OFFLINE Exam Solutions 1

Thursday, 21 March 2024 10:58 AM

1) See lecture Notes & Nesterov book

2) 6.4.4 in Beck's book eg. 6.35

3) $f(x) = \max_{y \in C} x^T y \Leftrightarrow f^*(x) = I_C$.

a) $\text{Prox}_{f^*}(x) = \Pi_C(x)$.

By Moreau decomposition, $\text{Prox}_f(x) = x - \text{prox}_{f^*}(x) = x - \Pi_C(x)$.

↳ Theorem 6.44 in Beck's book.

b) $f(x) = \frac{1}{2}\|x - \Pi_C(x)\|^2 = \min_{y \in C} \frac{1}{2}\|x-y\|^2$

$\text{Prox}_f(x) = \arg\min_z \frac{1}{2}\|z-x\|^2 + \min_{y \in C} \frac{1}{2}\|z-y\|^2 \longrightarrow = \frac{x+\Pi_C(x)}{2}$

Consider $\min_{z,y} \frac{1}{2}\|z-x\|^2 + \frac{1}{2}\|z-y\|^2$

$= \min_{y \in C} \min_z \frac{1}{2}\|z-x\|^2 + \frac{1}{2}\|z-y\|^2 \longrightarrow z^* = \frac{x+y}{2}$

$= \min_{y \in C} \frac{1}{2}\left\|\frac{x+y}{2} - x\right\|^2 + \frac{1}{2}\left\|\frac{x+y}{2} - y\right\|^2$

$= \min_{y \in C} \frac{1}{4}\|x-y\|^2 \longrightarrow y^* = \Pi_C(x)$

4) Consider $\Pi_C(\text{prox}_f(x)) \equiv \arg\min_{y \in C} \frac{1}{2}\|y - \text{prox}_f(x)\|^2 \equiv g(y)$

optimality condition

$\nabla g(\Pi_C(\text{prox}_f(x)))^T (z - \Pi_C(\text{prox}_f(x))) \geq 0 \ \forall z \in C$

$\Leftrightarrow (\Pi_C(\text{prox}_f(x)) - \text{prox}_f(x))^T (z - \Pi_C(\text{prox}_f(x))) \geq 0 \ \forall z \in C$

Choose $z \equiv \Pi_C(x)$.

CS5660_2024_Quizes

CS5660 Offline Exam1

20-Mar-2024 7pm-8:30pm

NOTE: Please write your ROLL NO. clearly on ALL answer sheets. Be extremely precise and formal in your derivations. Answers without justification will NOT be awarded marks. Don't use any advanced/sophisticated results not taught in this course.

1. For an L-smooth and convex function f, prove from first principles that:

$$\frac{1}{L}\|\nabla f(x) - \nabla f(y)\|_2^2 \leq (\nabla f(x) - \nabla f(y))^\top (x-y) \leq L\|x-y\|_2^2.$$

Clearly indicate which steps/results use convexity and which use smoothness etc. Hint: Consider functions $\phi_x(y) \equiv f(y) - \nabla f(x)^\top y$.

[1+ 3 Marks]

Using this inequality, present a convergence analysis of gradient descent for unconstrained minimization of $L-$smooth, convex objectives. Assume the step-size is $\frac{1}{L}$. Hint: Try to set-up recursion starting with r_{k+1}^2, then lower bounding $\|\nabla f(x^{(k)})\|$ and solving the recursion.

[4 Marks]

2. Attempt this sequence of questions from section 6.4.4 in Beck's book and (almost :) covered in the lectures:

(a) Let $f(x) \equiv -\log(x), x \in \mathbb{R}_{++}$. Derive a simplified expression for $prox_{\lambda f}(x)$ for a fixed (given, yet arbitrary) $x \in \mathbb{R}, \lambda > 0$.

[2 Marks]

(b) Consider the function $h(x) = -\sum_{i=1}^n \log(x_i), x \equiv \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. Derive a simplified expression for $prox_{\lambda h}(x)$ for a fixed (given, yet arbitrary) $x \in \mathbb{R}^n, \lambda > 0$.

[1 Mark]

(c) Consider the set $C_\alpha \equiv \left\{ x \equiv \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}_{++}^n \;\middle|\; x_1 x_2 \ldots x_n \geq \alpha \right\}$, where $\alpha > 0$. Express this set in terms of h.

[1 Mark]

(d) Present a *simple* algorithm for computing $\Pi_{C_\alpha}(x)$ that uses your simplified expression for $prox_{\lambda h}(x)$.

[4 Marks]

3. Let C be a closed convex set and let Π_C denote the projection onto C operator. Express the prox operator of f, where f is:

(a) the support function of C,

[2 Marks+2Marks]

(b) defined by $f(x) = \frac{1}{2}\|x - \Pi_C(x)\|_2^2$ (projection error),

[2 Marks]

in terms of Π_C. In case you use any theorem/result from this course, then repeat it's proof/derivation. In case you use any (advanced) theorem/result outside this course, then clearly write down the formal statement (no need to prove).

4. For any closed convex set $C \subset \mathbb{R}^n$ and any function $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that it's prox operator is well-defined, show that

$$(\Pi_C(prox_f(x)) - prox_f(x))^\top (\Pi_C(x) - \Pi_C(prox_f(x))) \geq 0 \ \forall\, x \in \mathbb{R}^n.$$

[2+2 Marks]