Fundamentals of Linear Algebra and Matrix Theory

Vector space

$+ : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$, $\cdot : \mathbb{F} \times \mathbb{V} \to \mathbb{V}$

$(\mathbb{V}, +, \cdot)$ over $\mathbb{F}$: Nonempty set $\mathbb{V}$ st $\forall\, \underline{v}_1, \underline{v}_2, \underline{v}_3 \in \mathbb{V}$

$\downarrow$ field

$\alpha, \beta \in \mathbb{F}$,

① $\underline{v}_1 + \underline{v}_2 = \underline{v}_2 + \underline{v}_1$

② $\underline{v}_1 + (\underline{v}_2 + \underline{v}_3) = (\underline{v}_1 + \underline{v}_2) + \underline{v}_3$

③ $\exists\, \underline{0} \in \mathbb{V}$ st $\underline{0} + \underline{v}_1 = \underline{v}_1$

④ for each $\underline{v}_1 \in \mathbb{V}$, $\exists\, \underline{\bar{v}} \in \mathbb{V}$ st $\underline{v} + \underline{\bar{v}} = \underline{0}$

⑤ $(\alpha \cdot \beta) \cdot \underline{v}_1 = \alpha \cdot (\beta \cdot \underline{v}_1)$

⑥ $1 \cdot \underline{v}_1 = \underline{v}_1$

⑦ $(\alpha + \beta) \cdot \underline{v}_1 = \alpha \cdot \underline{v}_1 + \beta \cdot \underline{v}_1$

$\alpha(\underline{v}_1 + \underline{v}_2) = \alpha \underline{v}_1 + \alpha \underline{v}_2$

You should be able to answer the following:

- What is a subspace of a vector space? Given a subset S of a vector space V, do you need to test whether all 7 propertiesare satisfied? Is there an easier test?

- How do you define linear combinations of vectors?

- What do you mean by linear independence?

$$\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \cdots + \alpha_n v_n = \underline{0} \quad \text{iff} \quad \alpha_i = 0 \ \forall i$$

- What do you mean by the span of vectors?

- What is a spanning set of a vector space?

- When are vectors said to be linearly independent?

- What is a basis of a vector space? Is it unique?

- What is the dimension of a vector space?

- What are the four fundamental subspaces associated with a matrix?

- What is the rank of a matrix, and what is its nullity?

- How do you compute the rank or nullity of a given matrix? What is the computational complexity of doing so?

- You should know what elementary row operations are, how to convert a matrix into the row reduced echelon form (RREF), and the QR decomposition of a matrix

Is the following a vector space? If yes, what is its dimension?

① $\mathbb{F}^n$ over $\mathbb{F}$ for any field $\mathbb{F}$

② Set of all $m \times n$ matrices with elements from $\mathbb{R}$ (over $\mathbb{R}$)

$$A^{(i,j)} = \begin{bmatrix} 0 & & \vdots & \\ \vdots & 1 & 0 & \end{bmatrix}_{(i,j)} \text{ entry}$$

③ $\mathbb{Q}^n$ — Not a VS

$\sqrt{2}\, \underline{v} \notin \mathbb{Q}^n$ for any $\underline{v} \in \mathbb{Q}^n \setminus \{0\}$

(4) Set of all polynomials of degree ≤ 3 & coefficients from $\mathbb{R}$.

$a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$\{1, x, x^2, x^3\}$ dim $= 4$

(5) Set of all polynomials with coefficient from $\mathbb{R}$

Yes. dim $= \infty$

(6) Set of all continuous functions

$\mathcal{F} = \{ f : \mathbb{R} \to \mathbb{R},\ f \text{ continuous} \}$ $\quad$ 0 : $f(x) = 0$ $\forall x \in \mathbb{R}$

(1) $\mathbb{R}$ $\mathbb{F} = \mathbb{Q}$

Yes. $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}$

Vectors and matrices:

If $\mathbb{V}$ is finite dimensional V.S, can represent any vector as $n\times 1$ matrix

$$\underline{v} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \cdots + \alpha_n \underline{v}_n$$

$$\underline{v} \equiv \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \text{w.r.t basis } \{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n\}$$

* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear iff

$$f(\alpha \underline{v}_1 + \beta \underline{v}_2) = \alpha f(\underline{v}_1) + \beta f(\underline{v}_2)$$

* A linear transformation can be represented by an $m \times n$ matrix.

Similarity

$A : n \times n$

$B : n \times n$

A is similar to B if $\exists$ an invertible P st

$$A = P B P^{-1}$$

$$A\underset{\sim}{x} = \underbrace{P}_{\text{revert to old basis}} B \underbrace{(P^{-1} \underset{\sim}{x})}_{\text{rep of } \underset{\sim}{x} \text{ in terms of cols of } P}$$

$\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_n\}$ $\quad \underset{\sim}{w} \rightarrow$ Standard ord. basis

$\downarrow$

New basis $\quad \underset{\sim}{w} = \alpha_1 \underset{\sim}{v}_1 + \alpha_2 \underset{\sim}{v}_2 + \dots + \alpha_n \underset{\sim}{v}_n$

$$\underset{\sim}{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$ $\rightarrow$ representation of $\underset{\sim}{w}$ but in terms of new basis

$$\underset{\sim}{w} = \overbrace{\begin{bmatrix} \underset{\sim}{v}_1 & \underset{\sim}{v}_2 & \dots & \underset{\sim}{v}_n \end{bmatrix}}^{P} \underset{\sim}{\alpha}$$

$$\underset{\sim}{\alpha} = P^{-1} \underset{\sim}{w}$$

Determinant

Permutation of $[n] = \{1, 2, 3, \dots, n\}$

$\sigma : \quad (1, 2, 3, 4) \mapsto (3, 1, 2, 4)$

$\sigma(1) = 3, \quad \sigma(2) = 1, \quad \sigma(3) = 2, \quad \sigma(4) = 4$

Pairwise exchanges

$$\sigma : \quad (1,2,3,4) \longmapsto (2,1,4,3)$$

$$\downarrow$$

$$(1,2,4,3) \to (1,2,3,4)$$

$$(2,1,4,3) \to (1,2,4,3) \to (1,4,2,3) \to (1,4,3,2)$$

$$\downarrow$$

$$(1,2,3,4)$$

Theorem: # of pairwise exchanges required to go from $\sigma[n]$ to $[n]$ is always either odd or even

Sign of σ: $\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if even} \\ -1 & \text{if odd} \end{cases}$

Determinant

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) \cdot a_{1,\sigma(1)}\, a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

$$= \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

Examples

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\operatorname{sgn}(1, 2) = +1$$

$$\operatorname{sgn}(2, 1) = -1$$

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i, \sigma(i)}$$

$$= (+1)\, a_{11}\, a_{22} + (-1)\, a_{12}\, a_{21}$$

$$= a_{11}\, a_{22} - a_{12}\, a_{21}$$

Another expression for the determinant

$$\det(A) = \sum_i a_{ij} \, \mathrm{cof}_{ij}(A) \quad \text{for any } j$$

$$= \sum_j a_{ij} \, \mathrm{cof}_{ij}(A) \quad \text{for any } i$$

Computing the determinant

$$A \mapsto \begin{bmatrix} 1 & \ddots & \\ 0 & & 1 \end{bmatrix} \to B$$

- Bring A to RRF
- Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the row multipliers
- Let β be # of row exchanges

$$\det(A) = (-1)^{\beta} \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_k} \det(B)$$

Similarity

$$A\underline{x} = \lambda \underline{x}$$

eigenvalue

eigenvector

$$\det(A - \lambda I) = 0$$

$$(A - \lambda I)\underline{v} = \underline{0}$$

Gram-Schmidt orthogonalization

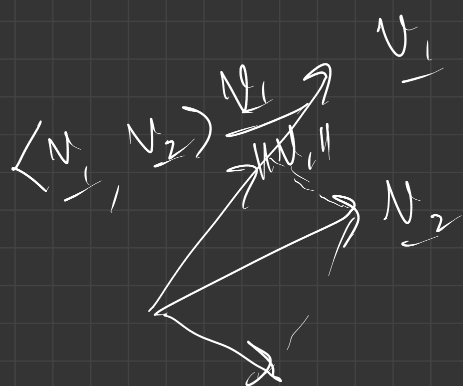
$\underline{u}, \underline{v}$ are orthogonal if $\underline{u}^T \underline{v} = 0$

orthonormal if " & $\|\underline{u}\| = \|\underline{v}\| = 1$

$\begin{bmatrix}1\\2\end{bmatrix}$ $\begin{bmatrix}2\\1\end{bmatrix}$

Given basis $\{\underline{v}_1, \dots, \underline{v}_n\}$

- Take $\underline{v}_1$ & $\underline{u}_1 = \dfrac{\underline{v}_1}{\|\underline{v}_1\|}$



$\underline{v}_2 - \dfrac{\langle \underline{v}_1, \underline{v}_2 \rangle}{\|\underline{v}_1\|} \underline{u}_1$

- For $i = 2, 3, 4, \dots, n$

① $\underline{\tilde{u}}_i = \underline{v}_i - \sum_{j=1}^{i-1} \langle \underline{v}_i, \underline{u}_j \rangle \underline{u}_j$

② $\underline{u}_i = \dfrac{\tilde{u}_i}{\|\tilde{u}_i\|}$

$A = QR$

$\downarrow$

full col. rank

Eigenspace

$A\underline{v} = \lambda \underline{v}$ → eigenvector

λ ↓ eigenvalue

$\det(A - \lambda I) = 0$

distinct

$\lambda_1, \lambda_2 \sim \lambda_k$ → eigenvalues

↓

λ_i has Multiplicity α_i

called algebraic / arithmetic multiplicity of λ_i

Algebraic multiplicity and geometric multiplicity, linear independence of eigenspaces

$$V_i = \{ \underline{v} : A\underline{v} = \lambda_i \underline{v} \}$$

$$A(\alpha \underline{v}_1 + \beta \underline{v}_2) = \alpha A v_1 + \beta A v_2$$
$$= \alpha \lambda_i v_1 + \beta \lambda_i v_2$$
$$= \lambda_i (\alpha v_1 + \beta v_2)$$
$$\Uparrow$$
$$V_i$$

$\dim(V_i)$ = Geometric multiplicity of λ_i

If we can construct a basis for $\mathbb{R}^n$ out of eigenvectors $\{\underline{v}_1, \ldots \underline{v}_n\}$, then A rep in terms of $\downarrow$ is diagonal

$A = PBP^{-1}$

$\searrow$ cols of P are new basis vectors

linear transform wrt P.

$A = PDP^{-1}$.

A is diagonalizable iff

① $\sum_{i=1}^{k} a_i = n$

② Geom multip$(\lambda_i) = a_i \quad \forall i$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\lambda = 1$

$\alpha_1 = 2$

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Geom multiplicity $= \dim NS(A - \lambda I)$

$= 1 \neq 2$

$1 \leq$ Geom Mult $\leq$ Arith Mult.

Projection matrices and spectral decomposition

$$A = \lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_k E_k$$

$\downarrow$

projection matrix

project $\mathbb{R}^n$ into the eigenspace

corresp to λ_i

$$E^2 = EE = E$$

Symmetric matrices and their eigenvectors

$$A^T = A$$

$\lambda_1 \neq \lambda_2$ $\quad v_1, v_2$ corresp to λ_1, λ_2

$$v_1^T A v_2 = \lambda_2 v_1^T v_2$$

$$(v_1^T A v_2)^T = \lambda_2 v_2^T v_1 = \lambda_2 v_1^T v_2$$

$$\|$$

$$v_2^T A^T v_1 = v_2^T A v_1 = \lambda_1 v_2^T v_1 = \lambda_1 v_1^T v_2$$

$$\Rightarrow v_1^T v_2 = 0$$

Eigenvectors corresp to distinct eigenvalues are orthogonal

Spectral theorem

A real matrix A is symmetric if & only if it is orthogonally diagonalizable.

$$A = PDP^{-1} = PDP^{T}$$

If $A = PDP^{T}$, then $A^{T} = (PDP^{T})^{T}$

$$= PD^{T}P^{T}$$

$$= PDP^{T}$$

If A is symmetric.

If $n=1$, trivial.

Suppose all symmetric $(n-1)\times(n-1)$ matrices are orthogonally diagonalizable.

A $\quad \lambda_1$ $\quad \underline{v}_1 \to$ unit norm eigenvector

$\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\} \to$ orthonormal basis

$$A = P_1 \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \underbrace{\begin{bmatrix} & & \\ & & \end{bmatrix}}_{B} & \\ 0 & & & \end{bmatrix} P^T = \bar{P} D \bar{P}^T$$

Symmetric

$$A(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$$

$$B \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \lambda \begin{bmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\|$$

$$\alpha_1 Av_1 + \alpha_2 Av_2 + \cdots + \alpha_n Av_n$$

$$A\underline{v} = A \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Positive Semidefinite (PSD) and Positive Definite (PD) matrices

A real symmetric A is positive semidefinite if all eigenvalues are ≥ 0.

$A \succeq 0$ $\qquad\qquad$ $A \succeq B$ iff $(A-B) \succeq 0$

A is positive definite if all eigenvalues > 0

$A \succ 0$ $\qquad\qquad$ $A \succ B$ iff $(A-B) \succ 0$

$$\underline{x}^T A \underline{x} \qquad \underline{x} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \cdots + \alpha_n \underline{v}_n$$

$$\|$$

$$\underline{v}^T \left(\alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \cdots + \alpha_n \lambda_n v_n \right)$$

$$= \alpha_1^2 \lambda_1 + \alpha_2^2 \lambda_2 + \alpha_3^2 \lambda_3 + \cdots + \alpha_n^2 \lambda_n \geq 0$$

$$\lambda_i \geq 0 \quad \forall i$$

$$A \text{ is PSD} \iff \underline{x}^T A \underline{x} \geq 0 \quad \forall \underline{x} \in \mathbb{R}^n$$

Square root of PSD matrix

$$A = PDP^T$$

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$A^{1/2} = P \begin{bmatrix} \sqrt{\lambda_1} & & & 0 \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\lambda_n} \end{bmatrix} P^T$$

$$A^{1/2} A^{1/2} = P \Lambda^{1/2} P^T P \Lambda^{1/2} P^T$$

$$= P \Lambda^{1/2} \Lambda^{1/2} P^T = PDP^T = A$$

Singular Value Decomposition (SVD)

$A : m \times n$ $\quad$ $\text{rank}(A) = r$

$A = U \Lambda V^T$

U: $m \times r$ $\quad$ Λ: $r \times r$ $\quad$ $V : n \times r$

$A^T A = V \Lambda^2 V^T$

$\downarrow$

PSD

$x^T A^T A x = \|Ax\|^2 \geq 0$

Singular values : Square root of eigenvalues of $A^T A$

$$A^T A = (U \Lambda V^T)^T U \Lambda V^T$$

$$= V \Lambda U^T U \Lambda V^T$$

$$= V \Lambda^2 V^T$$

$$A A^T = U \Lambda^2 U^T$$

Goal: Solve minimization problems

1. Unconstrained minimization

   - closed form solutions

   - numerical methods