)$



Claim : $x^* = \underset{x \in A}{\operatorname{argmin}} \; \lambda^T x$ for every $\lambda \succ_{K^*} 0$

Proof : $x^* \preceq_K x \quad \forall x \in A$

$\Rightarrow \lambda^T x^* \leq \lambda^T x \quad \forall x \in A$



Claim : If $\lambda \succ_{K^*} 0$ &

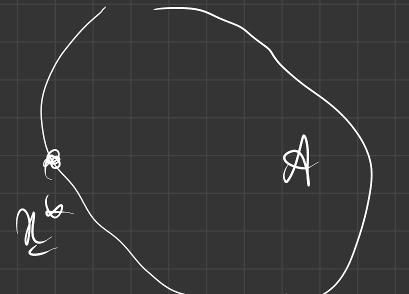
$x^* = \underset{x \in A}{\operatorname{argmin}} \; \lambda^T x$, then x^* is minimal.

Proof : If not, $\exists x \in A$ $x^* \succeq_K x \Rightarrow \lambda^T x^* > \lambda^T x$

Consider the componentwise inequality

$x \in A$ is a minimum vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix}$$

$$\begin{bmatrix} x_1^* \\ x_2 \\ x_3^* \end{bmatrix} \in A$$

A minimal point is called a Pareto optimal point

P_1 \ P_2	C	NC
C	(3, 3)	(0, 4)
NC	(4, 0)	(1, 1)

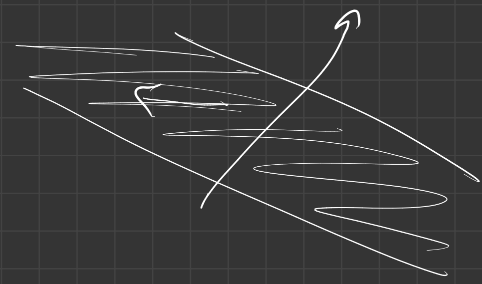
(3, 3) → dominant strategy but not Pareto optimal

(0, 4), (4, 0), (1, 1) → all Pareto optimal

Problem: $\{\underline{x} : \underline{a}_1^T \underline{x} = b_1\} = \mathcal{H}_1$

$\{\underline{x} : \underline{a}_2^T \underline{x} = b_2\} = \mathcal{H}_2$

When are these parallel?



$\underline{x}_1, \underline{x}_2 \in \mathcal{H}_1$

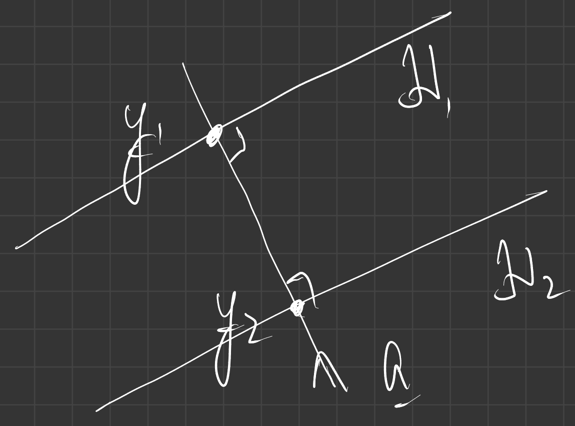
$a_1^T (\underline{x}_2 - \underline{x}_1) = 0$

→ vector that is "along" $\mathcal{H}_1$

$\underline{a}_1$ is normal to $\mathcal{H}_1$

$\mathcal{H}_1$ & $\mathcal{H}_2$ are parallel if $\underline{a}_1 = \alpha\, \underline{a}_2$ for some nonzero α

Suppose $\mathcal{H}_1 \parallel \mathcal{H}_2$. What is the distance b/w $\mathcal{H}_1$ & $\mathcal{H}_2$?



$$y_1 \in \text{ray}(\underline{a})$$

$$y_1 = \alpha_1 \underline{a}$$

$$\& \; y_1 \in \mathcal{H}_1 \Rightarrow \underline{a}^T y_1 = b_1$$

$$\Rightarrow \alpha_1 = \frac{b_1}{\|a\|^2}$$

$$y_1 = \frac{b_1}{\|\underline{a}\|^2} \underline{a} \qquad y_2 = \frac{b_2}{\|a\|^2} \underline{a}$$

$$\|y_2 - y_1\| = \frac{|b_2 - b_1|}{\|a\|^2} \|a\| = \frac{|b_2 - b_1|}{\|\underline{a}\|}$$

Given $C \subseteq \mathbb{R}^n$, finite

$\underline{x} \in C \rightarrow$ transmitted

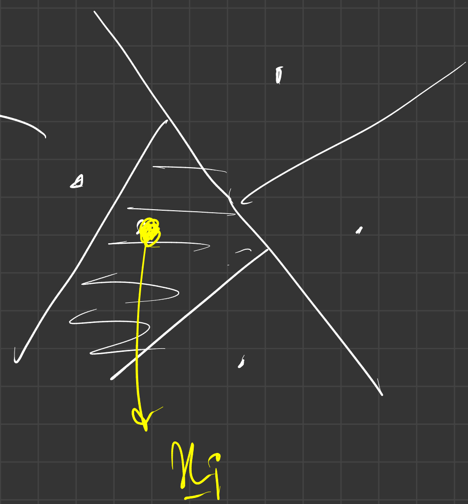
$y = \underline{x} + \underline{e} \rightarrow$ received.

Decision rule: Given $y \in \mathbb{R}^n$,

choose $\hat{\underline{x}} = \underset{\underline{x} \in C}{\operatorname{argmin}} \| y - \underline{x} \|$

$V_i = \{ y \in \mathbb{R}^n : \| y - \underline{x}_i \| \leq \| y - x_j \| \ \forall j \neq i \}$

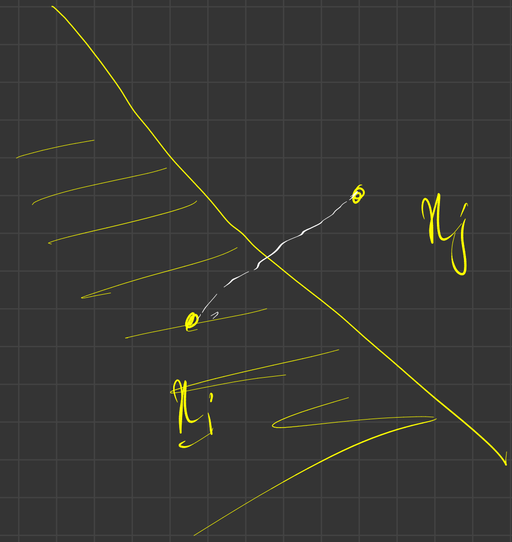
Voronoi region



Voronoi region is a polyhedron

Let $\mathcal{H}$ be the set of all pts in $\mathbb{R}^n$ which are closer to $\underline{x}_i$ than x_j.

Show that this is a half space.



If $\underline{x} \in \mathcal{H}$, $\quad \|\underline{x} - \underline{x}_i\|^2 \leq \|\underline{x} - x_j\|^2$

$$(\underline{x} - \underline{x}_i)^T(\underline{x} - \underline{x}_i) \leq (\underline{x} - \underline{x}_j)^T(\underline{x} - \underline{x}_j)$$

$$\cancel{\underline{x}^T\underline{x}} - 2\underline{x}_i^T\underline{x} + \|x_i\|^2 \leq \cancel{\underline{x}^T\underline{x}} - 2x_j^T\underline{x} + \|x_j\|^2$$

$$(x_i - x_j)^T\underline{x} \geq \frac{\|x_i\|^2 - \|x_j\|^2}{2}$$

Let $\mathcal{H}_{ij}$ is the halfspace of points in $\mathbb{R}^n$ that are closer to x_i than x_j

$$\mathcal{H}_{ij} = \{ \underline{x} \in \mathbb{R}^n : \quad \| \underline{x} - x_i \| \leq \| \underline{x} - x_j \| \}$$

$$V_i = \{ \underline{x} \in \mathbb{R}^n : \| \underline{x} - x_i \| \leq \| x - x_j \| \quad \forall j \neq i \}$$

$$= \bigcap_{j \neq i} \mathcal{H}_{ij}$$

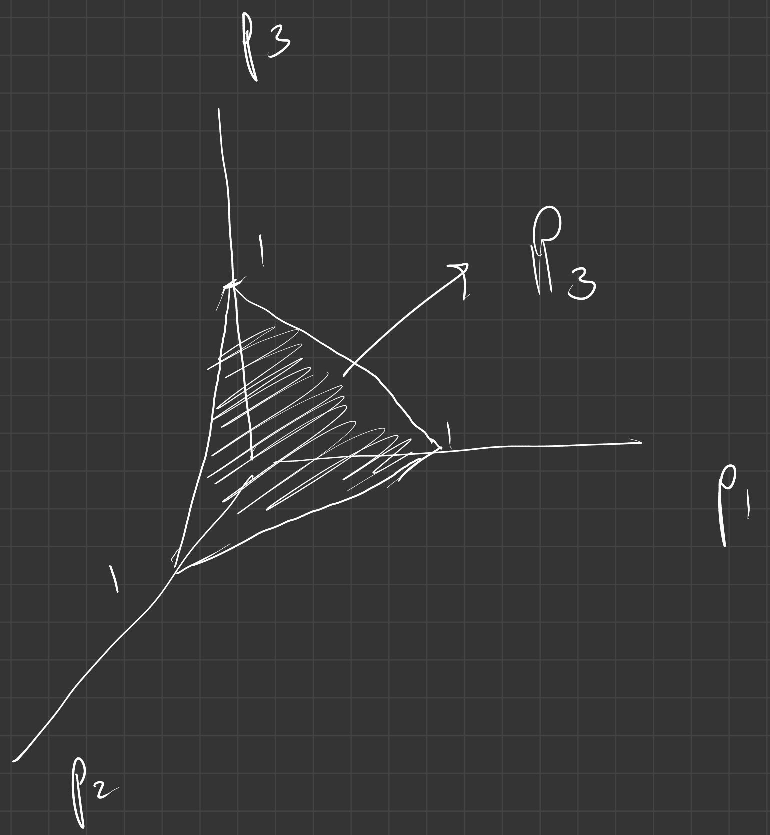
$\therefore$ V_i is a polyhedron

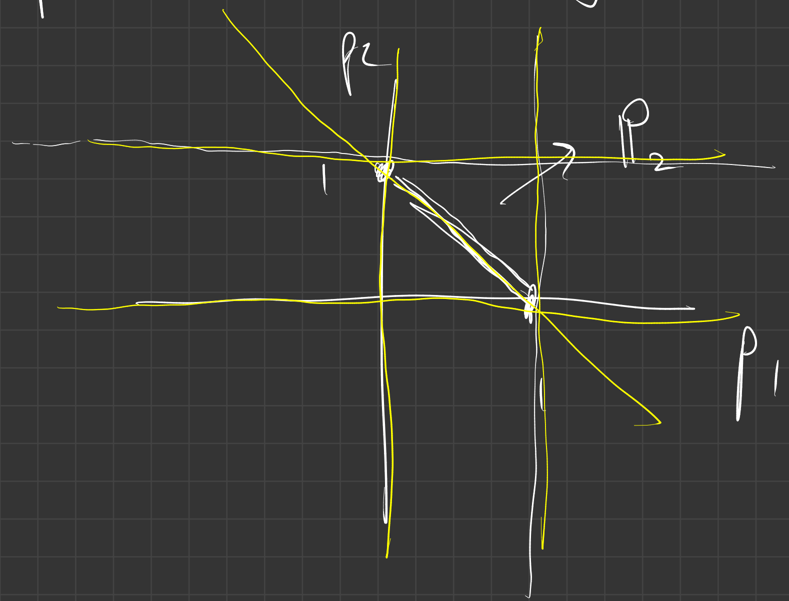
③ pmf on $[n] = \{1, 2, 3 \ldots n\}$

$\mathbb{P}_n$ = set of all pmfs on $[n]$

$\mathbb{P}_2 = \{ p \in \mathbb{R}^2 : p_1 + p_2 = 1$

$0 \le p_i \le 1 \quad i = 1, 2 \}$





$$\mathcal{H}_i = \{ p \in \mathbb{R}^n : p_i \leq 1 \} \quad n \rightarrow \text{Halfspace}$$

$$p_i = e_i^T p \leq 1$$

$$\{ p \in \mathbb{R}^n : p_i \geq 0 \} \quad n \rightarrow \text{Halfspace}$$

$$\{ p \in \mathbb{R}^n : \sum_{i=1}^{n} p_i = 1 \} \quad 2$$

Π_n is a polytope.

$$\mathcal{A} = \{ p \in \mathbb{P}_n : \underline{\mathbb{E}X^2 \le \alpha}, \quad X \sim p \}$$

$$\| $$

$$\sum_{i=1}^{n} i^2 p_i \le \alpha$$

$$\|$$

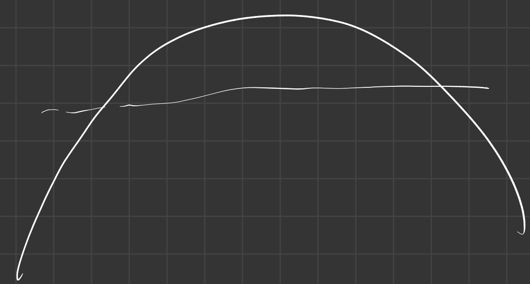
$$[1 \quad 2^2 \quad 3^2 \cdots n^2] \, p \le \alpha$$

$\Rightarrow$ $\mathcal{A}$ is a polytope.

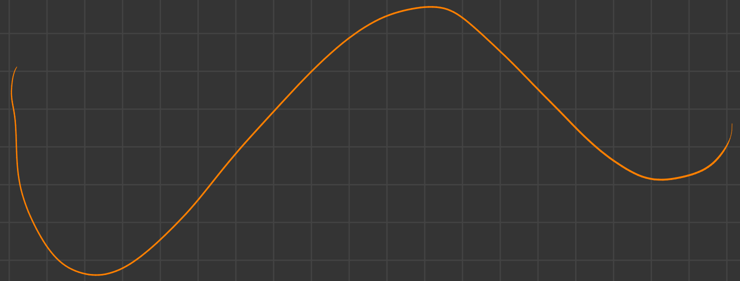
$A = \{ p \in P_n : H(p) \geq \alpha \}$

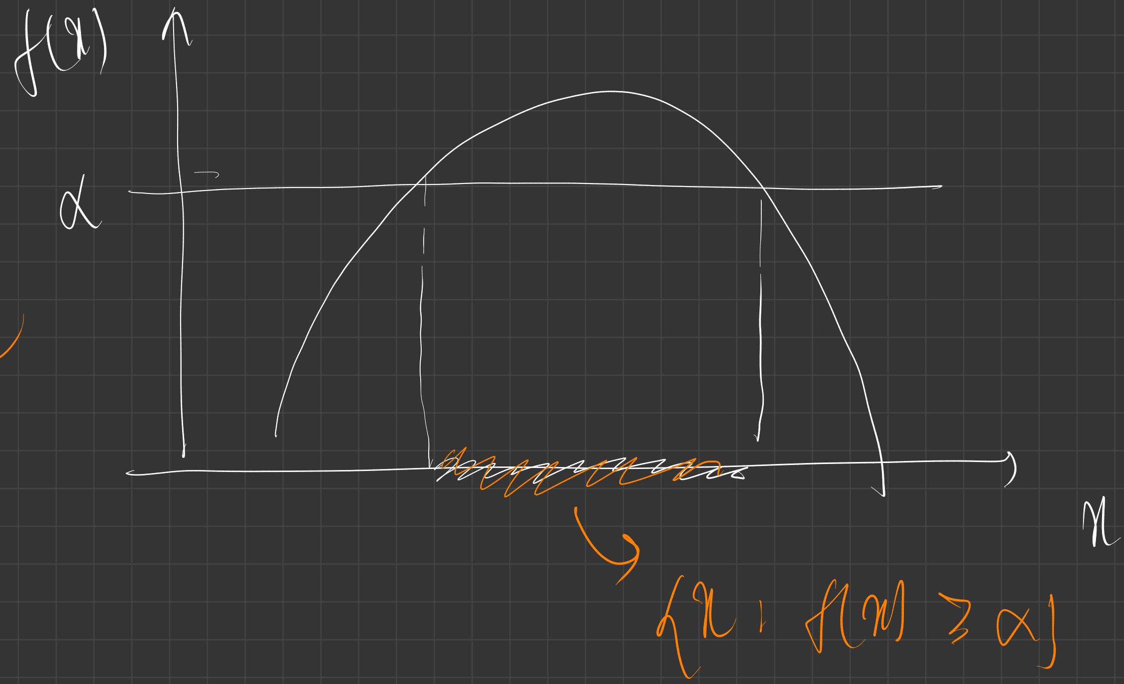
$$H(p) = \sum_{i=1}^{n} p_i \log \frac{1}{p_i}$$

$\{ \underline{x} : f(\underline{x}) \geq \alpha \}$ is a convex set if f is concave



$\{ \underline{x} : f(\underline{x}) \leq \alpha \}$ is a convex set if f is convex





f(x)
α
x
{x : f(x) ≥ α}

(A) Consider the set of all copositive matrices

$$C = \{ A \in S^n : \quad \underline{x}^T A \underline{x} \geq 0 \quad \text{for all } \underline{x} \geq \underline{0} \}$$

Q1: Is this a cone?

If $A \in C$, then $\alpha A \in C$ for $\alpha \geq 0$

Q2: Is this convex?

Q3: Is C closed?

The intersection of any number of closed sets is closed.

Fix an $\underline{x} \geq 0$. $\mathcal{H}_{\underline{x}} = \{ A : \underline{x}^T A \underline{x} \geq 0 \}$

↓

linear in A

$tr(xx^T A) \geq 0$

$\mathcal{H}_x$ is a halfspace for any $\underline{x}$

$C = \bigcap_{\underline{x} \geq 0} \mathcal{H}_x$ → intersection of infinitely many halfspaces

Q4:

Q4: Does C have nonempty interior?

$C \supseteq S^n_+$ & S^n_+ has a nonempty interior

Q5: Is C pointed?

Yes

$A \in C \Rightarrow -A \notin C$ (if $A \neq 0$)

C is a proper cone.

HW: Find the dual cone of C.