JOINTLY TYPICAL SETS

EE 6317

This lecture

1. Typical sequences

2. Jointly typical sequences

3. Properties of jointly typical sequences

Typical sequences revisited

$$X^n = (X_1, \dots, X_n) \sim \text{iid}(p_X) \qquad X_i \in \mathcal{X}$$

Strong typicality

$$\mu_{X^n}(a) \xrightarrow{P} p_X(a)$$

$$\mu_{X^n}(a) = \frac{1}{n} |\{i : X_i = a\}|$$

$$|\mu_{X^n}(a) - p_X(a)| \leq \epsilon p_X(a) \qquad \forall a \in \mathcal{X}$$

Weak typicality

$$\frac{1}{n} \log_2 \frac{1}{p_{X^n}(X^n)} \xrightarrow{P} H(X)$$

$$\frac{1}{n}\log\frac{1}{P_{X^n}(x^n)}$$

$$P_{X^n}(x^n) = \prod_{i=1}^{n} P_X(x_i)$$

$$= \prod_{a\in\mathcal{X}} \left(P_X(a)\right)^{n\mu_{x^n}(a)}$$

$$= \frac{1}{n}\log\frac{1}{\prod_{a\in\mathcal{X}}\left(P_X(a)\right)^{n\mu_{x^n}(a)}}$$

$$= \frac{1}{n}\sum_{a\in\mathcal{X}} n\,\mu_{x^n}(a)\log\frac{1}{P_X(a)}$$

$$= \sum_{a\in\mathcal{X}} \mu_{x^n}(a)\log\frac{1}{P_X(a)} \xrightarrow{P} H(X)$$

$$\Pr\left[\left|\sum_{a\in\mathcal{X}} \mu_{x^n}(a)\log\frac{1}{P_X(a)} - H(X)\right| > \epsilon\right] \to 0$$

as $n\to\infty$

Joint typicality

(X^n, Y^n) For each i, $(X_i, Y_i) \sim p_{XY}$

$(X_i\ Y_i) \perp\!\!\!\perp (X_j\ Y_j)$ $i \neq j$

$$X_1 Y_1,\ X_2 Y_2,\ X_3 Y_3 \ \cdots \ X_n Y_n$$

$$\begin{matrix} \| & \| & \| & & \| \\ Z_1 & Z_2 & Z_3 & & Z_n \end{matrix} \rightarrow \text{iid}$$

$$Z_i = (X_i\ Y_i) \in \mathcal{X} \times \mathcal{Y}.$$

$$\sim p_{XY}$$

$$\mu_{x^n y^n}(a, b) = \frac{1}{n} \#\ i\text{'s}\ X_i = a \ \&\ Y_i = b,$$

$$p_{XY}(a, b)$$

Joint Typicality

x^n, y^n is ϵ - jointly strongly typical if

$$\left| \mu_{x^n y^n}(a, b) - p_{XY}(a, b) \right| \leq \epsilon \, p_{XY}(a, b) \quad \forall (a, b)$$

$$A^{(n)}_{\epsilon, STR}(p_{XY}) = \Big\{ (x^n, y^n) \text{ st } \left| \mu_{x^n y^n}(a, b) - p_{XY}(a, b) \right| \leq \epsilon \, p_{XY}(a, b) \;\; \forall a, b \in \mathcal{X}\mathcal{Y} \Big\}$$

$$A_\epsilon^{(n)}(P_{XY}) = \Big\{ (x^n, y^n) :$$

$$H(X) - \epsilon \leq \frac{1}{n} \log \frac{1}{P_{X^n}(x^n)} \leq H(X) + \epsilon$$

$$H(Y) - \epsilon \leq \frac{1}{n} \log \frac{1}{P_{Y^n}(y^n)} \leq H(Y) + \epsilon$$

$$H(X,Y) - \epsilon \leq \frac{1}{n} \log \frac{1}{P_{X^nY^n}(x^n, y^n)} \leq H(X,Y) + \epsilon \Big\}$$

$$2^{-n(H(X,Y)+\epsilon)} \leq P_{X^nY^n}(x^n, y^n) \leq 2^{-n(H(X,Y)-\epsilon)}$$

$$
A_\epsilon^{(n)}(p_{XY}) = \Bigg\{ (x^n, y^n) : \begin{array}{l}
H(X) - \epsilon \le \displaystyle\sum_{a \in \mathcal{X}} \mu_{x^n}(a) \log \frac{1}{p_X(a)} \le H(X) + \epsilon \\[2ex]
H(Y) - \epsilon \le \displaystyle\sum_{b \in \mathcal{Y}} \mu_{y^n}(b) \log \frac{1}{p_Y(b)} \le H(Y) + \epsilon \\[2ex]
H(XY) - \epsilon \le \displaystyle\sum_{a,b} \mu_{x^n y^n}(a,b) \log \frac{1}{p_{XY}(a,b)} \le H(XY) + \epsilon
\end{array} \Bigg\}
$$

PROPERTIES OF $A_\epsilon^{(n)}(p_{XY})$

$(X^n, Y^n) \sim$ iid p_{XY} $\qquad X^n \sim$ iid p_X

$\qquad\qquad\qquad\qquad\qquad\qquad Y^n \sim$ iid p_Y

$$\Pr\left[\left| \frac{1}{n} \log \frac{1}{p_{X^n}(X^n)} - H(X) \right| > \epsilon \right] = \delta_X(n)$$

$$\Pr\left(\left| \frac{1}{n} \log \frac{1}{p_{Y^n}(y^n)} - H(Y) \right| > \epsilon \right] = \delta_Y(n)$$

$$\Pr\left[\left| \frac{1}{n} \log \frac{1}{p_{X^n Y^n}(X^n y^n)} - H(X,Y) \right| > \epsilon \right] = \delta_{XY}(n)$$

$$\frac{1}{n} \log \frac{1}{p_{X^n Y^n}(X^n Y^n)} = \sum_{i=1}^{n} \frac{1}{n} \log \frac{1}{p_{XY}(X_i Y_i)}$$

$$\xrightarrow{p} H(XY)$$

$$\Pr\left[(X^n, Y^n) \notin A_\epsilon^{(n)}(p_{XY}) \right] \le \delta_X(n) + \delta_Y(n) + \delta_{XY}(n)$$

$$\to 0$$

as $n \to \infty$

Bounds on $A_\epsilon^{(n)}(p_{xy})$

$$1-\delta(n) \leq \Pr\left[(X^n, Y^n) \overset{\sim p_{XY}}{\in} A_\epsilon^{(n)}(p_{xy})\right] \leq 1$$

$$\shortparallel$$

$$1-\delta(n) \leq \sum_{(x^n, y^n)\in A_\epsilon^{(n)}(p_{xy})} p_{X^nY^n}(x^n, y^n) \leq 1$$

$$1-\delta(n) \leq \sum_{(x^n y^n)\in A_\epsilon} 2^{-n(H(X,Y)-\epsilon)} \quad / \quad \sum_{(x^n y^n)\in A_\epsilon} 2^{-n(H(X,Y)+\epsilon)} \leq 1$$

$$2^{-n(H(XY)+\epsilon)} \, |A_\epsilon| \leq 1$$

$$|A_\epsilon^{(n)}(p_{xy})| \geq (1-\delta(n))\, 2^{n(H(XY)-\epsilon)} \quad / \quad |A_\epsilon^{(n)}(p_{xy})| \leq 2^{n(H(XY)+\epsilon)}$$

$A_\epsilon^{(n)}(p_{XY})$

$X^n Y^n \sim \text{iid}(p_{XY}) \Rightarrow (X^n Y^n) \in A_\epsilon$ whp.

$\tilde{X}^n \tilde{Y}^n \sim \text{iid}(p_X p_Y)$

$X_i\ Y_i$ are independent.

$$\Pr\left[(\tilde{X}^n, \tilde{Y}^n) \in A_\epsilon(p_{XY})\right] = \sum_{(x^n, y^n) \in A_\epsilon(p_{XY})} p_{X^n}(x^n)\, p_{Y^n}(y^n)$$

$$\leq \sum_{(x^n, y^n) \in A_\epsilon(p_{XY})} 2^{-n(H(X)-\epsilon)}\, 2^{-n(H(Y)-\epsilon)}$$

$$\leq 2^{-n(H(X)+H(Y)-2\epsilon)}\, 2^{n(H(XY)+\epsilon)}$$

$$\leq 2^{-n(H(X)+H(Y)-H(XY)-3\epsilon)}$$

$$\leq 2^{-n(I(X;Y)-3\epsilon)}$$

Properties:

$(X^n, Y^n) \sim \text{iid}(P_{XY}) \quad (\tilde{X}^n \tilde{Y}^n) \sim \text{iid}(P_X P_Y)$

① $\Pr[(X^n, Y^n) \notin A_\epsilon(P_{XY})] = \delta(n) \to 0$

as $n \to \infty$

② $(1-\delta(n))\, 2^{n(H(XY)-\epsilon)} \leq |A_\epsilon(P_{XY})| \leq 2^{n(H(XY)+\epsilon)}$

③ $\Pr[(\tilde{X}^n \tilde{Y}^n) \in A_\epsilon(P_{XY})] \leq 2^{-n(I(X;Y)-3\epsilon)}$

$\sim$ iid $(P_X P_Y)$

Revisit strong typicality

$$T_{\epsilon, str}(p_X) = \{ x^n : \quad |\mu_{x^n}(a) - p_X(a)| \leq \epsilon p_X(a) \quad \forall a \in \mathcal{X} \}$$

$$p_X(a)(1-\epsilon) \leq \mu_{x^n}(a) \leq p_X(a)(1+\epsilon)$$

Suppose $X^n \sim \text{iid}(q_X)$

We guess p_X

What is $\Pr[X^n \in T_{\epsilon, str}(p_X)]$?

$$\Pr[X^n \in T_{\epsilon,str}(p_X)] = \sum_{x^n \in T_{\epsilon,str}(p_X)} \prod_{i=1}^{n} q_X(x_i)$$

$X^n \sim \text{iid}(q_X)$

$$= \sum_{x^n \in T_{\epsilon,str}(p_X)} \prod_{a \in \mathcal{X}} (q_X(a))^{n\mu_{x^n}(a)}$$

$$\leq \sum_{x^n \in T_{\epsilon,str}(p_X)} \prod_{a \in \mathcal{X}} 2^{np_X(a)(1-\epsilon)\log q_X(a)}$$

$$\leq \prod_{a \in \mathcal{X}} 2^{np_X(a)(1-\epsilon)\log q_X(a)} \; 2^{nH(X)(1+\epsilon)}$$

$$= 2^{\sum_{a\in\mathcal{X}} np_X(a)(1-\epsilon)\log q_X(a)} \; 2^{nH(X)(1+\epsilon)}$$

$$= 2^{n\sum_{a\in\mathcal{X}}\left[p_X(a)(\log q_X(a))(1-\epsilon) + p_X(a)\log\frac{1}{p_X(a)}(1+\epsilon)\right]}$$

$$= 2^{n\left[\sum_{a\in\mathcal{X}} p_X(a)\log\frac{q_X(a)}{p_X(a)} + \epsilon\,\alpha_{p,q}\right]}$$

$$= 2^{-nD(p_X\|q_X) + n\epsilon\alpha_{p,q}}$$

$$X^n, Y^n \sim \text{iid}(q_{XY})$$

$$\Pr\left[(X^n, Y^n) \in A_{\epsilon,str}(p_{XY}) \right] \leq 2^{-n D(p_{XY} \| q_{XY}) + n\epsilon \alpha_{pq}}$$

$$\downarrow$$

$$q_{XY} = p_X p_Y \Rightarrow \quad 2^{-n D(p_{XY} \| p_X p_Y) + \epsilon(\ )}$$

$$2^{-n I(X;Y) + n\epsilon(\cdot)}$$