Low-Density Parity-Check (LDPC) Codes

$$\arg\max_{(x_1 \ldots x_n) \in C} p(y_1 \ldots y_n \mid x_1 \ldots x_n)$$

$$= \arg\max_{(x_1 \ldots x_n) \in \{0,1\}^n} \left(\prod_{i=1}^{n} p(y_i \mid x_i) \right) \prod_{j=1}^{n-k} \mathbb{1}_{\{\underline{h}_j \underline{x}^T = 0\}}$$

$$\arg\max_{x_i} p(y_1 \ldots y_n \mid x_i) = \arg\max_{x_i} \max_{(x_1 \ldots x_{i-1}, x_{i+1} \ldots x_n)} p(y^n \mid x^n)$$

$$= \arg\max_{x_i} \max_{(x_1 \ldots x_n) \in \{0,1\}^n} \left(\prod_{i=1}^{n} p(y_i \mid x_i) \right) \prod_{j=1}^{n-k} \mathbb{1}_{\{\underline{h}_j \underline{x}^T = 0\}}$$

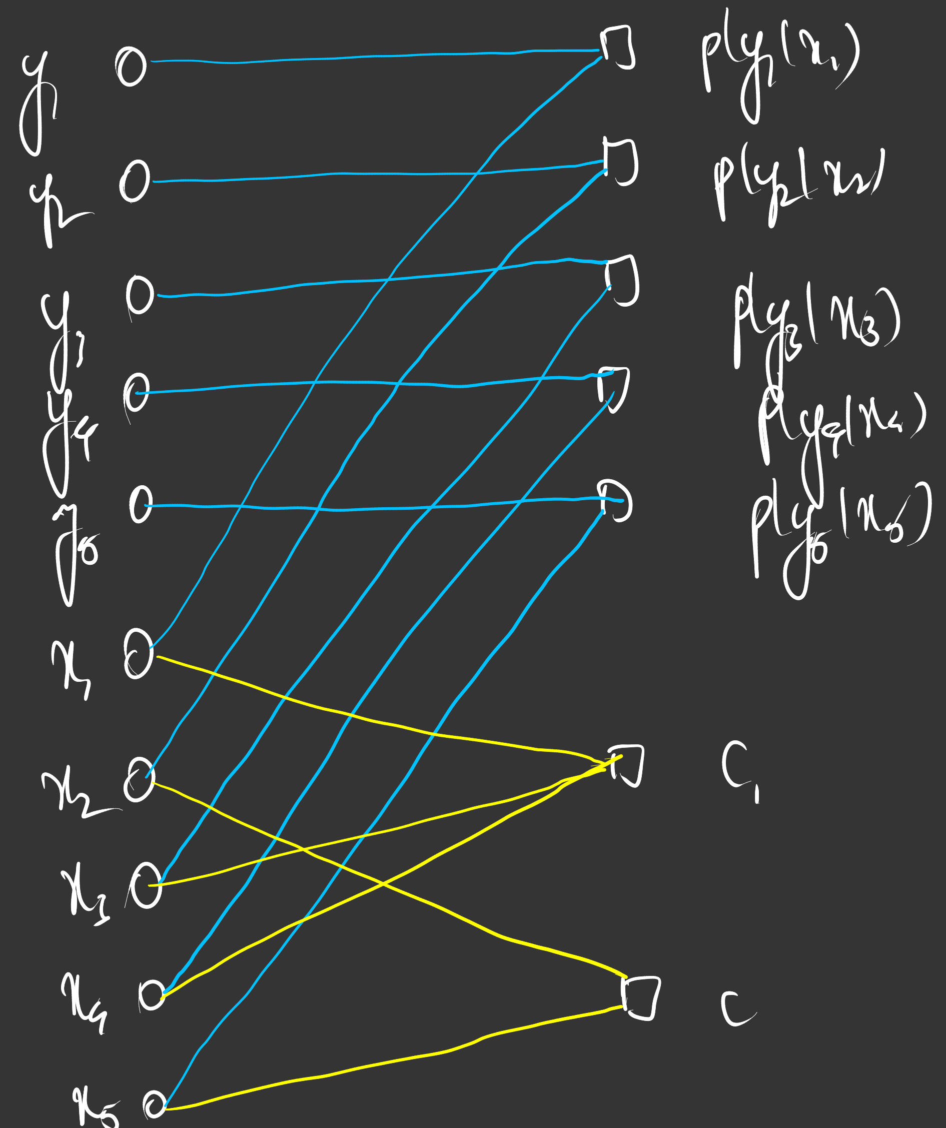
$$\arg\max_{x_i} p(y_1 \dots y_n, x_i) = \arg\max_{x_i} \sum_{\substack{x_1 \dots x_{i-1} \\ x_{i+1} \dots x_n}} p(y_1 \dots y_n\, x_1 \dots x_n)$$

$$= \arg\max_{x_i} \sum_{\substack{x_1 \dots x_{i-1} \\ x_{i+1} \dots x_n}} p(y_1 \dots y_n \mid x_1 \dots x_n)\, \mathbb{1}_{\{x^T \in C\}}$$

$$\hat{x}_i = \arg\max_{x_i} \sum_{\substack{x_1 \dots x_{i-1} \\ x_{i+1} \dots x_n}} \left(\prod_{i=1}^{n} p(y_i \mid x_i) \right) \prod_{j=1}^{n-k} \mathbb{1}_{\{ \underline{h_j}\, \underline{x}^T = 0 \}}$$

$$(\hat{x}_1, \hat{x}_2 \dots \hat{x}_n)$$

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



Factor graph

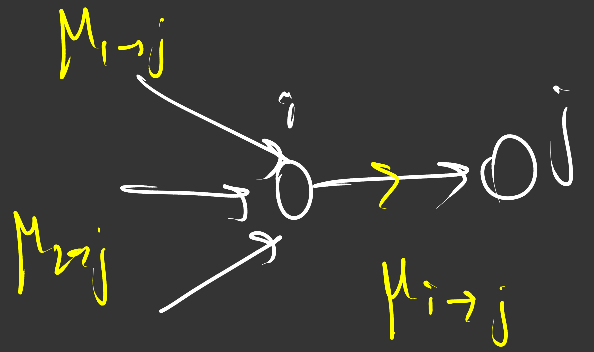
|||

Junction tree

if FG has no cycles.

Update rules:

$$\mu_{i \to j}(x_{S_j \cap S_i}) = \sum_{x_{S_i \setminus S_j}} \alpha(x_i) \prod_{l \in N_i \setminus j} \mu_{l \to i}(x_{S_l \cap S_i})$$



Algorithm : Belief propagation :

(1) Initialization : Given $y_1 - y_n$,

compute $\left(p(y_i|x_i) : x_i \in \{0,1\}\right)$

$$\mu^{(0)}_{v_j \to c_i}(x_j) = p(y_j|x_j)$$

(2) Until convergence : do

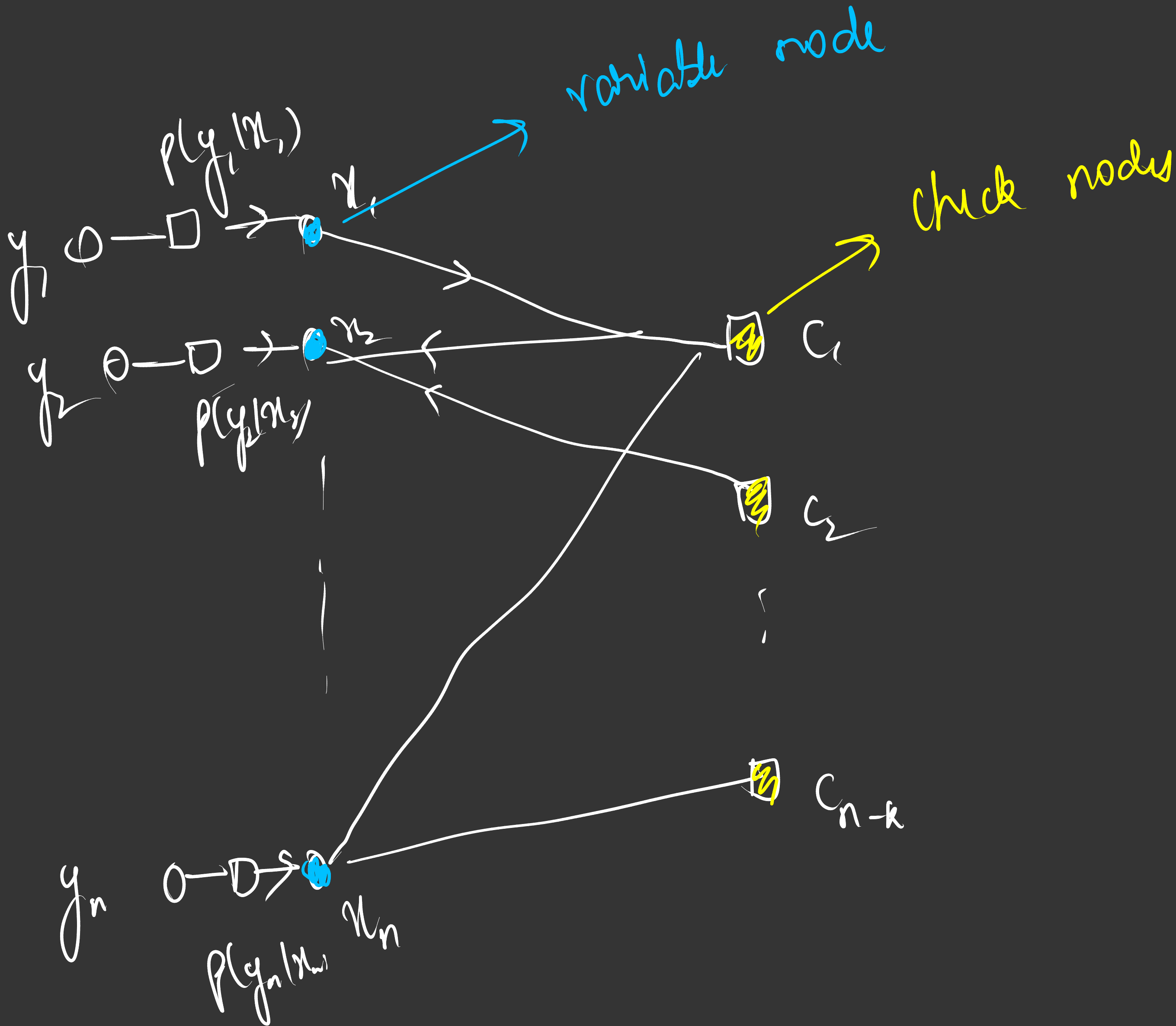
(i) Check nodes to variable nodes

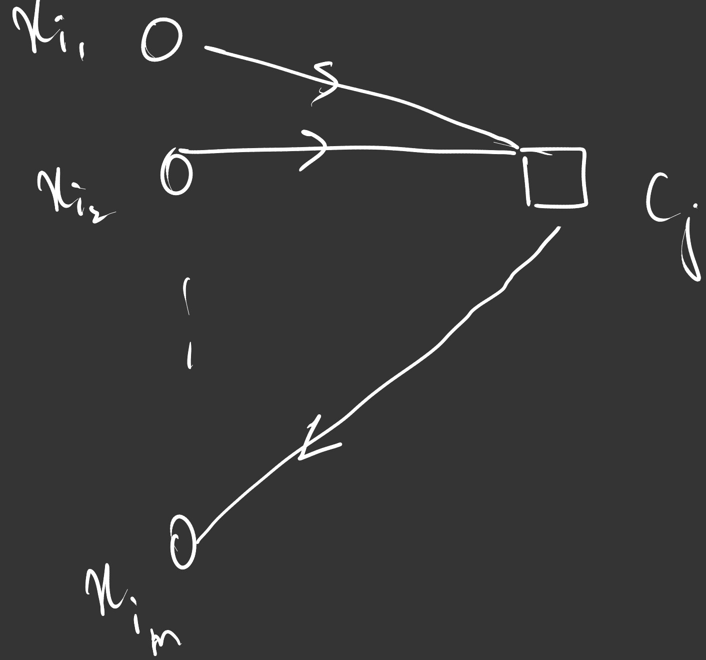
$$\mu^{(t)}_{c_i \to v_j}(x_j) = \sum_{x_{\mathcal{N}(c_i)\setminus j}} \alpha(x_{\mathcal{N}(c_i)}) \prod_{l \in \mathcal{N}(c_i)} \mu^{(t-1)}_{v_l \to c_i}(x_l)$$

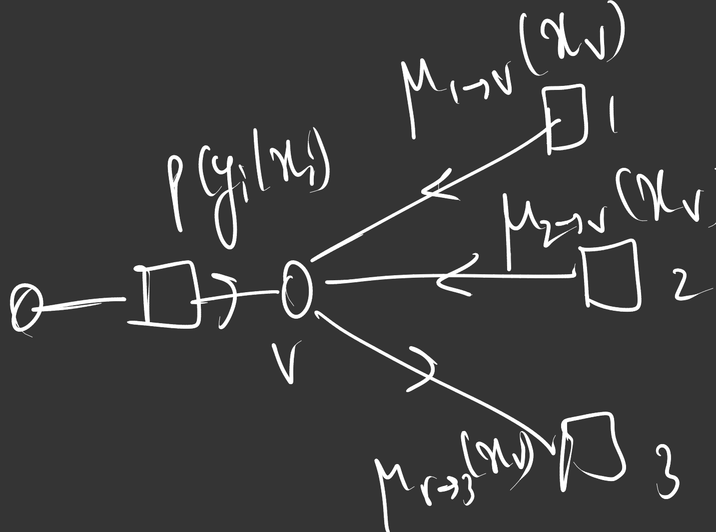
$$\mathbb{1}\{\sum_{l \in \mathcal{N}(c_i)} x_l = 0\}$$

(ii) Variable node to check node

$$\mu^{(t)}_{v_j \to c_i}(x_j) = p(y_j \mid x_j) \prod_{l \in N(v_j)} \mu^{(t)}_{c_l \to v_j}(x_j)$$


variable node
check nodes
$p(y_1|x_1)$
x_1
y_1
y_2
x_2
$p(y_2|x_2)$
c_1
c_2
c_{n-k}
y_n
x_n
$p(y_n|x_n)$





$\mu_{1\to v}(x_v)\mu_{2\to v}(x_v)$

$p(y_v|x_v)$

Simplified Algorithm for BEC:

① Initialization: Label all variable node with recieved value

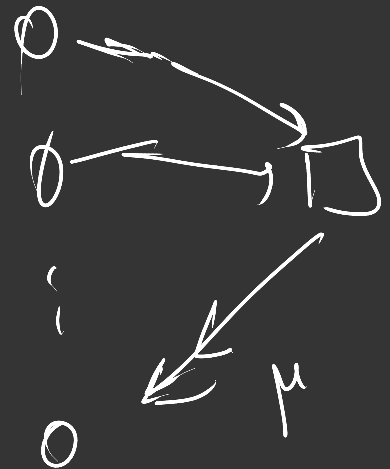
$$x_i^{(0)} = y_i$$

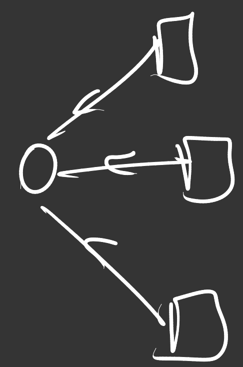
② Iteration:

- $$\mu^{(t)}_{v_i \to c_j} = x_i^{(t)}$$

- $$\mu^{(t)}_{c_j \to v_i} = \begin{cases} \displaystyle\sum_{l \in \mathcal{N}(c_j)\setminus v_i} \mu^{(t)}_{v_i \to c_j} & \text{if all msg} \in \{0,1\} \\ e & \text{else} \end{cases}$$

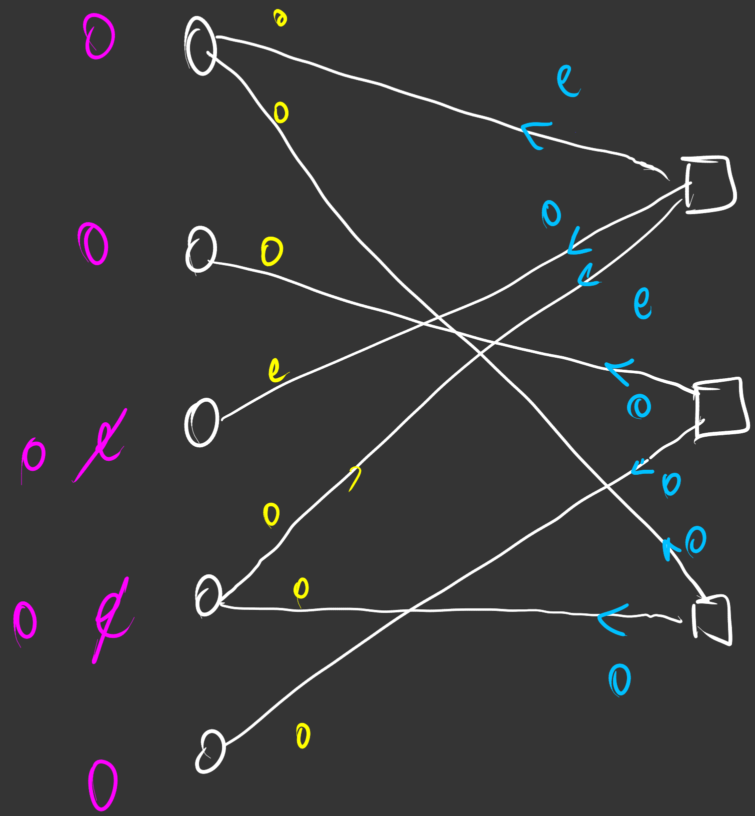
- $x_i^{(t+1)} = \begin{cases} \mu_{c_j \to v_i} & \text{if any of incoming msg} \neq e \end{cases}$

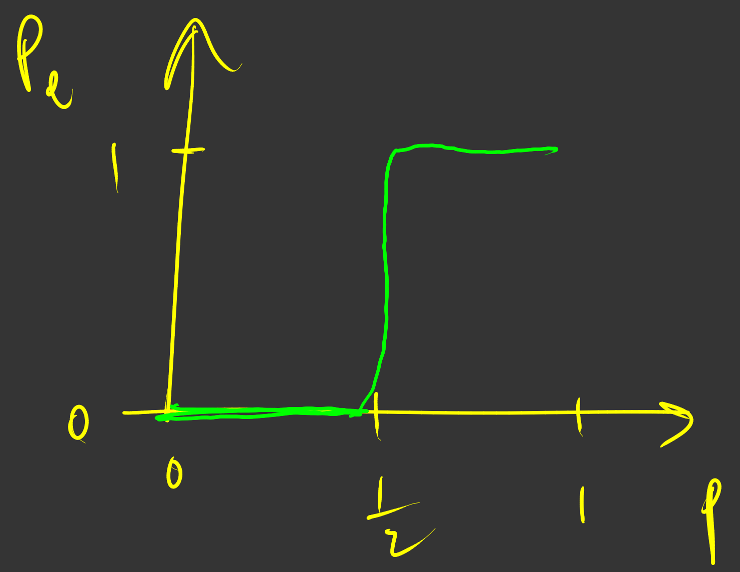




Homework: Show that this is equivalent to general algorithm

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$





P_L
1
0
0
$\frac{1}{2}$
1
p

Regular LDPC codes

Factor graph is "Tree-like" $\longrightarrow$ No short cycles

Then, BP gives good performance!

(d_v, d_c) - regular LDPC codes: Factor graph is s.t.

* each variable node is connected to d_v check nodes

* each check node is connected to d_c var nodes

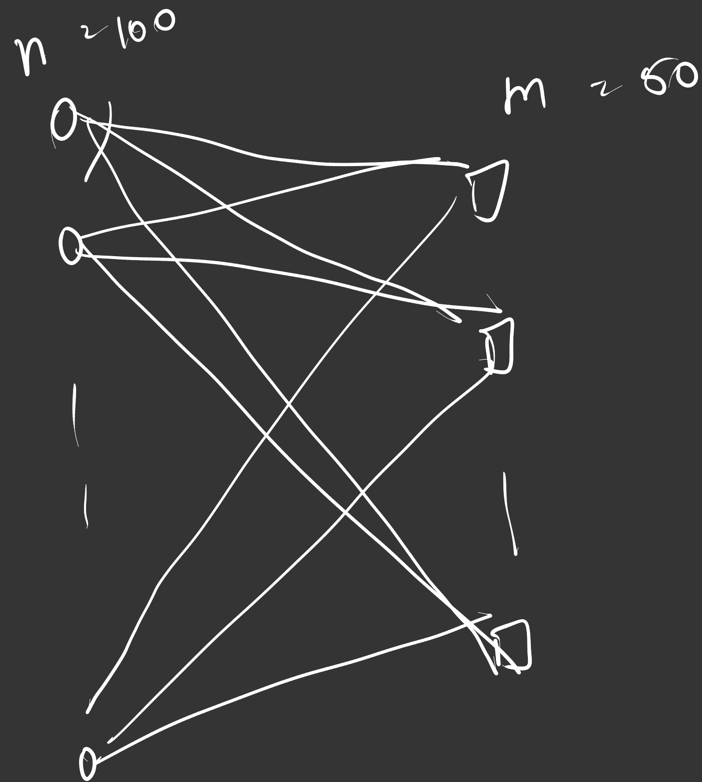
$\downarrow$

smallest cycle in FG is $\Theta(\log n)$

$H = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \rightarrow d_c$

$d_v \quad\quad d_v$

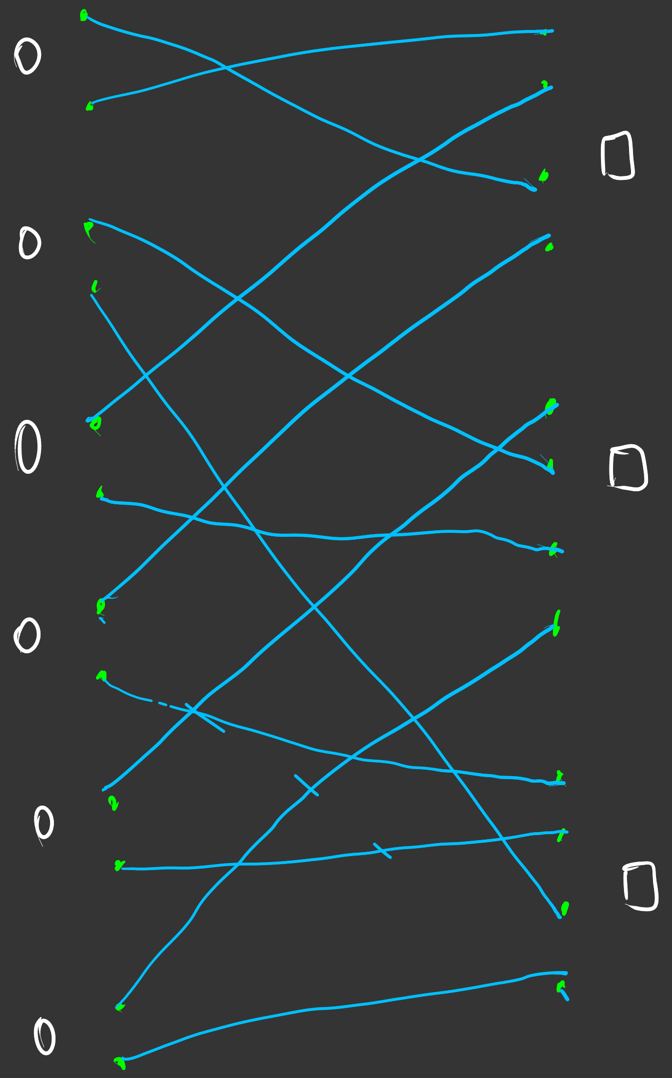
$$\boxed{d_c = \frac{n d_v}{m}}$$



$d_v = 3$

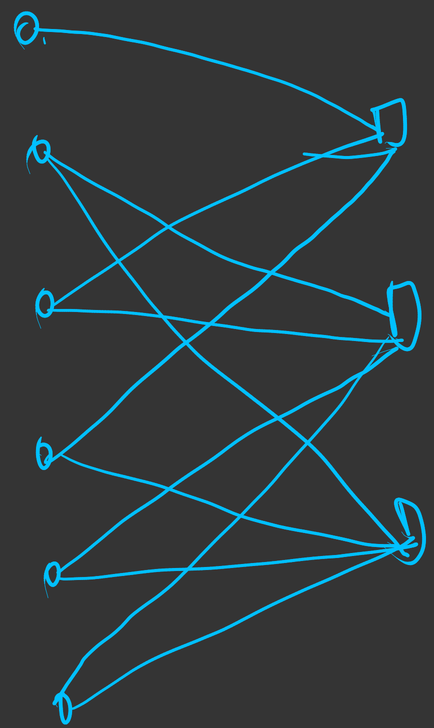
$d_c = 6$

Configuration model / Standard ensemble



$(2, 4)$

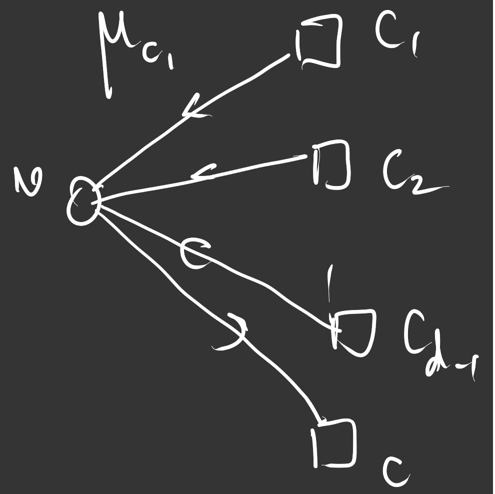
$n d_v = m d_c$



Recall

Message update rules at variable node

$$\mu_{v\to c}(0) = \prod_{i=1}^{d-1} \mu_{c_i\to v}(0)$$



$$\mu_{v\to c}(1) = \prod_{i=1}^{d-1} \mu_{c_i\to v}(1)$$

$$r_{v\to c} = \frac{\mu_{v\to c}(0)}{\mu_{v\to c}(1)} = \prod_{i=1}^{d-1} \frac{\mu_{v_i\to c}(0)}{\mu_{v_i\to c}(1)}$$

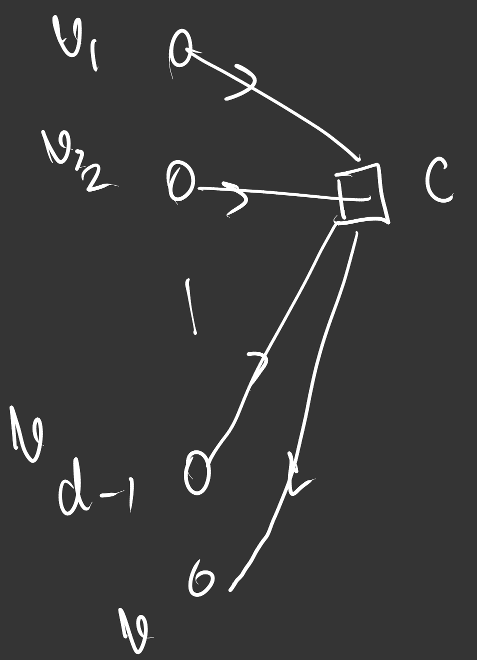
Beliefs

$$= \prod_{i=1}^{d-1} r_{v_i\to c}$$

$$l_{v\to c} = \ln r_{v\to c} = \sum_{i=1}^{d-1} l_{v_i\to c}$$

Message updates at check nodes

$$\mu_{c\to v}(x) = \sum_{x_1 \ldots x_d} \mathbb{1}_{\{\sum_{j=1}^{d} x_j = 0\}} \prod_{i=1}^{d-1} \mu_{v_i \to c}(x_i)$$



x : $0 \mapsto +1$

$1 \mapsto -1$

$$\sum_{i=1}^{d} x_i \longmapsto \prod_{i=1}^{d} x_i$$

$0 \oplus 0 = 0$

$0 \oplus 1 = 1$

$1 \oplus 0 = 1$

$1 \oplus 1 = 0$

x_1	x_2	
1	1	1
1	-1	-1
-1	1	-1
-1	-1	1

$$\mu_{c\to v}(x_i) = \sum_{\substack{x_1 \dots x_d \\ \in \{1,-1\}^d : \\ x_d = x}} \mathbb{1}\left\{ \prod_{i=1}^{d} x_i = 1 \right\} \prod_{i=1}^{d-1} \mu_{v_i \to c}(x_i)$$

$$\prod_{i=1}^{d} x_i = 1 \Rightarrow x \prod_{i=1}^{d-1} x_i = 1$$

$$\Leftrightarrow \prod_{i=1}^{d-1} x_i = x$$

$$\mu_{c\to v}(x) = \sum_{\substack{x_1 \dots x_{d-1} : \\ \prod_{i=1}^{d-1} x_i = x}} \prod_{j=1}^{d-1} \mu_{v_j \to c}(x_j)$$

$$\mu_{c\to v} = \frac{\displaystyle\sum_{\substack{x_1 \dots x_{d-1}:\\ \prod_{i=1}^{d-1} x_i = +1}} \prod_{j=1}^{d-1} \frac{\mu_{v_j\to c}(x_j)}{\mu_{v_j\to c}(-1)}}{\displaystyle\sum_{\substack{x_1 \dots x_{d-1}:\\ \prod_{i=1}^{d-1} x_i = -1}} \prod_{j=1}^{d-1} \frac{\mu_{v_j\to c}(x_j)}{\mu_{v_j\to c}(-1)}}$$

$$= \frac{\displaystyle\sum_{\substack{x_1 \dots x_{d-1}:\\ \prod_i x_i = 1}} \prod_{j=1}^{d-1} \mu_{v_j\to c}^{\left(\frac{1+x_j}{2}\right)}}{\displaystyle\sum_{x_1 \dots x_{d-1}:\ \prod x_i = -1} \prod_{j=1}^{d-1} \mu_{v_j\to c}^{(1+x_j)/2}}$$

$$= \frac{\prod_{j=1}^{d-1} (r_{v_j \to c} + 1) + \prod_{j=1}^{d-1} (r_{v_j \to c} - 1)}{\prod_{j=1}^{d-1} (r_{v_j \to c} + 1) - \prod_{j=1}^{d-1} (r_{v_j \to c} - 1)}$$

$$\prod_{i=1}^{d-1} (\alpha_i + 1) = 1 + \alpha_1 + \alpha_2 + \cdots + \alpha_{d-1} + \alpha_1\alpha_2 + \alpha_1\alpha_3 \cdots$$

$$+ \alpha_1\alpha_2\alpha_3 \cdots\cdots + \prod_{i=1}^{d-1} \alpha_i$$

$$(\alpha_1+1)(\alpha_2+1)(\alpha_3+1) = 1 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$$

$$+ \alpha_1\alpha_2\alpha_3$$

$$(\alpha_1 - 1)(\alpha_2 - 1)(\alpha_3 - 1) = -1 + \alpha_1 + \alpha_2 + \alpha_3 - \alpha_1\alpha_2 - \alpha_2\alpha_3 - \alpha_1\alpha_3 + \alpha_1\alpha_2\alpha_3$$

$$+ \frac{}{2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3)}$$

$$\sum_{\substack{x_1 \dots x_{d-1} \,; \\ \prod_{i=1}^{d-1} x_i = 1}} \underbrace{\prod_{j=1}^{d-1} \alpha_j^{\frac{1+x_j}{2}}}$$

of j's for which α_j not present should be even

$$\prod_{i=1}^{d-1}(\alpha_i+1) = \sum_{(x_1 \cdots x_{d-1}) \in \{1,-1\}^{d-1}} \prod_{j=1}^{d-1} \alpha_j^{\frac{1+x_j}{2}} \, 1^{\frac{1-x_j}{2}}$$

$$\prod_{i=1}^{d-1}(\alpha_i-1) = \sum_{(x_1 \cdots x_{d-1}) \in \{1,-1\}^{d-1}} \prod_{j=1}^{d-1} \alpha_j^{\frac{1+x_j}{2}} \, x_j$$

$+$

$$= \sum_{(x_1 \cdots x_{d-1})} \prod_{j=1}^{d-1} \alpha_j^{\frac{1+x_j}{2}} \underbrace{\left[1 + \prod_{j=1}^{d-1} x_j \right]}$$

$$= \begin{cases} +2 & \text{if } \prod_{j=1}^{d-1} x_j = +1 \\ 0 & \text{if } \prod_{j=1}^{d-1} x_j = -1 \end{cases}$$

$$r_{c\to v} = \frac{\prod_{j=1}^{d-1}\left(r_{v_j\to c}+1\right) + \prod_{j=1}^{d-1}\left(r_{v_j\to c}-1\right)}{\prod_{j=1}^{d-1}\left(r_{v_j\to c}+1\right) - \prod_{j=1}^{d-1}\left(r_{v_j\to c}-1\right)}$$

$$= \frac{1 + \prod_{j=1}^{d-1}\left(\frac{r_{v_j\to c}-1}{r_{v_j\to c}+1}\right)}{1 - \prod_{j=1}^{d-1}\left(\frac{r_{v_j\to c}-1}{r_{v_j\to c}+1}\right)}$$

$$\ln r_{v\to c} = l_{v\to c}$$

$$r_{v\to c} = e^{l_{v\to c}}$$

$$e^{l_{c\to v}} = \frac{1 + \prod_{j=1}^{d-1}\left(\frac{e^{l_{v_j\to c}} - 1}{e^{l_{v_j\to c}} + 1}\right)}{1 - \prod_{j=1}^{d-1}\left(\frac{e^{l_{v_j\to c}} - 1}{e^{l_{v_j\to c}} + 1}\right)}$$

$$\frac{e^{x} - 1}{e^{x} + 1} = \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} = \tanh(x/2)$$

$$e^{l_{c\to v}} = \frac{1 + \prod_{j=1}^{d-1} \tanh\left(\frac{l_{v_j\to c}}{2}\right)}{1 - \prod_{j=1}^{d-1} \tanh\left(\frac{l_{v_j\to c}}{2}\right)}$$

$$\left[\ \beta = \frac{1+\alpha}{1-\alpha} \Rightarrow \alpha = \frac{\beta-1}{\beta+1}\ \right]$$

$$\prod_{j=1}^{d-1} \tanh\left(\frac{l_{v_j\to c}}{2}\right) = \frac{e^{l_{c\to v}}-1}{e^{l_{c\to v}}+1} = \tanh\left(\frac{l_{c\to v}}{2}\right)$$

$$l_{c\to v} = 2\tanh^{-1}\left(\prod_{j=1}^{d-1}\tanh\left(\frac{l_{v_j\to c}}{2}\right)\right)$$

BP for LDPC Codes:

① Initialize:

$$l^{(0)}_{v_i\to c_j} = \ln\frac{p(y_i|0)}{p(y_i|1)}$$

② Do: for $t = 1, 2, \ldots$

$$l^{(t)}_{c_j\to v_i} = 2\tanh^{-1}\left(\prod_{v\in\mathcal{N}(c_j)\setminus v_i}\tanh\left(\frac{l^{(t-1)}_{v\to c_j}}{2}\right)\right)$$

$$l^{(t)}_{v_i\to c_j} = \sum_{c\in\mathcal{N}(v_i)\setminus c_j} l_{c\to v_i}$$

③ Find estimate:

$$l_{v_i} = \sum_{c \in N(v_i)} l_{c \to v_i}$$

Declare $\hat{x}_i = \begin{cases} 0 & \text{if } l_{v_i} > 0 \\ 1 & \text{if } l_{v_i} \leq 0 \end{cases}$

Overall Computational complexity: $\Theta(n \times T)$

↳ # of iterations

$T = \Theta(\log n)$

Ensembles of LDPC codes

Irregular LDPC codes: n var $\quad n/2 \quad 4, \quad n/2 \quad 5$

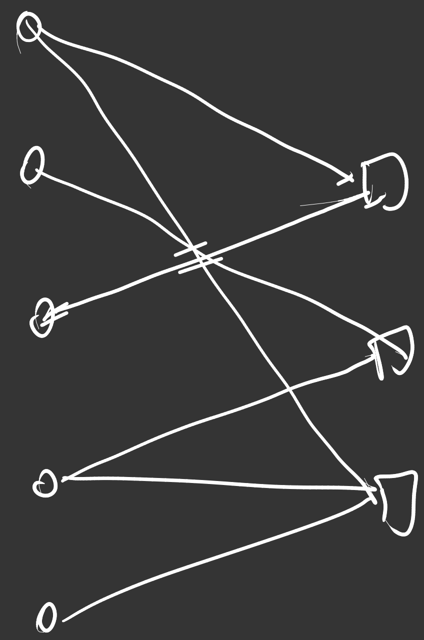
m check $\quad m/2 \quad 10 \quad m/2 \quad 12$

$$\Lambda(x) = \sum_{i=1}^{l_{max}} \Lambda_i x^i$$

Λ_i → # of variable nodes having degree i

$$P(x) = \sum_{i=1}^{r_{max}} P_i x^i$$

P_i → # of check nodes w/ degree i



$$\Lambda(x) = 3x^1 + 2x^2$$

$$P(x) = 2x^2 + x^3$$

* for any Λ, P,

$$\Lambda(1) = n$$

$$P(1) = m$$

$$\Lambda'(x) = \Lambda_1 + 2\Lambda_2 x^1 + 3\Lambda_3 x^2 + \cdots = \sum_{i=1}^{l_{max}} i \Lambda_i x^{i-1}$$

* $\Lambda'(1) = P'(1) =$ # edges

Normalized degree distribution (from a vertex perspective)

$$L(x) = \frac{\Lambda(x)}{\Lambda(1)} \quad \& \quad R(x) = \frac{P(x)}{P(1)}$$

Coefficient of

x^i is the fraction of nodes with degree i

Normalized degree distribution from an edge perspective

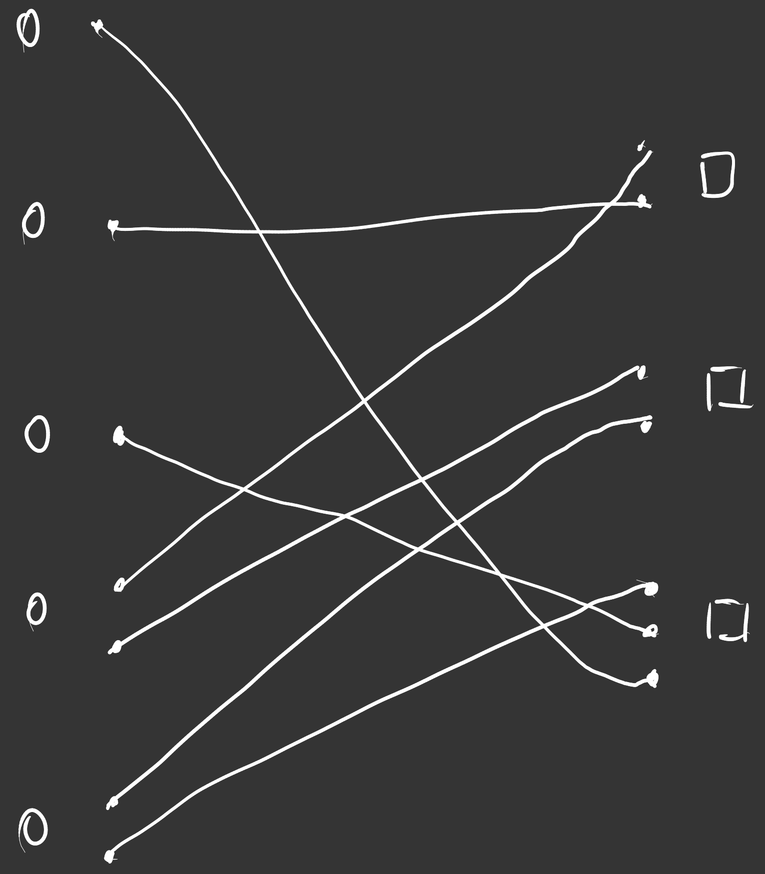
(Variable) $\lambda(x) = \sum_{i=1}^{l_{max}} \lambda_i x^{i-1}$ (Check) $\rho(x) = \sum_{i=1}^{r_{max}} \rho_i x^{i-1}$

$$= \frac{\Lambda'(x)}{\Lambda'(1)} \qquad = \frac{P'(x)}{P'(1)}$$

λ_i : fraction of edges that connect to var node of degree i

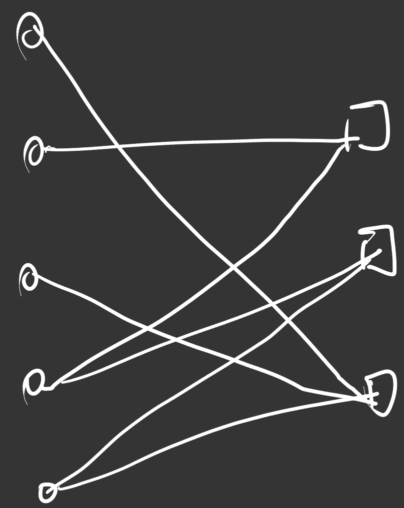
$$\lambda_i = \frac{i \Lambda_i}{\Lambda'(1)}$$

$n(x) = 3x + 2x^2,$ $\quad p(x) = 2x^2 + x^3$



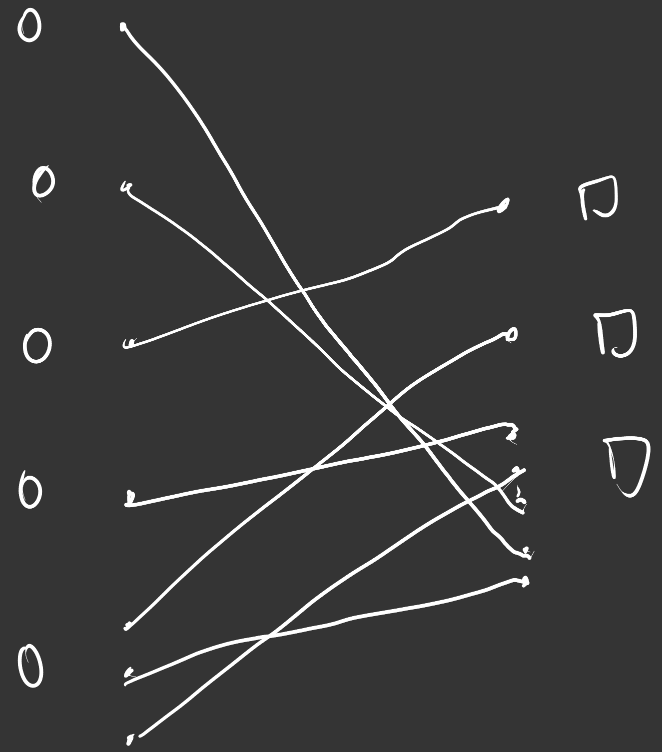
1 2 3 4 5 6 7

7 2 6 1 3 4 5



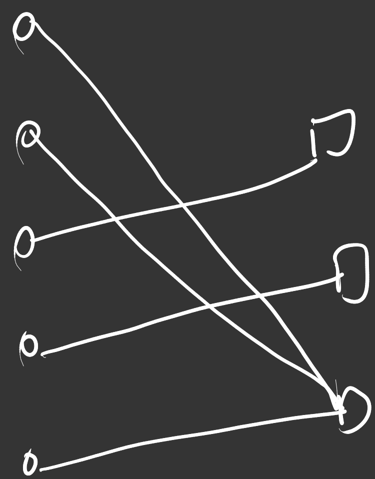
$n(x) = 4x + x^3$

$p(x) = 2x + x^5$



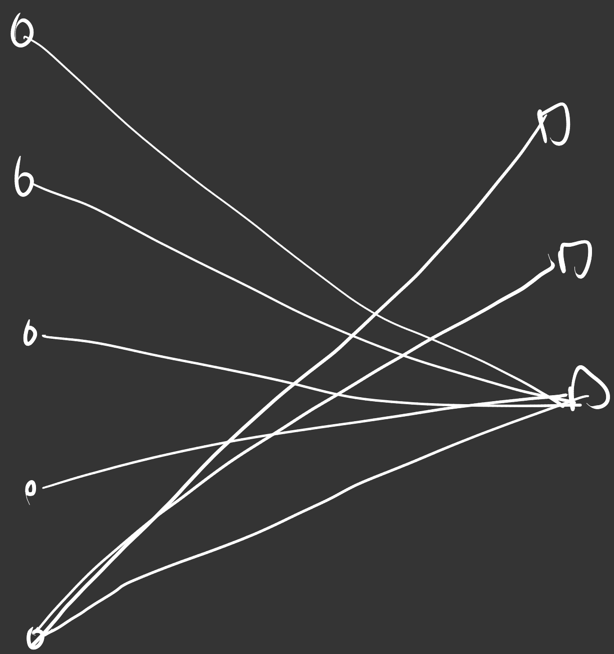
1 2 3 4 5 6 7

6 5 1 3 2 7 4



$n(x) = 5x$

$p(x) = 2x + x^3$



* If $n \to \infty$, then $\Pr[\text{parallel edges}] \to 0$

* for $\lambda(x)$ & $\rho(x)$, $\Pr[\text{smallest cycle} = o(\log n)]$
≈ 0

$n = \Lambda(1)$ $\qquad$ $m = P(1)$

$n, \; L(x), \; R(x)$

* Design rate $= \dfrac{n-m}{n}$

In general, rate $\geq$ design rate

Under some (mild) conditions on λ & ρ

$$\Pr\Big[\underset{\text{actual rate}}{r(G)} > \underbrace{r(\lambda, \rho)}_{\text{design rate}} (1+\delta) \Big] = o(1)$$

For regular LDPC codes,

$$\Pr\left[\text{rr}(y) > n\,r(d_v, d_c) + \nu \right] = o(1)$$

actual dimension of code

$$\nu = \begin{cases} 0 & \text{if } d_v \text{ is odd} \\ 1 & \text{if } d_v \text{ is even} \end{cases}$$

$$H = \begin{bmatrix} \quad \end{bmatrix}_{m \times n}$$

$n - m + 1$

RECAP

$$\Lambda(x) = \sum_{i=1}^{l_{max}} \Lambda_i x^i$$

$\Lambda_i \rightarrow$ # of variable nodes having degree i

$$L(x) = \frac{\Lambda(x)}{\Lambda(1)}$$

$$P(x) = \sum_{i=1}^{r_{max}} P_i x^i$$

$P_i \rightarrow$ # of check nodes w/ degree i

$$R(x) = \frac{P(x)}{P(1)}$$

$$\lambda(x) = \frac{\Lambda'(x)}{\Lambda'(1)} \qquad \rho(x) = \frac{P'(x)}{P'(1)}$$

$$L(x) = \frac{\int_0^x \lambda(z)\,dz}{\int_0^1 \lambda(z)\,dz} \qquad R(x) = \frac{\int_0^x \rho(z)\,dz}{\int_0^1 \rho(z)\,dz}$$

$$\lambda(x) = \sum_{i=1}^{l_{max}} \lambda_i\, x^{i-1}$$

$$\rho(x) = \sum_{i=1}^{n_{max}} \rho_i\, x^{i-1}$$

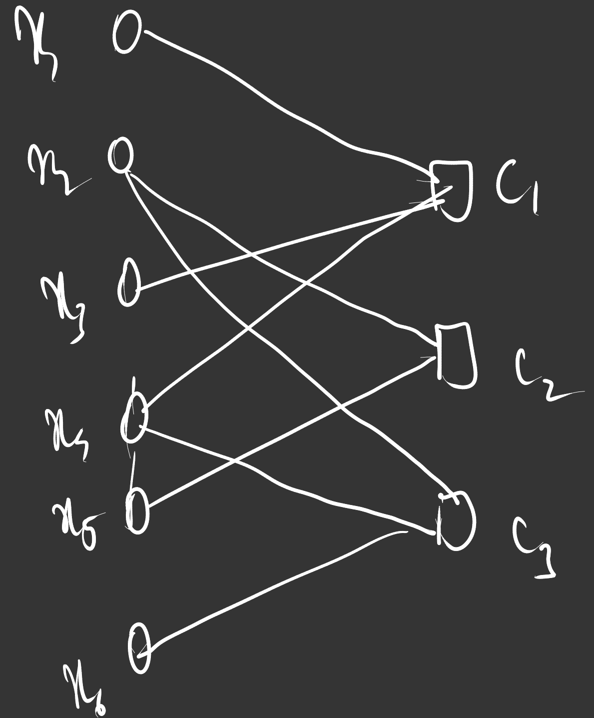
ASSUMPTIONS

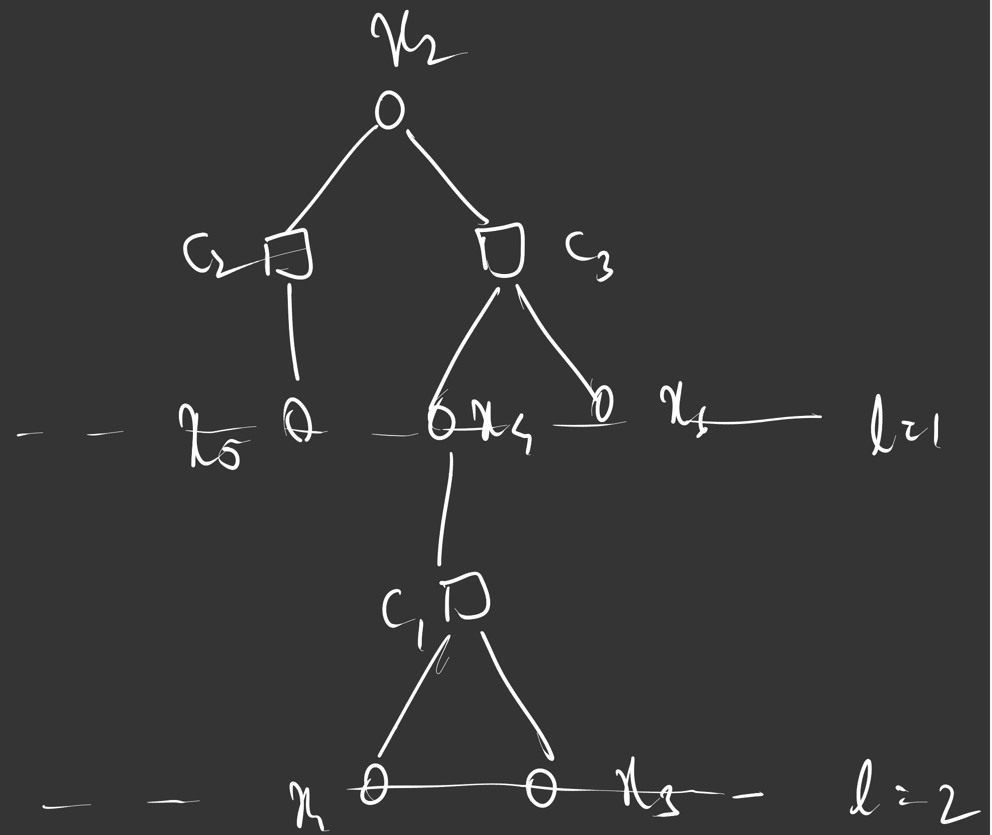
① Restriction to all-0's codeword : Prob of error for BP algo is the same irrespective of which cw is sent

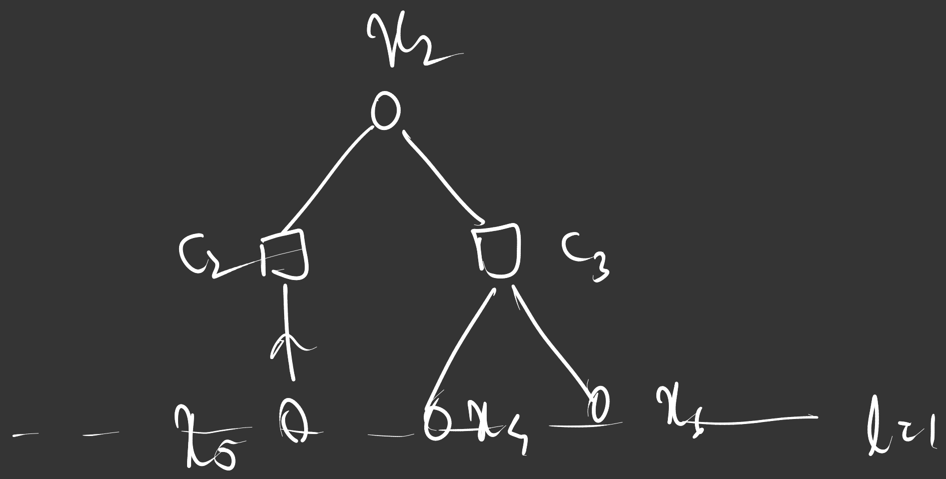
② Study the performance (codeword bit error probability) for random codes

$$\Pr\left[\left| P_b^{BP}(y(\lambda, \rho)) - \mathbb{E}_y P_b^{BP}(y) \right| > \delta \right] \le e^{-\alpha n}$$

To study $\mathbb{E}_y P_b^{BP}(y)$, we look at the "computation graph"







$$\Pr[\hat{x}_2 = e]$$

$$= \Pr[y_2 = e] \times \Pr[\mu_{c_2 \to x_2} = e,\ \mu_{c_3 \to x_2} = e]$$

$$= p \times \Pr[\mu_{x_5 \to c_2} = e] \ \Pr[\mu_{x_4 \to c_3} = e \text{ OR } \mu_{x_6 \to c_3} = e]$$

$$= p \times p\,\left(1 - (1-p)^2\right)$$

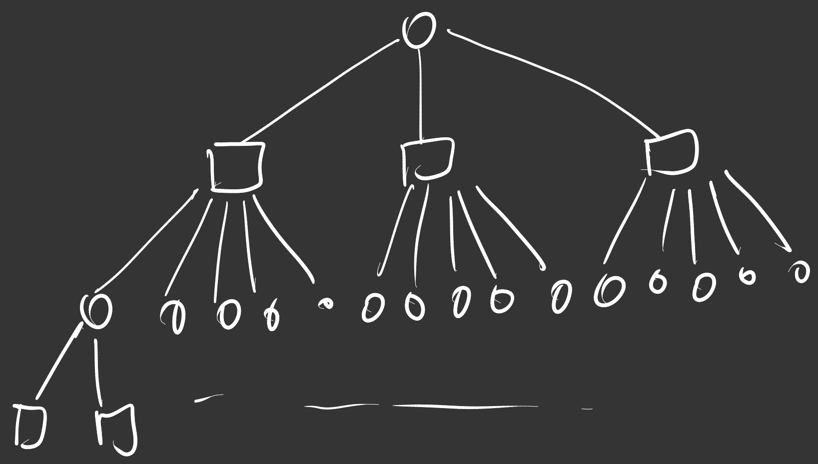
$$= p^2\left(1 - (1-p)^2\right)$$

For <u>random</u> $LDPC(\lambda, \rho)$, as $n \to \infty$,

$Pr[\text{Computation graph for any } x_i = \text{Tree}] \to 1$

(for any finite l)

$$LDPC\ C \xrightarrow{\text{Fix } l} \text{Computation graph} \xrightarrow{n \gg l} \text{Computation Tree}$$



(3, 6)

Tree Ensembles

$\vec{T}_\ell(\lambda, \rho)$ → edge perspective

$\mathring{T}_\ell(\lambda, \rho)$ → node perspective

Iterative

$\lambda(1)$

$\vec{T}_\ell(\lambda, \rho)$: $\ell = 0$ → $\vec{T}_0(\lambda, \rho)$ = one vertex (variable node)

For gen ℓ :

- Pick $T \in \vec{T}_{\ell-1}(\lambda, \rho)$
- for each leaf (var nodes), connect i new check nodes with prob λ_{i+1}
- for each check connect j new var nodes w/p ρ_{j+1}

Eg 1

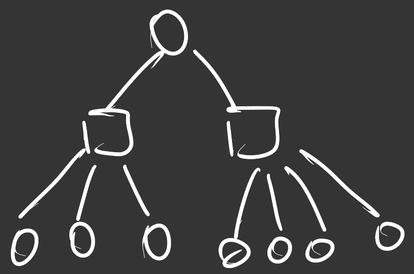
$$\lambda(x) = \frac{1}{2}x + \frac{1}{2}x^2 = \sum_i \lambda_i x^{i-1}$$

$$\rho(x) = \frac{x^3}{5} + \frac{4x^4}{5} = \sum_i \rho_i x^{i-1}$$

$l=1$

$\vec{T}_l(\lambda, \rho)$

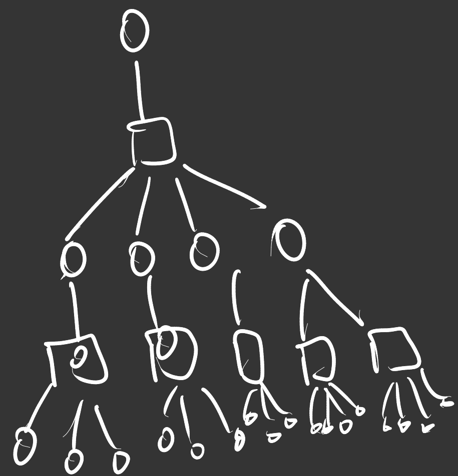
$\frac{1}{5}$



$\frac{4}{5}$

$\frac{1}{2}$

$\frac{2}{25}$



$$\frac{1}{2} \times \frac{4}{5} = \frac{2}{5}$$

$\mathring{T}_1(\lambda, \ell)$: Same as $T_1(\lambda, \ell)$ but:

i children of $\mathring{T}_0(\lambda, \ell)$ are drawn from $\{L_i\}$ w.p L_i

$$\lambda(x) = \frac{1}{2}x + \frac{1}{2}x^2$$

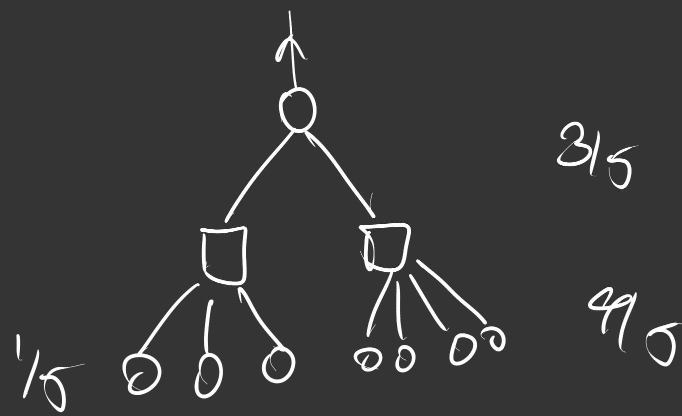
$$L(x) = \frac{\frac{x^2}{4} + \frac{x^3}{6}}{1/4 + 1/6}$$

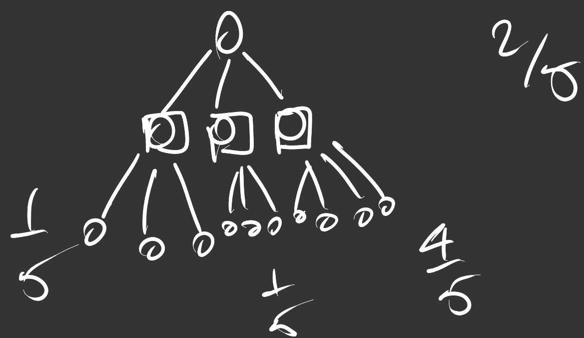
$$\rho(x) = \frac{x^3}{5} + \frac{4x^4}{5}$$

$$= \frac{6x^2 + 4x^3}{10}$$

$$= \frac{3x^2}{5} + \frac{2x^3}{5}$$

$$\mathring{T}_1(\lambda, \rho)$$





Theorem 1

$$\lim_{n\to\infty} \mathbb{E}_{LDPC}\, P_b^{BP}(\lambda, \rho, p) = P_b^{BP}(\mathring{T}_l(\lambda, \rho), p)$$

Theorem 2 Fix any λ, ρ

Define $x_{-1} = 1$ & $x_l = p \times \lambda(1 - \rho(1 - x_{l-1}))$

for $l = 0, 1, 2, \ldots$

Then, for any $l \geq 0$,

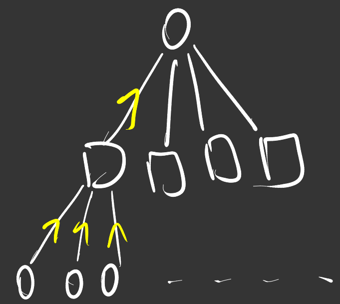
$$P_b^{BP}(\mathring{T}_l(\lambda, \rho), p) = p \times L(1 - \rho(1 - x_{l-1}))$$

Eg: for (d_v, d_c) reg. LDPC code,

$$\lambda(x) = x^{d_v-1} \qquad \rho(x) = x^{d_c-1}$$

$$L(x) = x^{d_v}$$

$$x_\ell = p \times \left(1 - (1-x_{\ell-1})^{d_c-1}\right)^{d_v-1}$$



$$x_0 = p \times \left(1 - (1-1)^{d_c-1}\right)^{d_v-1} = p$$

$$x_1 = p \times \left(1 - (1-p)^{d_c-1}\right)^{d_v-1}$$

$$x_2 = p \times \left(1 - (1-x_1)^{d_c-1}\right)^{d_c-1}$$

$$x_\ell = p \times \left(1 - (1-x_{\ell-1})^{d_c-1}\right)^{d_v}$$

Consider :

$$\underbrace{x_t = p \times \lambda\left(1 - e\left(1 - x_{t-1}\right)\right)}$$

$$x_t = f(p, x_{t-1})$$

We say that x^* is a fixed pt of this equation if

$$x^* = f(p, x^*)$$

- 0 is a fixed point

* Recursion converges to 0 iff 0 is the only fixed pt.

Threshold: (BP threshold) – the largest p for which the recursion $\to 0$

```
LPoly[-1] = 2/5
    LPoly[2] = 3/5
    RPoly = np.zeros(dcmax+1)
    RPoly[-1] = 16/21
    RPoly[4] = 5/21
```

$$L(x) = \frac{2}{5}x^3 + \frac{3}{5}x^2$$

$$R(x) = \frac{16}{21}x^5 + \frac{5}{21}x^4$$

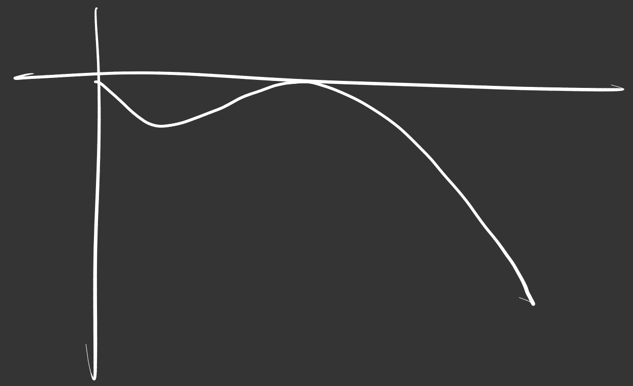
$$\varepsilon_{BP}(\lambda, \rho) = \max\{\varepsilon : x = f(\varepsilon, x) \text{ has no NONZERO solution}\}$$

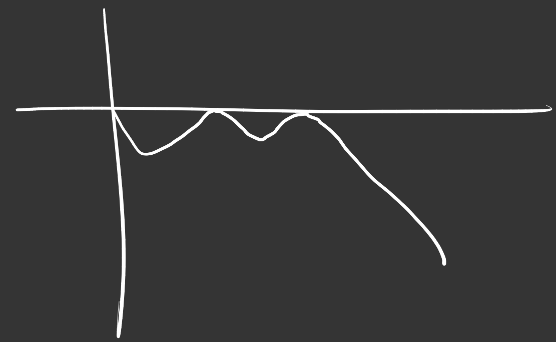
Critical point: $(\lambda, \rho, \varepsilon)$ → erasure prob

We say that x_{BP} is a critical pt if

$$f(\varepsilon_{BP}, x_{BP}) = x_{BP} \quad \& \quad \left.\frac{\partial f(\varepsilon_{BP}, x)}{\partial x}\right|_{x_{BP}} = 1$$

$$\left.\frac{\partial\left(f(\varepsilon_{BP}, x) - x\right)}{\partial x}\right|_{x_{BP}} = 0$$





$f(x) < x$

Stability condition

If $\quad \varepsilon \lambda'(0) \rho'(1) > 1 \quad$ then $\quad x_\ell \to 0 \quad$ as $\ell \to \infty$

$\varepsilon \lambda'(0) \rho'(1) < 1 \quad$ then $\quad x_\ell \not\to 0$

$$\varepsilon_{BP} \leq \frac{1}{\lambda'(0) \rho'(1)}$$

$$x = f(\rho, x)$$

$$= \Sigma \lambda\left(1 - \rho(1-x)\right)$$

$$= \nu_\Sigma\left(c(x)\right)$$

$$\nu_\Sigma(y) = \Sigma \lambda(y)$$

$$c(x) = 1 - \rho(1-x)$$

$$x_t = \nu_\Sigma\left(c(x_{t-1})\right)$$

$$\nu_\Sigma\left(c(x)\right) < x$$

Condition for success $\quad c(x) < \nu_\Sigma^{-1}(x)$

Eg:

$$\lambda(x) = x^2 \qquad \rho(x) = x^5$$

$$c(x) = 1 - \rho(1-x)$$

$$= 1 - (1-x)^5$$

$$v_\varepsilon(x) = \varepsilon\,\lambda(x) = \varepsilon x^2 = y$$

$$v_\varepsilon^{-1}(x) = \sqrt{\frac{x}{\varepsilon}}$$

EXIT

Visual depiction of density evolution