Dual Decomposition

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Last time: coordinate descent

Consider the problem

$$\min_x \ f(x)$$

where $f(x) = g(x) + \sum_{i=1}^n h_i(x_i)$, with g convex and differentiable and each h_i convex. Coordinate descent: let $x^{(0)} \in \mathbb{R}^n$, and repeat

$$x_i^{(k)} = \operatorname*{argmin}_{x_i} \ f\big(x_1^{(k)}, \ldots, x_{i-1}^{(k)}, x_i, x_{i+1}^{(k-1)}, \ldots, x_n^{(k-1)}\big), \qquad i = 1, \ldots, n$$

for $k = 1, 2, 3, \ldots$

- Very simple and easy to implement
- Careful implementations can achieve state-of-the-art
- Scalable, e.g., don't need to keep full data in memory

Reminder: conjugate functions

Recall that given $f : \mathbb{R}^n \to \mathbb{R}$, the function

$$f^*(y) = \max_x \; y^T x - f(x)$$

is called its conjugate

- Conjugates appear frequently in dual programs, since

$$-f^*(y) = \min_x \; f(x) - y^T x$$

- If f is closed and convex, then $f^{**} = f$. Also,

$$x \in \partial f^*(y) \iff y \in \partial f(x) \iff x \in \operatorname*{argmin}_z \; f(z) - y^T z$$

- If f is strictly convex, then $\nabla f^*(y) = \operatorname*{argmin}_z \; f(z) - y^T z$

Outline

Today:

- Dual ascent
- Dual decomposition
- Augmented Lagrangians
- A peak at ADMM

Dual first-order methods

Even if we can't derive dual (conjugate) in closed form, we can still use dual-based gradient or subgradient methods

Consider the problem

$$\min_x \; f(x) \;\; \text{subject to} \;\; Ax = b$$

Its dual problem is

$$\max_u \; -f^*(-A^T u) - b^T u$$

where f^* is conjugate of f. Defining $g(u) = -f^*(-A^T u) - b^T u$, note that

$$\partial g(u) = A \partial f^*(-A^T u) - b$$

Dual subgradient method

Therefore, using what we know about conjugates

$$\partial g(u) = Ax - b \quad \text{where} \quad x \in \operatorname*{argmin}_z \; f(z) + u^T Az$$

The dual subgradient method (for maximizing the dual objective) starts with an initial dual guess $u^{(0)}$, and repeats for $k = 1, 2, 3, \ldots$

$$\begin{aligned} x^{(k)} &\in \operatorname*{argmin}_x \; f(x) + (u^{(k-1)})^T Ax \\ u^{(k)} &= u^{(k-1)} + t_k(Ax^{(k)} - b) \end{aligned}$$

Step sizes t_k, $k = 1, 2, 3, \ldots$, are chosen in standard ways

Dual gradient ascent

Recall that if f is strictly convex, then f^* is differentiable, and so this becomes dual gradient ascent, which repeats for $k = 1, 2, 3, \ldots$

$$x^{(k)} = \underset{x}{\operatorname{argmin}} \; f(x) + (u^{(k-1)})^T A x$$
$$u^{(k)} = u^{(k-1)} + t_k (A x^{(k)} - b)$$

(Difference is that each $x^{(k)}$ is unique, here.) Again, step sizes t_k, $k = 1, 2, 3, \ldots$ are chosen in standard ways

Lastly, proximal gradients and acceleration can be applied as they would usually

Curvature and conjugates

Assume that f is a closed and convex function. Then f is strongly convex with parameter $m \iff \nabla f^*$ Lipschitz with parameter $1/m$

Proof of "$\implies$": Recall, if g strongly convex with minimizer x, then

$$g(y) \geq g(x) + \frac{m}{2}\|y - x\|_2, \quad \text{for all } y$$

Hence defining $x_u = \nabla f^*(u)$, $x_v = \nabla f^*(v)$,

$$f(x_v) - u^T x_v \geq f(x_u) - u^T x_u + \frac{m}{2}\|x_u - x_v\|_2^2$$
$$f(x_u) - v^T x_u \geq f(x_v) - v^T x_v + \frac{m}{2}\|x_u - x_v\|_2^2$$

Adding these together, using Cauchy-Schwartz, rearranging shows that $\|x_u - x_v\|_2 \leq \|u - v\|_2/m$

Proof of "$\Longleftarrow$": for simplicity, call $g = f^*$ and $L = 1/m$. As ∇g is Lipschitz with constant L, so is $g_x(z) = g(z) - \nabla g(x)^T z$, hence

$$g_x(z) \le g_x(y) + \nabla g_x(y)^T (z - y) + \frac{L}{2}\|z - y\|_2^2$$

Minimizing each side over z, and rearranging, gives

$$\frac{1}{2L}\|\nabla g(x) - \nabla g(y)\|_2^2 \le g(y) - g(x) + \nabla g(x)^T (x - y)$$

Exchanging roles of x, y, and adding together, gives

$$\frac{1}{L}\|\nabla g(x) - \nabla g(y)\|_2^2 \le (\nabla g(x) - \nabla g(y))^T (x - y)$$

Let $u = \nabla f(x)$, $v = \nabla g(y)$; then $x \in \partial g^*(u)$, $y \in \partial g^*(v)$, and the above reads $(x - y)^T (u - v) \ge \|u - v\|_2^2 / L$, implying the result

Convergence guarantees

The following results hold from combining the last fact with what we already know about gradient descent:[1]

- If f is strongly convex with parameter m, then dual gradient ascent with constant step sizes $t_k = m$ converges at sublinear rate $O(1/\epsilon)$
- If f is strongly convex with parameter m and ∇f is Lipschitz with parameter L, then dual gradient ascent with step sizes $t_k = 2/(1/m + 1/L)$ converges at linear rate $O(\log(1/\epsilon))$

Note that this describes convergence in the dual. (Convergence in the primal requires more assumptions)

[1]This is ignoring the role of A, and thus reflects the case when the singular values of A are all close to 1. To be more precise, the step sizes here should be: $m/\sigma_{\max}(A)^2$ (first case) and $2/(\sigma_{\max}(A)^2/m + \sigma_{\min}(A)^2/L)$ (second case).

Dual decomposition

Consider

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad Ax = b$$

Here $x = (x_1, \ldots, x_B) \in \mathbb{R}^n$ divides into B blocks of variables, with each $x_i \in \mathbb{R}^{n_i}$. We can also partition A accordingly

$$A = [A_1 \ldots, A_B], \ \text{where} \ A_i \in \mathbb{R}^{m \times n_i}$$

Simple but powerful observation, in calculation of (sub)gradient, is that the minimization decomposes into B separate problems:

$$x^+ \in \operatorname*{argmin}_x \sum_{i=1}^B f_i(x_i) + u^T Ax$$

$$\iff x_i^+ \in \operatorname*{argmin}_{x_i} f_i(x_i) + u^T A_i x_i, \quad i = 1, \ldots, B$$

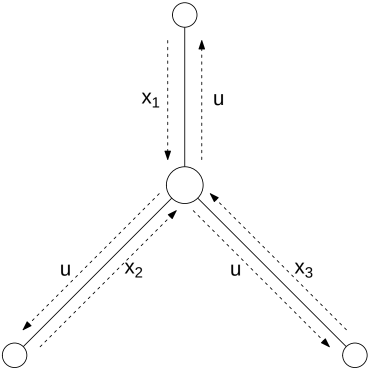
Dual decomposition algorithm: repeat for $k = 1, 2, 3, \ldots$

$$x_i^{(k)} \in \underset{x_i}{\operatorname{argmin}} \; f_i(x_i) + (u^{(k-1)})^T A_i x_i, \quad i = 1, \ldots, B$$

$$u^{(k)} = u^{(k-1)} + t_k \left(\sum_{i=1}^{B} A_i x_i^{(k)} - b \right)$$

Can think of these steps as:

- Broadcast: send u to each of the B processors, each optimizes in parallel to find x_i
- Gather: collect $A_i x_i$ from each processor, update the global dual variable u



Inequality constraints

Consider

$$\min_x \sum_{i=1}^B f_i(x_i) \quad \text{subject to} \quad \sum_{i=1}^B A_i x_i \le b$$

Dual decomposition, i.e., projected subgradient method:

$$x_i^{(k)} \in \operatorname*{argmin}_{x_i} \; f_i(x_i) + (u^{(k-1)})^T A_i x_i, \quad i = 1, \ldots, B$$

$$u^{(k)} = \left(u^{(k-1)} + t_k \left(\sum_{i=1}^B A_i x_i^{(k)} - b \right) \right)_+$$

where u_+ denotes the positive part of u, i.e., $(u_+)_i = \max\{0, u_i\}$, $i = 1, \ldots, m$

Price coordination interpretation (Vandenberghe):

- Have B units in a system, each unit chooses its own decision variable x_i (how to allocate its goods)
- Constraints are limits on shared resources (rows of A), each component of dual variable u_j is price of resource j
- Dual update:

$$u_j^+ = (u_j - ts_j)_+, \quad j = 1, \ldots, m$$

  where $s = b - \sum_{i=1}^B A_i x_i$ are slacks

  - Increase price u_j if resource j is over-utilized, $s_j < 0$
  - Decrease price u_j if resource j is under-utilized, $s_j > 0$
  - Never let prices get negative

Augmented Lagrangian method

(also known as: method of multipliers)

Dual ascent disadvantage: convergence requires strong conditions. Augmented Lagrangian method transforms the primal problem:

$$\begin{aligned} \min_x \quad & f(x) + \frac{\rho}{2}\|Ax - b\|_2^2 \\ \text{subject to} \quad & Ax = b \end{aligned}$$

where $\rho > 0$ is a parameter. Clearly equivalent to original problem. Strongly convex if A has full column rank. Dual gradient ascent:

$$\begin{aligned} x^{(k)} &= \underset{x}{\operatorname{argmin}}\; f(x) + (u^{(k-1)})^T Ax + \frac{\rho}{2}\|Ax - b\|_2^2 \\ u^{(k)} &= u^{(k-1)} + \rho(Ax^{(k)} - b) \end{aligned}$$

Notice step size choice $t_k = \rho$ in dual algorithm. Why? Since $x^{(k)}$ minimizes $f(x) + (u^{(k-1)})^T Ax + \frac{\rho}{2}\|Ax - b\|_2^2$ over x, we have

$$\begin{aligned} 0 &\in \partial f(x^{(k)}) + A^T\Big(u^{(k-1)} + \rho(Ax^{(k)} - b)\Big) \\ &= \partial f(x^{(k)}) + A^T u^{(k)} \end{aligned}$$

This is the stationarity condition for original primal problem; under mild conditions $Ax^{(k)} - b \to 0$ as $k \to \infty$, so KKT conditions are satisfied in the limit and $x^{(k)}, u^{(k)}$ converge to solutions

- Advantage: augmented Lagrangian gives better convergence
- Disadvantage: lose decomposability! (Separability is ruined)

Alternating direction method of multipliers

Alternating direction method of multipliers or ADMM: try for best of both worlds. Consider the problem

$$\min_{x,z}\ f(x) + g(z) \ \text{ subject to } \ Ax + Bz = c$$

As before, we augment the objective

$$\begin{aligned} \min_{x} \quad & f(x) + g(z) + \frac{\rho}{2}\|Ax + Bz - c\|_2^2 \\ \text{subject to} \quad & Ax + Bz = c \end{aligned}$$

for a parameter $\rho > 0$. We define augmented Lagrangian

$$L_\rho(x, z, u) = f(x) + g(z) + u^T(Ax + Bz - c) + \frac{\rho}{2}\|Ax + Bz - c\|_2^2$$

ADMM repeats the steps, for $k = 1, 2, 3, \ldots$

$$x^{(k)} = \underset{x}{\operatorname{argmin}} \; L_\rho(x, z^{(k-1)}, u^{(k-1)})$$
$$z^{(k)} = \underset{z}{\operatorname{argmin}} \; L_\rho(x^{(k)}, z, u^{(k-1)})$$
$$u^{(k)} = u^{(k-1)} + \rho(Ax^{(k)} + Bz^{(k)} - c)$$

Note that the usual method of multipliers would have replaced the first two steps by a joint minimization

$$(x^{(k)}, z^{(k)}) = \underset{x,z}{\operatorname{argmin}} \; L_\rho(x, z, u^{(k-1)})$$

Convergence guarantees

Under modest assumptions on f, g (these do not require A, B to be full rank), the ADMM iterates satisfy, for any $\rho > 0$:

- Residual convergence: $r^{(k)} = Ax^{(k)} - Bz^{(k)} - c \to 0$ as $k \to \infty$, i.e., primal iterates approach feasibility
- Objective convergence: $f(x^{(k)}) + g(z^{(k)}) \to f^\star + g^\star$, where $f^\star + g^\star$ is the optimal primal objective value
- Dual convergence: $u^{(k)} \to u^\star$, where $u^\star$ is a dual solution

For details, see Boyd et al. (2010). Note that we do not generically get primal convergence, but this is true under more assumptions

Convergence rate: roughly, ADMM behaves like first-order method. Theory still being developed, see, e.g., in Hong and Luo (2012), Deng and Yin (2012), Iutzeler et al. (2014), Nishihara et al. (2015)

Scaled form ADMM

Scaled form: denote $w = u/\rho$, so augmented Lagrangian becomes

$$L_\rho(x, z, w) = f(x) + g(z) + \frac{\rho}{2}\|Ax + Bz - c + w\|_2^2 - \frac{\rho}{2}\|w\|_2^2$$

and ADMM updates become

$$\begin{aligned} x^{(k)} &= \underset{x}{\operatorname{argmin}}\ f(x) + \frac{\rho}{2}\|Ax + Bz^{(k-1)} - c + w^{(k-1)}\|_2^2 \\ z^{(k)} &= \underset{z}{\operatorname{argmin}}\ g(z) + \frac{\rho}{2}\|Ax^{(k)} + Bz - c + w^{(k-1)}\|_2^2 \\ w^{(k)} &= w^{(k-1)} + Ax^{(k)} + Bz^{(k)} - c \end{aligned}$$

Note that here kth iterate $w^{(k)}$ is just a running sum of residuals:

$$w^{(k)} = w^{(0)} + \sum_{i=1}^{k} \big(Ax^{(i)} + Bz^{(i)} - c\big)$$

Example: alternating projections

Consider finding a point in intersection of convex sets $C, D \subseteq \mathbb{R}^n$:

$$\min_x \; I_C(x) + I_D(x)$$

To get this into ADMM form, we express it as

$$\min_{x,z} \; I_C(x) + I_D(z) \;\; \text{subject to} \;\; x - z = 0$$

Each ADMM cycle involves two projections:

$$\begin{aligned} x^{(k)} &= \operatorname*{argmin}_x \; P_C\big(z^{(k-1)} - w^{(k-1)}\big) \\ z^{(k)} &= \operatorname*{argmin}_z \; P_D\big(x^{(k)} + w^{(k-1)}\big) \\ w^{(k)} &= w^{(k-1)} + x^{(k)} - z^{(k)} \end{aligned}$$

Compare classic alternating projections algorithm (von Neumann):

$$\begin{aligned} x^{(k)} &= \underset{x}{\operatorname{argmin}}\ P_C\big(z^{(k-1)}\big) \\ z^{(k)} &= \underset{z}{\operatorname{argmin}}\ P_D\big(x^{(k)}\big) \end{aligned}$$

Difference is ADMM utilizes a dual variable w to offset projections. When (say) C is a linear subspace, ADMM algorithm becomes

$$\begin{aligned} x^{(k)} &= \underset{x}{\operatorname{argmin}}\ P_C\big(z^{(k-1)}\big) \\ z^{(k)} &= \underset{z}{\operatorname{argmin}}\ P_D\big(x^{(k)} + w^{(k-1)}\big) \\ w^{(k)} &= w^{(k-1)} + x^{(k)} - z^{(k)} \end{aligned}$$

Initialized at $z^{(0)} = y$, this is equivalent to Dykstra's algorithm for finding the closest point in $C \cap D$ to y

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