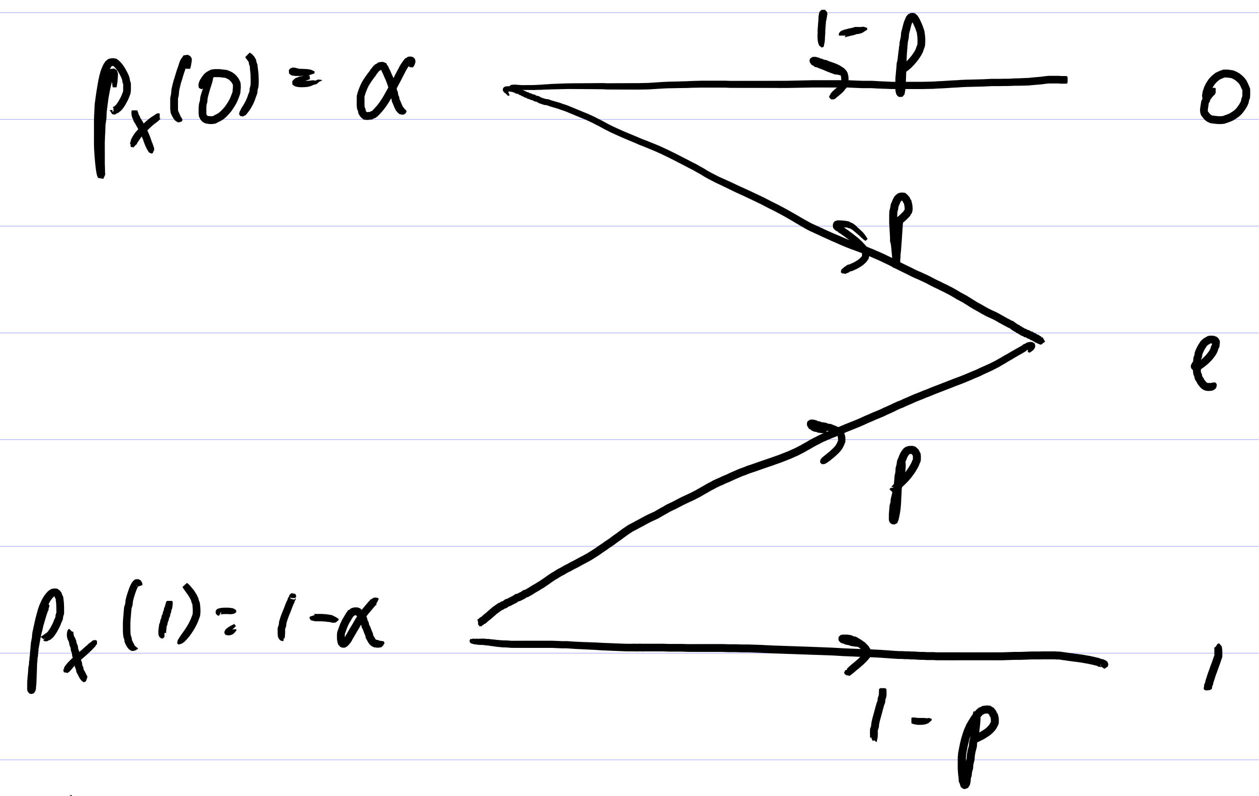
Capacity of the BEC



$$P_Y(0) = \alpha(1-p)$$
$$P_Y(1) = (1-\alpha)(1-p)$$
$$P_Y(e) = p$$

Recall that $E = \begin{cases} 1 & \text{if } Y = e \\ 0 & \text{if } Y \neq e \end{cases}$

$$P_E(1) = P_Y(e) = p.$$

$\therefore \; H(E) = H_2(p)$

Now compute $H(Y|E)$

$$H(Y|E) = H(Y|E=0)\,P_E(0) + H(Y|E=1)\,P_E(1)$$

$$P_{Y|E}(0|1) = 0$$
$$P_{Y|E}(1|1) = 0$$
$$P_{Y|E}(e|1) = 1$$

$\therefore \; H(Y|E=1) = 0.$

$$p_{Y|E}(0|0) = \alpha$$
$$p_{Y|E}(1|0) = 1-\alpha$$
$$p_{Y|E}(e|0) = 0$$

(since $p_{Y|E}(0|0) = p_{X|E}(0|0)$
$$= \frac{p_{XE}(00)}{p_E(0)}$$
$$= \alpha$$
similar for $p_{Y|E}(1|0)$)

$$\therefore H(Y|E=0) = \alpha \log_2 \frac{1}{\alpha} + (1-\alpha) \log_2 \frac{1}{1-\alpha}$$
$$= H_2(\alpha)$$

$$H(Y|E) = 0 + (1-p) H_2(\alpha)$$

$$H(Y) = H_2(p) + (1-p) H_2(\alpha)$$

$$\Rightarrow I(X;Y) = H_2(p) + (1-p) H_2(\alpha) - H_2(p)$$
$$= (1-p) H_2(\alpha)$$

$$\Rightarrow \max_{0 \le \alpha \le 1} I(X;Y) = 1-p$$