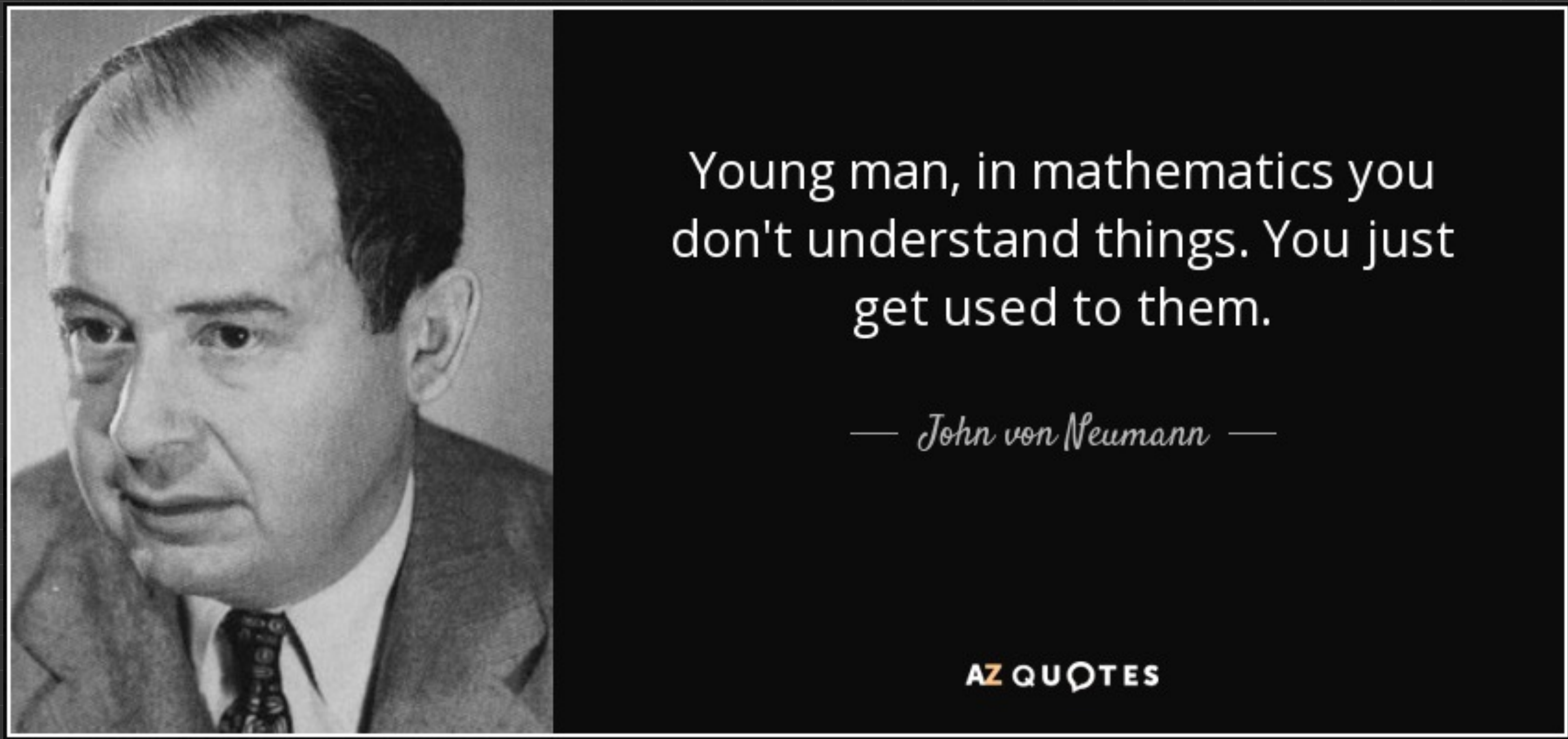
Basic Concepts in Topology



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Subsets of the reals: sup, inf, max and min

$A \subseteq \mathbb{R}$

- Upper bound : $u \geq x \quad \forall x \in A$
- Least upper bound/ Supremum : $u^* = \sup(A)$

  if ① u^* is an u.b of A

  ② If u is any ub of A, $u^* \leq u$

- If $u^* = \sup(A) \in A$, then u^* is the maximum of A

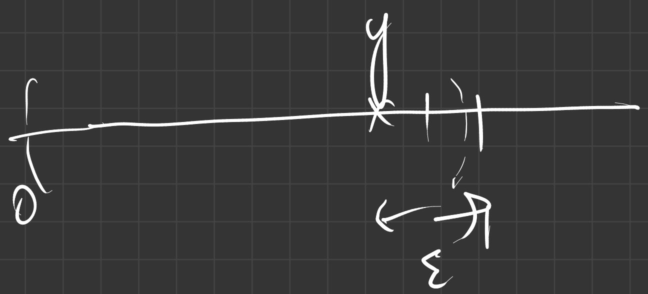
$A = (0, 1) \longrightarrow \sup(A) = 1$

$\downarrow$

$0 < x < 1$

Take any $y < 1$

$\varepsilon = 1 - y > 0$



$y < y + \frac{\varepsilon}{2} < 1$

$\in$

A

Lower bound: - l is a lb for A if

$$l \leq x \quad \forall x \in A$$

- Greatest lower bound/ Infimum of A if

① l^* is a l.b for A

② If l is any other lb for A, then $l \leq l^*$

- If $l^* \in A$ then l^* is the minimum of A.

Eg $A \subseteq \mathbb{Q}$

$$A = \{x \in \mathbb{Q} : x^2 < 2\}$$

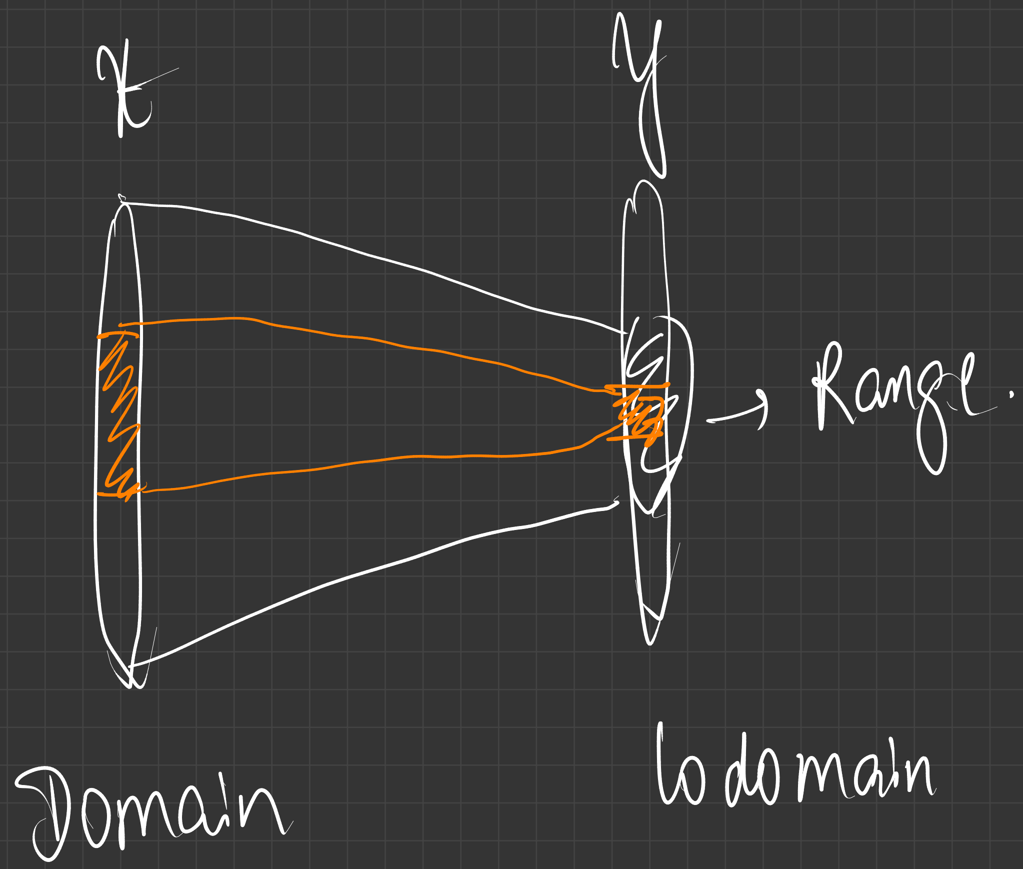
In $\mathbb{R}$, $\sup(A) = \sqrt{2}$

In $\mathbb{Q}$, $\sup(A)$ does not exist

Functions: definitions

$$f : \mathcal{X} \to \mathcal{Y}$$

$$\text{Range} = f(\mathcal{X}) = \{ f(x) : x \in \mathcal{X} \}$$



Take $\mathcal{A} \subseteq \mathcal{X}$

$f(\mathcal{A}) = \{ f(x) : x \in \mathcal{A} \}$ → Image of $\mathcal{A}$

Take $B \subseteq Y$

$$f^{-1}(B) = \{ x : f(x) \in B \}$$ Inverse Image of B.

- One-one / Injective if $|f^{-1}(\{x\})|$ is 0 or 1.
- Onto / Surjective if $f(X) = Y$
- Bijective if one-one & onto.

How big is your set?

- Finite
- Infinite — Countable — $\exists$ a 1-1 map from A to $\mathbb{Z}_{\geq 0}$.
- Infinite — Uncountable

$\mathbb{Z}$ – C , $\mathbb{Z}^n$ – C , $(0,1)$ – U

$\mathbb{Q}$ – C

$$f(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ -2x+1 & \text{if } x < 0 \end{cases}$$

Metric

$\mathcal{X}$, $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a metric / distance measure,

1. $d(x_1, x_2) \geq 0 \quad \forall x_1, x_2 \in \mathcal{X}$

   equal to 0 iff $x_1 = x_2$

2. $d(x_1, x_2) = d(x_2, x_1)$

3. $d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)$

Examples:

① $\mathcal{X} = \mathbb{R}$

$$d(x_1, x_2) = |x_1 - x_2|$$

② $\mathbb{R}^n$

$$d(\underline{x}_1, \underline{x}_2) = \| \underline{x}_1 - \underline{x}_2 \|$$

eg.

③ $\mathcal{X} = \{$ integrable real fns on $\mathbb{R} \}$

$$d(f_1, f_2) = \left(\int_{-\infty}^{\infty} |f_1(x) - f_2(x)|^p \, dx \right)^{1/p}$$

Neighborhood

$\mathbb{R}^n$

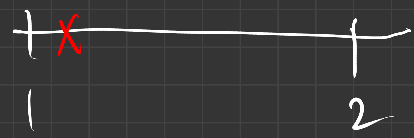
Given $(\mathcal{X}, d)$

* A neighborhood of radius $r > 0$ around $x \in \mathcal{X}$ is

$$N_r(x) = \{ y \in \mathcal{X} : d(x,y) < r \}$$

* If $A \subseteq \mathcal{X}$ then x is an interior point of A if $\exists\, \varepsilon > 0$ st $N_\varepsilon(x) \subseteq A$.

$(1-\varepsilon, 1+\varepsilon)$



Q Take any $1 < x < 2$. Is x an interior pt of $[1, 2]$

A Yes.

$$\varepsilon_1 = x - 1 > 0$$

$$\varepsilon_2 = 2 - x > 0$$

$$\varepsilon = \min(\varepsilon_1, \varepsilon_2)$$

$$(x - \varepsilon, x + \varepsilon) \subseteq [1, 2]$$

Interior point and the interior of a set, open sets

Interior of A = set of all interior pts of A.

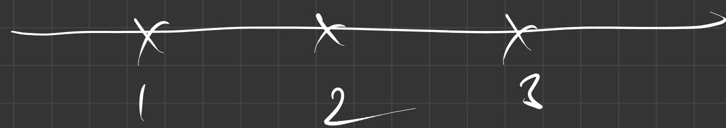
$\text{Int}([1, 2]) = (1, 2)$

$\text{Int}((1, 2)) = (1, 2)$

$\text{Int}(\{1, 2, 3\}) = \emptyset$

$(2-\varepsilon, 2+\varepsilon)$

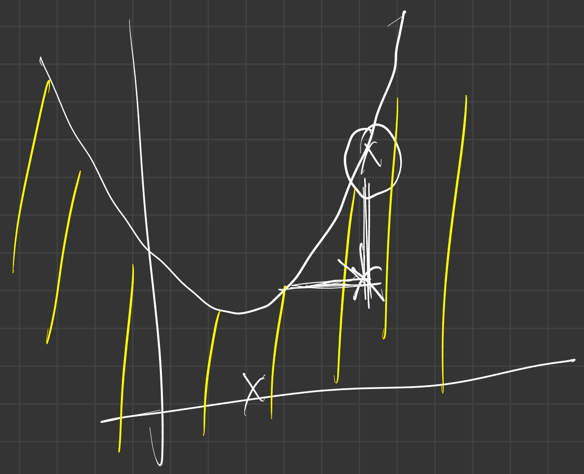
$\text{Int}(\text{countable set}) = \emptyset$



Set A is open if every point is an interior point

Countable subsets of $\mathbb{R}$ are not open

$\mathrm{Int}([a, b]) = (a, b) \neq [a, b]$ Not open



$\mathbb{R}^2$

$\{(x,y) : x^2 = y\}$

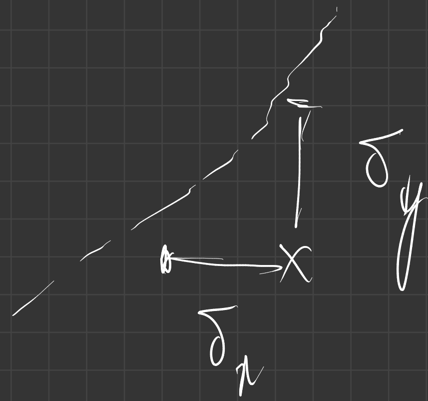
$S = \{(x,y) : x^2 > y\}$

Take any $(x,y) \in \mathbb{R}^2 : x^2 > y$

$\delta > 0 \quad x^2 = y + \delta_y$

$x - \delta_x = \sqrt{y}$

Take $\varepsilon = \dfrac{\min\{\delta_y \ \ \delta_x\}}{10}$



Claim: $N_\varepsilon((x,y)) \subseteq S$

$$d = \min \ d((x,y), \ c) > 0$$

$$N_\varepsilon(d/2)$$

$$\min_{y_1 = x_1^2} \sqrt{(x-x_1)^2 + (y-y_1)^2} = \min_{x_1} \sqrt{(x-x_1^2) + (y-x_1^2)^2} > 0$$

$$S = \{(x,y) : x^2 > y\}$$

Limit points, closure and closed sets, boundary points

We say that x is a limit point of A if every neighborhood $N_\varepsilon(x)$ contains a point in $A \setminus \{x\}$

$S = [1, 2]$

Any point in $[1, 2]$ is a limit point of S

$S = (1,2)$ Any pt in $(1,2)$ is a limit pt of $(1,2)$

$\{1, 2\}$ are also limit pts of S

$S = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$ No pt of S is a limit pt

$S_2 = \{1, \frac{1}{3}, \frac{1}{9}, \dots\}$ 0 is a limit pt

$x \in [1, 2]$ $(x - \delta_x, x + \delta_x)$

<u>Closed set</u>: A is closed if every limit pt of A belongs to A.

$(1, 2)$ — Not closed

$[1, 2]$ — Closed

$\{1, \frac{1}{2}, \frac{1}{4}, \ldots\} = \{\frac{1}{2^n} : n \in \mathbb{N}\}$ → Not closed

$\{1, 2, 3, 4\}$ Closed

Closure of a set A $\quad cl(A) = A \cup$ limit pts of A

$$cl([1,2]) = [1,2]$$

$$cl((1,2)) = [1,2]$$

$$cl\{1, \tfrac{1}{2}, \tfrac{1}{4}, \dots\} = \{0, 1, \tfrac{1}{2}, \tfrac{1}{4}, \dots\}$$

Boundary pts

$$bd(A) = cl(A) \setminus int(A)$$

Compactness

$S \subseteq \mathbb{R}^n$ is compact if it is closed & bounded

$$\sup_{x,y \in S} d(x,y) < \infty$$

Continuity and compactness

$f : X \to Y$

$f(x) = \lim_{y \to x} f(y)$, then f is continuous at x.

Theorem: If X is compact & $f : X \to \mathbb{R}$ is continuous, then, f has a maximum & a minimum achieved in X

① $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$ is continuous

$\mathcal{X} = \mathbb{R}$ is not compact

Max does not exist.

② $\mathcal{X}$ is not closed, bounded, f continuous

$f(x) = x^2 \qquad \mathcal{X} = (1, 2)$

③ $f(x) = \frac{1}{x}$; $x \in (0, 1)$

④ f not continuous, $\mathcal{X}$ is closed & bounded

$$f(x) = \begin{cases} x & \text{for } x < 1 \\ 0 & \text{for } x = 1 \end{cases} \qquad \mathcal{X} = [0, 1]$$

$T = \{ \underline{x}_1, \underline{x}_2, \cdots \underline{x}_m \} \qquad \underline{x}_i \in \mathbb{R}^n$

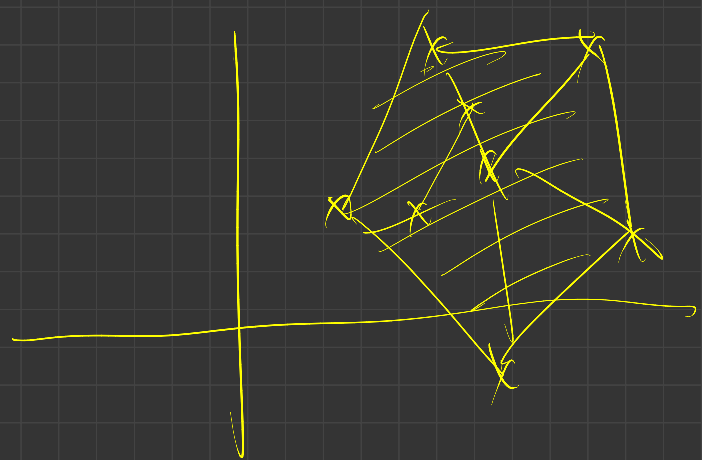
$$S = \left\{ \underline{x} = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m \text{ for some } \alpha_1, \alpha_2 - \alpha_m \quad 0 \le \alpha_i \le 1, \ \sum_{i=1}^{m} \alpha_i = 1 \right\}$$

$\downarrow$

Convex hull of T

- bounded

Exercise: Show that conv hull of $[x_1, x_5]$



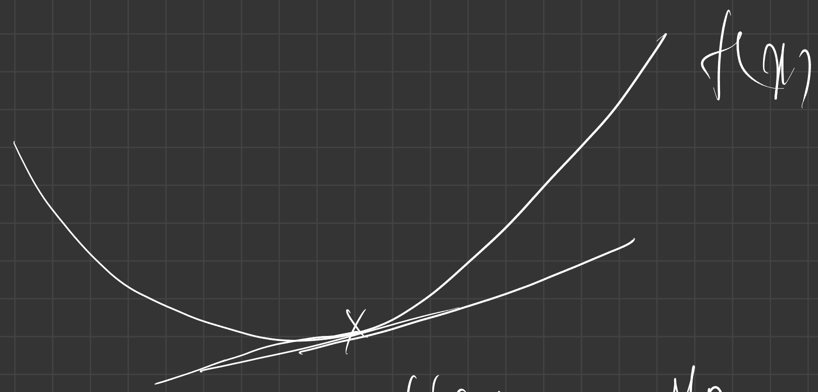
This is a closed set

$[1, 2)$

Derivative

$f : \mathbb{R} \to \mathbb{R}$

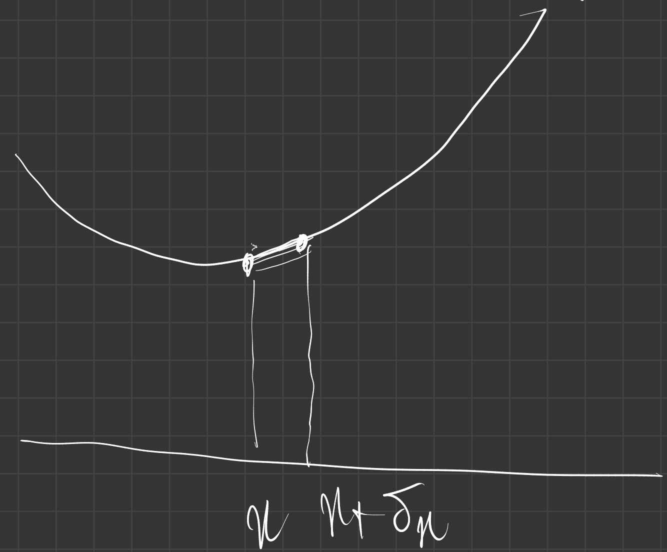
$$f'(x) = \lim_{\delta_x \to 0} \frac{f(x+\delta_x) - f(x)}{\delta_x}$$



$f'(x) \to$ slope of tangent

$$|||$$

$$f(x+\delta_x) = f(x) + \delta_x f'(x) + e(\delta_x)$$



$e(\delta_x)$ ↪ tends to 0 faster than δ_x

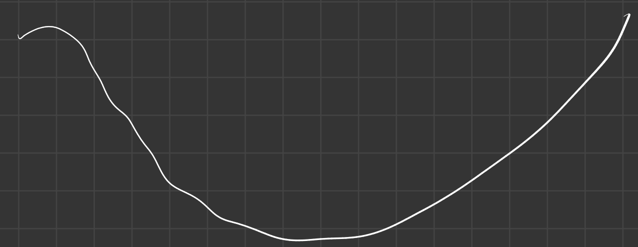
$$\lim_{\delta_x \to 0} \frac{e(\delta_x)}{\delta_x} = 0$$

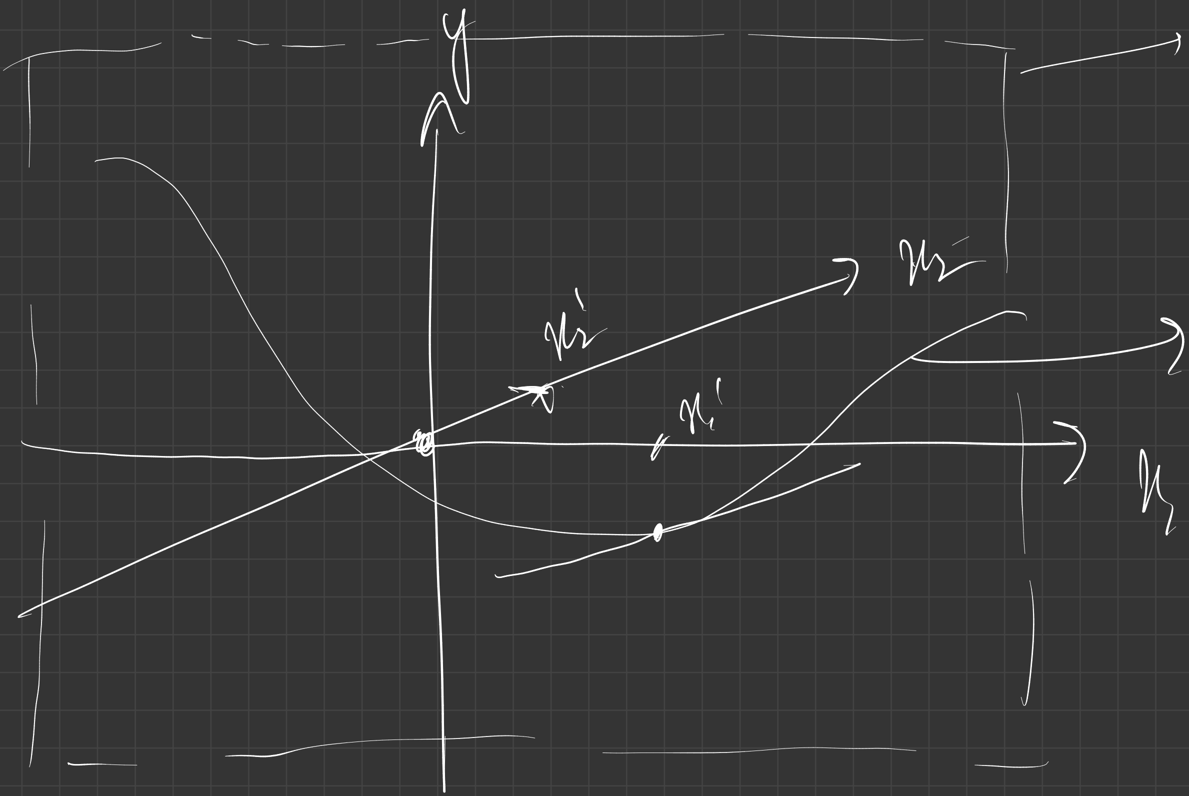
$f : \mathbb{R}^2 \to \mathbb{R}$

$f(x_1, x_2)$

$$\frac{\partial f}{\partial x_1}(x_1, x_2) = \lim_{\delta_{x_1} \to 0} \frac{f(x_1 + \delta_{x_1}, x_2) - f(x_1, x_2)}{\delta_{x_1}}$$

$$\frac{\partial f}{\partial x_2}(x_1 x_2) = \lim_{\delta_{x_2} \to 0} \frac{f(x_1, x_2 + \delta_{x_2}) - f(x_1 x_2}{\delta_{x_2}}$$





parallel to (x_1, y) plane

$f(x_1, x_2')$

$$\frac{\partial f}{\partial x_1}$$

$$\frac{\partial f}{\partial x_2}$$

Direction: unit vector

$\underline{u} \in \mathbb{R}^2$ $\qquad \|\underline{u}\|_2 = 1$

Directional derivative along $\underline{u}$

$$= \left\langle \underline{u}, \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2}\right] \right\rangle$$

$$\lim_{\delta \to 0} \frac{f(\underline{x} + \delta \underline{u}) - f(\underline{x})}{\delta}$$

$$= \frac{f(x_1 + \delta u_1, \ x_2 + \delta u_2) - f(x)}{\delta}$$

$$= \frac{f(x_1 + \delta u_1, \ x_2) + \frac{\partial f}{\partial x_2} \delta u_2 + e(\delta) - f(x_1, x_2)}{\delta}$$

$$= \frac{\cancel{f(x_1, x_2)} + \frac{\partial f}{\partial x_1} \delta u_1 + \frac{\partial f}{\partial x_2} \delta u_2 + e'(\delta) - \cancel{f(x_1, x_2)}}{\delta}$$

$$= \frac{\partial f}{\partial x_1} u_1 + \frac{\partial f}{\partial x_2} u_2$$

General definition $f: \mathbb{R}^n \to \mathbb{R}^m$

$$\lim_{\|h\| \to 0} \frac{\| f(\underline{x} + \underline{h}) - f(\underline{x}) - D_{f(x)} \underline{h} \|}{\|h\|} = 0$$

Then, we say that f is differentiable at $\underline{x}$

✓ The derivative is equal to $D_{f(\underline{x})}$

(Fréchet derivative)

$m \times n$ matrix

$$\left(D_{f(x)}\right)_{i,j} = \frac{\partial f_i}{\partial x_j}$$

(Jacobian)

1 $f: \mathbb{R} \to \mathbb{R}$

$$D_{f(x)} = \frac{df}{dx}$$

2 $f: \mathbb{R}^n \to \mathbb{R}$

$$D_{f(x)} = (\nabla f)^T$$

Properties

(1) f & g differentiable

$$D_{\alpha f+\beta g} = \alpha D_f + \beta D_g$$

(2) $g : R^m \to R^k$ $\quad f : R^n \to R^m$

$$h(x) = g(f(x))$$

$$D_{h(x)} = D_g(f(x)) \; D_f(x)$$

Example :

$$f(\underline{x}) = \underline{x}^T A \underline{x} \qquad f : \mathbb{R}^n \to \mathbb{R}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

$$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{n} a_{ij} x_j + \sum_{i=1}^{n} a_{ii} x_i$$

$$\nabla f = (A + A^T) \underline{x}$$

② $f : S^n_{++} \to \mathbb{R}$

↳ set of all $n \times n$ +ve definite matrices

$$f(X) = \log \det(X)$$

Positive definite matrices

(P1) Have n positive eigenvalues

(P2) $\exists$ some matrix $X^{1/2}$ invertible

$$X = X^{1/2} X^{1/2}$$

Take two points X, Z

$$f(Z) = f(X) + \langle D, Z-X \rangle + e(Z-X)$$

$$Z = X + \Delta X$$

$$\log\det(Z) = \log(\det(X + \Delta X))$$

$$= \log(\det(X^{1/2} X^{1/2} + \Delta X))$$

$$= \log(\det[X^{1/2}(I + X^{-1/2}(\Delta X)X^{-1/2})X^{1/2}]$$

$$= \log(\det X^{1/2} \times \det(I + X^{-1/2} \Delta X X^{-1/2}) \times \det(X^{1/2})]$$

$$= \log\det X^{1/2} + \log\det\left(I + X^{-1/2}\Delta X X^{-1/2}\right) + \log\det X^{1/2}$$

$$= \log\det(X) + \log\det\left(I + \underbrace{X^{-1/2}(\Delta X)X^{-1/2}}\right)$$

$\lambda_1, \lambda_2, \ldots, \lambda_n$

$$= \log\det X + \log\prod_{i=1}^{n}(1+\lambda_i)$$

$$= \log\det X + \sum_{i=1}^{n}\log(1+\lambda_i)$$

$$\approx \log\det X + \sum_{i=1}^{n}\lambda_i$$

$$\sum_{i=1}^{n} \lambda_i = \operatorname{trace}\left(X^{-1/2} \Delta X \; X^{-1/2} \right)$$

$$= \operatorname{trace}\left(X^{-1/2} X^{-1/2} \Delta X \right)$$

$$= \operatorname{trace}\left(X^{-1} \Delta X \right)$$

$$f(X + \Delta X) \approx f(X) + \langle X^{-1}, \Delta X \rangle$$

$$D_f(X) = X^{-1}$$

Second derivative

$$D_f^2(x) = D_{D_f}(x)$$

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$\nabla f : \mathbb{R}^n \to \mathbb{R}^n$$

$$D_f^2 : \mathbb{R}^n \to \mathbb{R}^{n \times n}$$

$$\boxed{(\nabla^2 f)_{ij} = \frac{\partial^2 f}{\partial x_i \, \partial x_j}}$$

Hessian matrix

<u>Chain rule</u>

$$h(x) = g(f(x))$$

$g: \mathbb{R} \to \mathbb{R}$, $f: \mathbb{R}^n \to \mathbb{R}$

$$\nabla^2 f(x) = g''(f(x)) (\nabla f(x)) (\nabla f(x))^T$$

$$+ g'(f(x)) \nabla^2 f(x)$$

$$\frac{f(x_1+\delta u_1,\ x_2+\delta u_2) - f(x_1, x_2+\delta u_2) + f(x_1,\ x_2+\delta u_2) - f(x_1,\ x_2)}{\delta}$$