CHANNEL CAPACITY

message | codeword | recieved seq | estimate

$$M^k \longrightarrow \boxed{ENC} \xrightarrow{X^n} \boxed{P_{Y|X}} \xrightarrow{Y^n} \boxed{DEC} \longrightarrow \hat{M}^k$$

$\in \{0,1\}^k$ $\qquad X_i \in \mathcal{X}$ $\qquad Y_i \in \mathcal{Y}$

$$P_{Y^n|X^n}$$

$$P_{Y^n|X^n}(y^n|x^n) = \prod_{i=1}^{n} P_{Y|X}(y_i|x_i)$$

$n \to$ # channel uses

$$P_e = P_e^{avg} = \Pr[\hat{M}^k \neq M^k]$$

$$\lim_{n\to\infty} P_e = 0$$

Suppose $Y^n = X^n$

In n channel uses, can tx $\lfloor \log |\mathcal{X}|^n \rfloor$

$= \lfloor n \log |\mathcal{X}| \rfloor$ bits

$|\mathcal{X}| = 2$ $k = 1$

$|\mathcal{X}| = 4$ $k = 2$

$k = O(n)$

$|\mathcal{X}|$

$$\boxed{R_n = \frac{k}{n}} \longrightarrow \text{rate}$$

$|\mathcal{X}| = 3$

$00 \to 1$

M = Message set

$01 \to 2$

$|M| = |\mathcal{X}|^n$

$10 \to 3$

$$R_n = \frac{\log |M|}{n} = \frac{n \log |\mathcal{X}|}{n}$$

Q: If we want $P_e \to 0$, how large can R be?

Defn: We say that $R > 0$ is ACHIEVABLE, if $\exists$ (ENC_n, DEC_n) for which:

$$\lim_{n \to \infty} P_e^{(n)} = 0$$

$$\lim_{n \to \infty} R_n = R$$

$\max\{R : R \text{ is achievable for } p_{Y|X}\} \to$ Channel capacity

Claim The maximum achievable R for $P_{Y|X}$ is

$$C = \max_{P_X} I(X;Y)$$

Information capacity

Capacity of $P_{Y|X}$

Converse :

If $R = \lim_{n\to\infty} R_n > C$, then

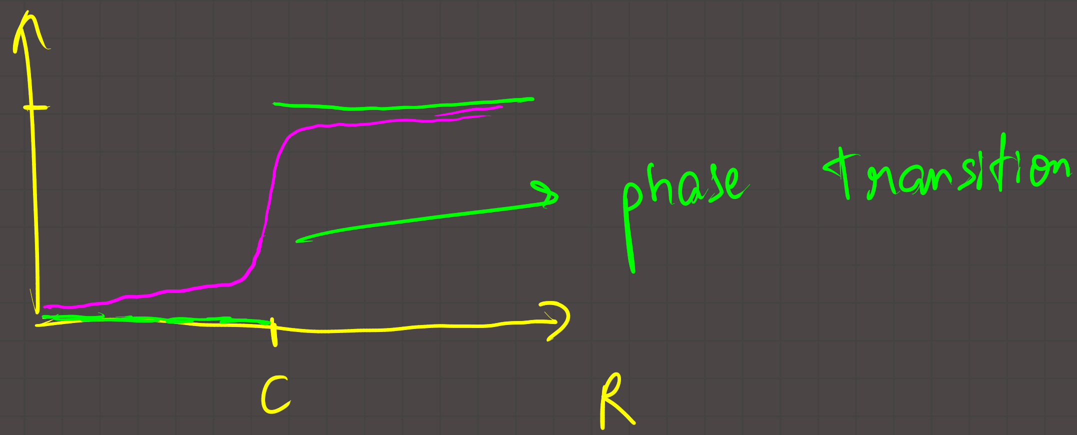
$$\lim_{n\to\infty} P_e > 0$$

(Weak Converse)

In fact, $P_e \to 1$ as $n \to \infty$

(Strong converse)

$\lim_{n \to \infty} P_e$



Fano's inequality:

$$H(m^k | y^n) \leq 1 + nR_n P_e^{(n)}$$

$$\parallel$$

$$H(X^n | y^n) \leq 1 + nR_n P_e^{(n)}$$

$$E = \begin{cases} 1 & \hat{m}^k \neq m^k \\ 0 & \hat{m}^k = m^k \end{cases}$$

$$\mathbb{E}(E) = 1 \times \Pr[E=1] + 0 \times \Pr[E=0]$$

$$= P_e^{(n)}$$

$$H(E, m^k | y^n) = H(m^k | y^n) + H(E | m^k, y^n)$$

$$= H(m^k | y^n) \quad \text{--- (1)}$$

Given m^k, y^n, you know if an error has occurred or not

$$H(E, M^k \mid Y^n) = \underbrace{H(E \mid Y^n)}_{\le H(E) \le \log 2} + H(M^k \mid E, Y^n)$$

$$\le 1 + \underbrace{H(M^k \mid Y^n, E=0)\, \Pr[E=0]}_{=0}$$

$$+ \underbrace{H(M^k \mid Y^n, E=1)}_{\le H(M^k) = k = nR_n} \underbrace{\Pr[E=1]}_{P_e^{(n)}}$$

$$=$$

$$\le 1 + nR_n P_e \quad \text{— (2)}$$

(1) & (2) $\Rightarrow$ $H(M^k \mid Y^n) \le 1 + nR_n P_e$

$\Rightarrow$ $H(X^n \mid Y^n) \le 1 + nR_n P_e$

Claim: $I(X^n;Y^n) \le nC$ for all P_{X^n}

for DMC $I(X^n;Y^n) \le \sum_{i=1}^{n} I(X_i;Y_i)$

$$I(X^n;Y^n) = H(Y^n) - H(Y^n|X^n)$$

$$= \sum_{i=1}^{n} H(Y_i|Y_1 \dots Y_{i-1})$$

$$- \sum_{i=1}^{n} H(Y_i|X^n, Y_1 \dots Y_{i-1})$$

$$\le \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i|X^n, Y_1 \dots Y_{i-1})$$

$$= \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i|X_i)$$

For ANY DMC,

$$H(Y_i | X_1 \dots X_i \dots X_n, Y_1 \dots Y_{i-1}) = H(Y_i | X_i)$$

$$= \sum_{i=1}^{n} I(X_i ; Y_i)$$

$$\leq nC$$

① $H(M^k \mid Y^n) \le 1 + nR_n P_e^{(n)}$

② $I(X^n; Y^n) \le nC$

$$
\begin{aligned}
nR_n = H(M^k) &= H(M^k \mid Y^n) + I(M^k; Y^n) \\
&\le 1 + nR_n P_e^{(n)} + I(M^k; Y^n) \\
&\le 1 + nR_n P_e^{(n)} + I(X^n; Y^n) \quad \Big| \text{ ENC is 1-1 map} \\
&\le 1 + nR_n P_e^{(n)} + nC
\end{aligned}
$$

$$R_n \leq \frac{1}{n} + R_n P_e^{(n)} + C$$

$$P_e^{(n)} \to 0 \quad \text{as } n \to \infty$$

$$\lim_{n \to \infty} R_n \leq C$$

For a DMC over finite alphabet,

$$C = \max_{p_X} I(X;Y)$$

$$Y = X + Z \qquad X, Z \in \mathbb{R}$$

If $\mathcal{X}$ is $\mathbb{R}$

$$C = \max_{f_X} I(X;Y)$$

Input constraints: $\frac{1}{n}\sum_{i=1}^{n} g(x_i) \le P$

power constraint: $\frac{1}{n}\sum_{i=1}^{n} x_i^2 \le P$

$$C = \max_{P_X:\ \mathbb{E}g(X)\le P} I(X;Y)$$

$$H(X) = \sum_x p(x) \log \frac{1}{p(x)}$$

Q: Can $H(X)$ be infinite?

Is there a $p(x)$: $x \in \{1, 2, 3, \ldots\}$

st $H(X) = \infty$?

$$p(x) = \frac{\alpha}{x^2} \qquad x = 1, 2, 3, \ldots$$

$$H(X) = \sum_{x=1}^{\infty} \frac{\alpha}{x^2} \log \frac{x^2}{\alpha}$$

$$= \beta \sum_{i=1}^{\infty} \frac{\log_2(x)}{x^2}$$

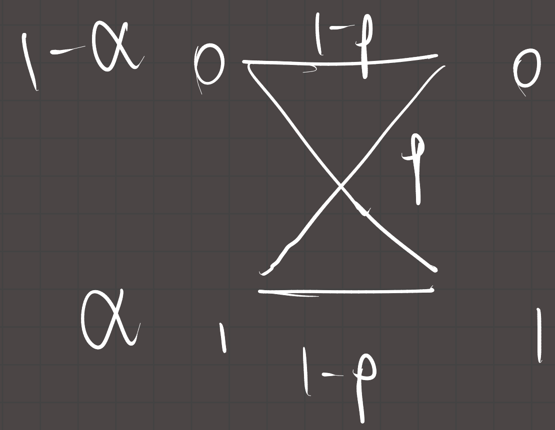
$$p(x) = \frac{\alpha}{x^\beta} \qquad p(x) = \frac{\alpha}{x \log x}$$

CHECK :

$$p(x) = \frac{\alpha}{x \log^2 x} \qquad x = 3, 3 \ldots$$

$$H(X) = \infty$$

BSC



$$C = \max_{p_X} I(X;Y)$$

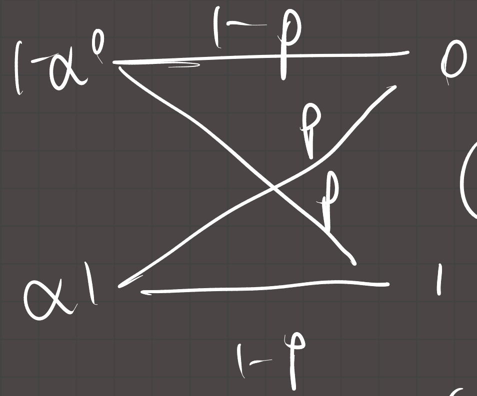
$p_X(1) = \alpha$, $\quad p_X(0) = 1-\alpha$

$$C = \max_{0 \le \alpha \le 1} I(X;Y)$$

$$I(X;Y) = H(Y) - H(Y|X)$$

$$H(Y|X) = \sum_{x,y} p(x,y) \log_2\left(\frac{1}{p(y|x)}\right)$$

$$= \sum_x p(x) \sum_y p(y|x) \log\left(\frac{1}{p(y|x)}\right)$$



$$p_Y(0) = (1-\alpha)(1-p)+\alpha p$$

$$= \sum_x p(x) \underbrace{\left[p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{1-p} \right]}_{H_2(p)}$$

$$p_Y(1) = (1-\alpha)p + \alpha(1-p)$$

$$= H_2(p)$$

$$C = \left(\max_{\alpha} H(Y) \right) - H_2(p) \qquad \le 1 - H_2(p)$$

$p_Y(0) = (1-\alpha)(1-p) + \alpha p \qquad p_Y(1) = (1-\alpha)p + (1-p)\alpha$

$\alpha = \frac{1}{2} \Rightarrow p_Y(1) = p_Y(0) = \frac{1}{2} \Rightarrow H(Y) = 1 \text{ bit}$

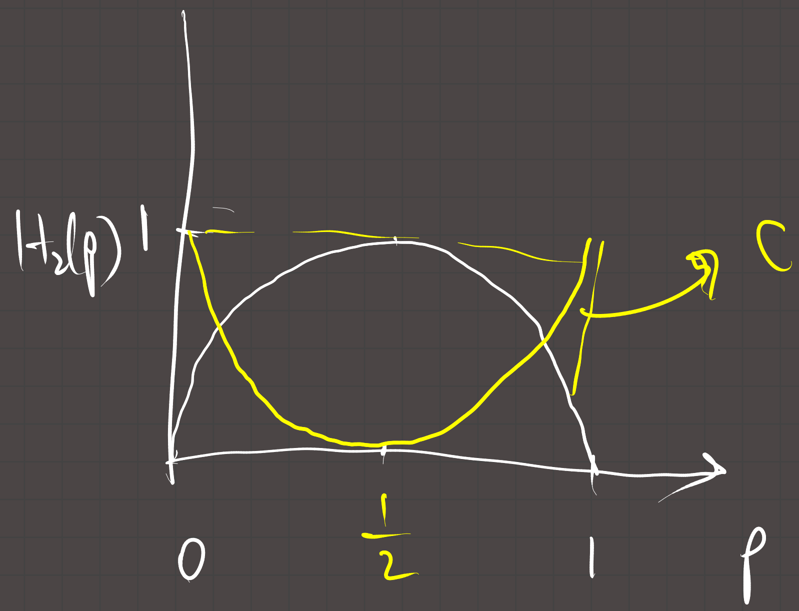
$C = 1 - H_2(p)$

$p = \frac{1}{2}$: $C = 0$

Y is independent of X

$p = 0$: $C = 1$

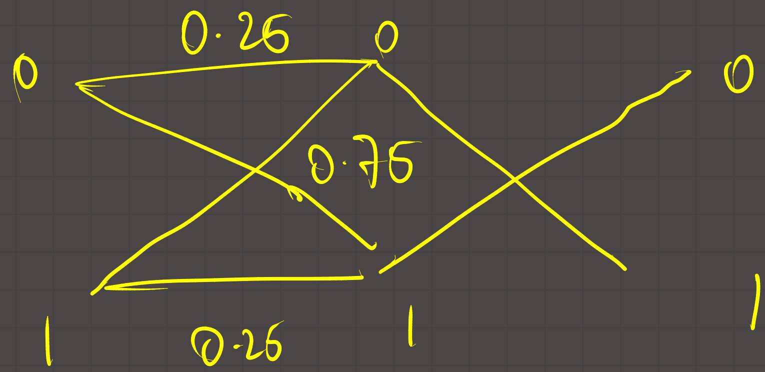
$p = 1$: $C = 1$



Take $0 < p < \frac{1}{2}$

(ENC, DEC) for $BSC(p)$: R, P_e

$BSC(1-p)$ R, P_e



$BSC(p) \to$ flip(NOT) o/p

$\Downarrow$

$BSC(1-p)$

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