BANACH ALGEBRAS

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Contents

1. Banach Algebras

The aim of this notes is to provide basic information about commutative Banach algebras. The final goal is to show that a unital, commutative complex Banach algebra $\mathcal{A}$ can be embedded as subalgebra of $C(\mathcal{M}_{\mathcal{A}})$, the algebra of continuous functions on a w^*-compact set $\mathcal{M}_{\mathcal{A}}$, known as the maximal ideal space or character space. Also, the non unital commutative complex Banach algebra can be embedded as a subalgebra of $\mathcal{C}_0(\Omega)$, the algebra of continuous functions on a w^*-locally compact (but not compact) set Ω, vanishing at ∞.

Definition 1.1 (Algebra)**.** *Let $\mathcal{A}$ be a non-empty set. Then $\mathcal{A}$ is called an algebra if*

(1) $(\mathcal{A}, +, .)$ *is a vector space over a field* $\mathbb{F}$
(2) $(\mathcal{A}, +, \circ)$ *is a ring and*
(3) $(\alpha a) \circ b = \alpha(a \circ b) = a \circ (\alpha b)$ *for every* $\alpha \in \mathbb{F}$*, for every* $a, b \in \mathcal{A}$

Usually we write ab instead of $a \circ b$ for notational convenience.

Definition 1.2. *An algebra $\mathcal{A}$ is said to be*

(1) ***real*** *or* ***complex*** *according to the field* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$ *respectively.*
(2) ***commutative*** *if* $(\mathcal{A}, +, \circ)$ *is commutative*

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Definition 1.3. *An algebra $\mathcal{A}$ is said to be **unital** if $(\mathcal{A}, +, \circ)$ has a unit, usually denoted by 1. Let $\mathcal{A}$ be unital and $a \in \mathcal{A}$. If there exists an element $b \in \mathcal{A}$ such that $ab = ba = 1$, then b is called an **inverse** of a.*

Remark 1.4. The unit element in a Banach algebra is unique. Also if an element has an in inverse, then it is unique.

Let $G(\mathcal{A}) := \{a \in \mathcal{A} : a \text{ is invertible in } \mathcal{A}\}$. Then $1 \in G(\mathcal{A})$ and $0 \notin G(\mathcal{A})$. The set $G(\mathcal{A})$ is a multiplicative group.

Definition 1.5. *Let $\mathcal{A}$ be an algebra and $\mathcal{B} \subseteq \mathcal{A}$. Then $\mathcal{B}$ is said to be a subalgebra if $\mathcal{B}$ it self is an algebra with respect to the operations of $\mathcal{A}$.*

Definition 1.6. *Let $I \subseteq \mathcal{A}$. Then I is called an **ideal** (two sided) if*

(i) *I is a subspace i.e., if $a, b \in I$ and $\alpha \in \mathbb{F}$, then $\alpha a + b \in I$*
(ii) *I is an ideal in the ring i.e., $a \in \mathcal{I}$ and $c \in \mathcal{A}$ implies that $ac, ca \in I$.*

*An ideal I is said to be **maximal** if $I \neq \{0\}$, $I \neq \mathcal{A}$ and if J is any ideal of $\mathcal{A}$ such that $I \subseteq J$, then either $J = I$ or $J = \mathcal{A}$.*

Remark 1.7. Every ideal is a subalgebra but a subalgebra need not be an ideal.

Definition 1.8 (normed algebra)**.** *If $\mathcal{A}$ is an algebra and $\|\cdot\|$ is a norm on $\mathcal{A}$ satisfying*

$$\|ab\| \leq \|a\|\|b\|, \quad \textit{for all } a, b \in \mathcal{A},$$

*then $\|\cdot\|$ is called an algebra norm and $(\mathcal{A}, \|\cdot\|)$ is called a **normed algebra**. A complete normed algebra is called a **Banach algebra**.*

Remark 1.9. In a normed algebra, the multiplication is both left and right continuous with respect to the algebra norm. That is if $(a_n) \subset \mathcal{A}$ is such that $a_n \to a$, then $a_n b \to ab$ and $ba_n \to ba$ as $n \to \infty$ for all $b \in \mathcal{A}$. In fact, the multiplication is jointly continuous. That is $a_n b_n \to ab$ as $n \to \infty$. In fact, the other way is also true.

Lemma 1.10. *Let $\mathcal{A}$ be an algebra such that $(\mathcal{A}, \|\cdot\|)$ is a Banach space and the multiplication is separately continuous. Then the multiplication is jointly continuous.*

Remark 1.11. We always denote the identity of a unital Banach algebra by 1 and assume that $\|1\| = 1$.

Here we illustrate the above definition with examples.

1.1. **Examples.** Here we give some examples of normed (Banach) algebras. In general, these can be classified into function algebras, operator algebras and group algebras according as multiplication is defined by point wise, by composition and by convolution.

1.1.1. *Function Algebras.* In these examples we consider algebras of functions. In all these examples the multiplication is point wise. These classes are known as function algebras.

(1) Let $\mathcal{A} = \mathbb{C}$. Then with respect to the usual addition, multiplication of complex numbers and the modulus, $\mathcal{A}$ is a commutative, unital Banach algebra.

(2) Let K be a compact Hausdorff space and $\mathcal{A} = C(K)$. Then with respect to the point wise multiplication of functions, $\mathcal{A}$ is a commutative unital algebra and with the norm $\|f\|_\infty = \sup\limits_{t\in K} |f(t)|$ is a Banach algebra.

(3) Let $S \neq \emptyset$ and $B(S) = \{f : S \to \mathbb{C} : f \text{ is bounded}\}$. For $f, g \in B(S)$, define

$$\begin{aligned}(f+g)(s) &= f(s) + g(s)\\ (\alpha f)(s) &= \alpha f(s), \text{for all } f, g \in B(S),\ \alpha \in \mathbb{C}\\ (fg)(s) &= f(s)g(s).\end{aligned}$$

Then $B(S)$ is an algebra with unit $f(s) = 1$ for all $s \in S$. With the norm

$$\|f\|_\infty = \sup\{|f(s)| : s \in S\},$$

$B(S)$ is a commutative Banach Algebra.

(4) Let Ω be a locally compact Hausdorff space and let

$$\mathcal{A} = C_b(\Omega) := \{f \in \mathcal{C}(\Omega) : f \ \text{ is bounded}\}.$$

Then $\mathcal{A}$ is a commutative unital Banach algebra.

(5) Let

$$\mathcal{A} = \mathcal{C}_0(\Omega) := \{f \in \mathcal{C}(\Omega) : f \text{ vanishes at } \infty\}.$$

We say $f \in \mathcal{C}(\Omega)$ vanishes at ∞ if and only if for every $\epsilon > 0$, there exists a compact subset K_ϵ of Ω such that $|f(t)| < \epsilon$ for every $t \in K_\epsilon^c$.

It can be verified easily that $\mathcal{A}$ is a commutative Banach algebra. Infact $\mathcal{A}$ is unital if and only if Ω is compact.

(6) Let

$$\mathcal{A} = \mathcal{C}_c(\Omega) := \{f \in \mathcal{C}(\Omega) : f \ \text{ has compact support}\}.$$

It can be seen that $\mathcal{A}$ is a commutative, normed algebra which is not a Banach algebra. And $\mathcal{A}$ is unital if and only if Ω is compact.

(7) Let $X = [0,1]$. Then $C'[0,1] \subset C[0,1]$ is an algebra and $\big(C'[0,1], \|\cdot\|_\infty\big)$ is not complete. Now define

$$\|f\| = \|f\|_\infty + \|f'\|_\infty,\ f \in C'[0,1].$$

Then $(C'[0,1], \|\cdot\|)$ is a Banach algebra.

(8) Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Consider

$$\mathcal{A}(\mathbb{D}) := \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \ \text{ is analytic}\}.$$

Then $\mathcal{A}(\mathbb{D})$ is a closed subalgebra of $C(\bar{\mathbb{D}})$. Hence it is a commutative, unital Banach algebra, known as the **disc algebra**.

(9) Let $H^\infty(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{C} : f \text{ is bounded and analytic}\}$ is a commutative, unital Banach algebra with respect to the point wise addition, multiplication of functions and usual scalar multiplication of functions and the supremum norm.

(10) Let (X, μ) be a finite measure space and

$$\mathcal{L}^{\infty}(\mu) := \{f : X \to \mathbb{C} : \text{ess sup}|f| < \infty\}.$$

With the point wise multiplication of functions and $\|f\|_{\infty} = \text{ess sup}|f|$, $\mathcal{L}^{\infty}(\mu)$ is a commutative, unital Banach algebra.

Remark 1.12. In $\mathcal{L}^1(\mathbb{R})$, there exists a sequence (e_n) with $\|e_n\| = 1$ for all n such that

$$\|e_n * f - f\|_1 \to 0 \text{ as } n \to \infty$$
$$\|f * e_n - f\|_1 \to 0 \text{ as } n \to \infty.$$

Such sequence (e_n) is called the approximation identity for $L^1(\mathbb{R})$.

But there are algebras without approximate identity also. For example, consider any Banach space $\mathcal{A}$ with the trivial multiplication, i.e $ab = 0$ for all $a, b \in \mathcal{A}$. Then $\mathcal{A}$ is a Banach algebra called the trivial Banach algebra and clearly it have no approximate identity.

1.1.2. *Operator Algebras.* Here we consider algebras whose elements are operators on a Banach space. In this case the multiplication is the composition of operators. These are known as operator algebras.

(1) Let $\mathcal{A} = \mathcal{M}_n(\mathbb{C}),\ (n \geq 2)$, the set of $n \times n$ matrices with matrix addition, matrix multiplication and with Frobenius norm defined by

$$\|A\|_F = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}$$

is a non-commutative unital Banach algebra.

(2) Let X be a complex Banach space with $\dim(X) \geq 2$ and $\mathcal{B}(X)$ is the Banach space of bounded linear operators on X with respect to the operator norm. With composition of operators as multiplication, $\mathcal{B}(X)$ is a non commutative, unital Banach algebra.

(3) $\mathcal{K}(X) = \{T \in \mathcal{B}(X) : T \text{ is compact}\}$ is a closed subalgebra of $\mathcal{B}(X)$. Hence it is a Banach algebra. It can be verified that $\mathcal{K}(X)$ is unital if and only if $\dim(X) < \infty$. Note that $\mathcal{K}(X)$ is an ideal in $\mathcal{B}(X)$.

Exercise 1.13. *Show that* $\|A\| = \max\limits_{1 \leq i,j \leq n} |a_{ij}|$ *defines a norm on* $\mathcal{M}_n(\mathbb{C})$ *but it is not an algebra norm.*

1.1.3. *Group Algebras.* Here is a third kind of examples of algebras which consists of functions but the multiplication is the convolution product. Usually this class is known as the group algebras.

(1) Let

$$\mathcal{A} = L^1(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} : f \text{ is measurable and } \int |f(t)|dt < \infty\}.$$

Define the multiplication and the norm by

$$(f * g)(x) := \int f(x-t)g(t)dm(t) \text{ for all } f, g \in L^1(\mathbb{R}),$$
$$\|f\|_1 := \int |f(t)|dt,$$

respectively. Using Fubini's theorem it can be shown that

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1 \text{ for all } f, g \in L^1(\mathbb{R}).$$

It is known that $\mathcal{A}$ is complete. Hence $\mathcal{A}$ is a Banach algebra. Also the convolution is commutative and we can show that $\mathcal{A}$ has no unit.

(2) Let $\ell^1(\mathbb{Z}) = \{(x_n)_{n\in\mathbb{Z}} : \sum_{n=-\infty}^{\infty} |x_n| < \infty\}$.

Define the multiplication and the norm respectively as: For $x = (x_n), y = (y_n) \in \ell^1(\mathbb{Z})$,

$$(x * y)_n := \sum_{k=-\infty}^{\infty} x_{n-k} y_k,$$
$$\|x\|_1 := \sum_{n=-\infty}^{\infty} |x_n|.$$

The element $\delta = (\delta_{0n})$, where $\delta_{0n} := \begin{cases} 1, \text{if } n = 0, \\ 0, \text{else} \end{cases}$ is the unit element in $\ell^1(\mathbb{Z})$. In fact, $e_n * e_m = e_{n+m}$ for all $n, m \in \mathbb{Z}$.

(3) Let $w : \mathbb{R} \to [0, \infty)$ be such that $w(s+t) \leq w(s) + w(t)$ for all $s, t \in \mathbb{R}$. Let $L^1_w(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} : f$ is measurable and $\int_{\mathbb{R}} |f(t)|w(t)dt < \infty\}$.

For $f \in L^1_w(\mathbb{R})$, define the norm

$$\|f\| = \int_{\mathbb{R}} |f(t)|w(t)dt.$$

Then $L^1_w(\mathbb{R})$ is Banach space and with the convolution it becomes a Banach Algebra.

(4) Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and

$$\mathcal{A} = L^1(\mathbb{T}) := \{f : \mathbb{T} \to \mathbb{C} : f \text{ is measurable and } \int_{\mathbb{T}} |f(t)|\mu(dt) < \infty\}.$$

Here μ denote the normalized Lebesgue measure on $\mathbb{T}$. For $f, g \in \mathcal{A}$ define the convolution by

$$(f * g)(t) = \int_{\mathbb{T}} f(t\tau^{-1})g(\tau)\mu(d\tau))$$

and the norm by

$$\|f\|_1 := \int_{\mathbb{T}} |f(t)|\mu(dt).$$

With these operations $\mathcal{A}$ is a non unital commutative Banach algebra.

(5) Let $\mathcal{B} := \{f : \mathbb{R} \to \mathbb{C} : f$ is measurable, $2\pi-$periodic and $\displaystyle\int_0^{2\pi} |f(t)|dt < \infty\}$.
With respect to the convolution: for $f, g \in \mathcal{B}$,

$$(f * g)(t) = \int_0^{2\pi} f(t-\tau)g(\tau)d\tau$$

and the norm

$$\|f\| := \frac{1}{2\pi}\int_0^{2\pi} |f(t)|dt,$$

$\mathcal{B}$ is a commutative, non unital Banach algebra.

Exercise 1.14. *Show that $\ell^1(\mathbb{N}\cup\{0\}) := \{x \in \ell^1(\mathbb{Z}) : x_n = 0 \; for \, n < 0\}$ is a subalgebra of $\ell^1(\mathbb{Z})$.*

Let G be a locally compact group. Then there exists a Borel measure μ on G called the Haar measure (left-invariant Haar measure) satisfying the following conditions:

(1) $\mu(xE) = \mu(E)$ for every $x \in G$ and for every measurable subset E in G
(2) $\mu(U) > 0$ for every non-empty open set U of G
(3) $\mu(K) < \infty$ for every compact subset K of G.

The above conditions determines the measure μ uniquely upto a positive scalar.

Definition 1.15. *The group algebra of G is the space*

$$L^1(G) := \{f : G \to \mathbb{C} : f \text{ is measurable and } \int_G |f(t)|\mu(dt) < \infty\}.$$

For $f, g \in L^1(G)$ define the multiplication and the norm by

$$(f * g)(t) = \int_G f(t\tau^{-1})g(\tau)\mu(d\tau)$$
$$\|f\|_1 := \int_G |f(t)|\mu(dt).$$

Then $L^1(G)$ is a Banach algebra. In fact it is commutative if and only if G is commutative. Also it is unital if and only if G is discrete.

1.2. **New Banach Algebras from old.** Here we present different ways of getting Banach algebras from the known ones.

(1) **Direct sum of algebras**
Case 1: Finite direct sum
Let $\{\mathcal{A}_i\}_{i=1}^n$ be Banach algebras. Then the direct sum of A_i' s is denoted by $\mathcal{A} := \oplus_{i=1}^n A_i$. We define the multiplication and the norm as follows:
Let $x = (x_1, x_2, \dots, x_n),\; y = (y_1, y_2, \dots, y_n) \in \mathcal{A}$. Then

$$x.y := (x_1y_1, x_2y_2, \dots, x_ny_n)$$
$$\|x\| := \max_{1\le i\le n} \|x_i\|.$$

Case 2: infinite sum
Let I be an index set and $\{\mathcal{A}_\alpha : \alpha \in I\}$ be Banach algebras. Let $\mathcal{A} =$

$\oplus_{\alpha\in I}\mathcal{A}_\alpha := \{(x_\alpha)_{\alpha\in I} : \sup_{\alpha\in I}\|x_\alpha\| < \infty\}$. For $x = (x_\alpha)$, $y = (y_\alpha) \in \mathcal{A}$. Define the multiplication and the norm respectively by

$$x.y := (x_\alpha.y_\alpha)_{\alpha\in I}$$
$$\|x\| := \sup_{\alpha\in I}\|(x_\alpha)\|.$$

Then $\mathcal{A}$ is a Banach algebra and is unital if each A_i is unital. A similar conclusion holds for commutativity.

(2) **Quotient algebras**:
Let $\mathcal{A}$ be a Banach algebra and I be a proper closed ideal in $\mathcal{A}$. We know that quotient space $\mathcal{A}/I$ is a Banach space with the norm $\|[x]\| := \inf\{\|y\| : y \in [x]\} = d(x, I)$, here $[x] = x + I$ is the equivalence class of $x \in \mathcal{A}$. If we define $[x][y] = [xy] = xy + I$ for all $x, y \in \mathcal{A}$, then $\mathcal{A}/I$ is a Banach algebra.

(3) **Complexification**
Let $\mathcal{A}$ be a real unital Banach algebra. Our aim is to imbed this algebra as a real subalgebra in a complex unital Banach algebra.
Let $\tilde{\mathcal{A}} := \{(x, y) : x, y \in \mathcal{A}\}$. Let $\alpha = a + ib \in \mathbb{C}$ and $(x_1, x_2), (y_1, y_2) \in \tilde{\mathcal{A}}$. Define

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$
$$\alpha(x_1, y_1) := (ax_1, by_1).$$

With these operations $\tilde{\mathcal{A}}$ is a vector space. Now we define the norm and the multiplication as follows:

$$\|(x, y)\| := \sup_{\theta\in\mathbb{R}}\{\|x\cos\theta - y\sin\theta\| + \|x\sin\theta + y\cos\theta\|\}$$
$$(x_1, y_1).(x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

With these operations $\tilde{\mathcal{A}}$ is a complex Banach algebra and the embedding $\mathcal{A} \mapsto \tilde{\mathcal{A}}$ given by $x \mapsto (x, 0)$ is a homeomorphic isomorphism. It can be checked easily that $\|x\| \le \|(x, 0)\| \le \sqrt{2}\|x\|$. The algebra $\tilde{\mathcal{A}}$ is called the complexification of $\mathcal{A}$.

(4) **Unitization**:
Let $\mathcal{A}$ be a non unital Banach algebra. Consider $\mathcal{A}_1 := \mathcal{A} \oplus \mathbb{C}$, the direct sum of $\mathcal{A}$ and $\mathbb{C}$. Let $(a, \lambda), (b, \mu) \in \mathcal{A}_1$ and $\alpha \in \mathbb{C}$. Define the addition, scalar multiplication, multiplication, and the norm by:

$$(a, \lambda) + (b, \mu) := (a + b, \lambda + \mu)$$
$$\alpha(a, \lambda) := (\alpha a, \alpha\lambda)$$
$$(a, \lambda)(b, \mu) := (ab + \mu a + \lambda b, \lambda\mu)$$
$$\|(a, \lambda)\| := \|a\| + |\lambda|.$$

It can be shown that $\mathcal{A}$ is a Banach algebra with unit $(0, 1)$. Also the map $a \to (a, 0)$ is an isometric from $\mathcal{A}$ into $\mathcal{A}_1$. Infact the algebra $\tilde{A} = \{(a, 0) : a \in \mathcal{A}\}$ is a closed two sided ideal of $\mathcal{A}_1$ such that codimension of

$\mathcal{A}$ in A_1 is one.

(5) **Multiplication with a weight**:
Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}$ be fixed. Define a new multiplication by
$$x \times_a y := axy, \text{ for all } x, y \in \mathcal{A}.$$
Then $\mathcal{A}$ is an algebra with this new multiplication. It is a Banach algebra if $\|a\| \leq 1$ (verify !)

(6) **Arens Multiplication**:
Let $\mathcal{A}$ be a Banach algebra. Fix $x \in \mathcal{A}$ and $f \in \mathcal{A}^*$. Define $\langle f, x\rangle : \mathcal{A} \to \mathbb{C}$ by $\langle f, x\rangle(y) = f(xy)$. It can be shown easily that $\langle f, x\rangle \in \mathcal{A}^*$ and $\|\langle f, x\rangle\| \leq \|f\|\|x\|$.
Now let $F \in \mathcal{A}^{**}$. Define $[F, f] : \mathcal{A} \to \mathbb{C}$ by $[F, f](y) = F(\langle f, y\rangle)$. Then $[F, f] \in \mathcal{A}^*$ with $\|[F, f]\| \leq \|F\|\|f\|$.
For $G \in \mathcal{A}^{**}$ define multiplication in $\mathcal{A}^{**}$ by $(FG)(f) = F[G, f]$ for all $f \in \mathcal{A}^*$. With the help of the above observations we can show that the norm on $\mathcal{A}^{**}$ is an algebra norm. Since $\mathcal{A}^{**}$ is complete, it is a Banach algebra.

1.2.1. *Examples of ideals.*

(1) Let H be a complex Hilbert space, then $\mathcal{K}(H)$ is an ideal of $\mathcal{B}(H)$.

(2) Let K be a compact, Hausdorff space and F be a closed subset of K. Then
$$I_F := \{f \in C(K) : f|_F = 0\}$$
is closed ideal. In fact, these are the only closed ideals in $C(K)$. Show that I_F is maximal if and only if F is a singleton set.

(3) The set of all $n \times n$ upper/lower triangular matrices is a subalgebra of $\mathcal{M}_n(\mathbb{C})$ but not an ideal.

(4) Let $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$ and $\mathcal{D} = \{(a_{ij}) \in \mathcal{A} : a_{ij} = 0,\ i \neq j\}$. Then $\mathcal{D}$ is a subalgebra of $\mathcal{M}_n(\mathbb{C})$ but not an ideal.

Definition 1.16. *Let $\mathcal{A}, \mathcal{B}$ be two algebras and $\phi : \mathcal{A} \to \mathcal{B}$ be a map. Then ϕ is said to be a homomorphism if*

(i) *ϕ is linear and*
(ii) *ϕ is multiplicative i.e., $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$*

*If ϕ is one-to-one, then ϕ is called an isomorphism. If $\mathcal{A}$ and $\mathcal{B}$ are normed algebras, then a homomorphism ϕ is called **isometric** if $\|\phi(a)\| = \|a\|$ for all $a \in \mathcal{A}$.*

Exercise 1.17. *If $\phi : \mathcal{A} \to \mathcal{B}$ is a homomorphism, then $ker(\phi)$ is an ideal.*

2. The spectrum

Throughout we assume that $\mathcal{A}$ to be a complex unital Banach algebra. Recall that $G(\mathcal{A}) := \{a \in \mathcal{A} :$ a is invertible in $\mathcal{A}\}$ is a multiplicative group. In this section we discuss the properties of this set.

Recall that if $z \in \mathbb{C}$ with $|z| < 1$, then $(1 - z)$ is invertible and

$$(1-z)^{-1} = \sum_{n=0}^{\infty} z^n.$$

This result can be generalized to elements in a unital Banach algebra.

Lemma 2.1. *If $a \in \mathcal{A}$ with $\|a\| < 1$. Then*

$$1 - a \in G(\mathcal{A}) \ \ \text{and} \ \ (1-a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Further more, $\|(1-a)^{-1}\| \leq \dfrac{1}{1 - \|a\|}$.

Proof. Let $s_n := 1 + a + a^2 + \cdots + a^n$. Then $\|s_n\| \leq \displaystyle\sum_{j=0}^{n} \|a\|^j$. Since $\|a\| < 1$, the sequence s_n is convergent. This shows that s_n is absolutely convergent. As $\mathcal{A}$ is a Banach algebra, the series $\displaystyle\sum_{n=0}^{\infty} a^n$ is convergent.

Let $b = \displaystyle\sum_{n=0}^{\infty} a^n$. Then

$$(1-a)b = \lim_{n\to\infty} (1-a)s_n = \lim_{n\to\infty} (1 - a^{n+1}) = 1.$$

Similarly we can show that $b(1-a) = 1$. Hence $(1-a)^{-1} = b$.

To get the bound, consider

$$\|b\| = \lim_{n\to\infty} \|s_n\| \leq \lim_{n\to\infty} \sum_{k=0}^{n} \|a\|^k = \lim_{n\to\infty} \sum_{k=0}^{n} \|a\|^k = \frac{1}{1 - \|a\|}.$$

□

Corollary 2.2. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in G(\mathcal{A})$.*

(1) *Let $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\|a\| < |\lambda|$. Then*

$$\lambda.1 - a \in G(\mathcal{A}) \ \text{and} \ (\lambda.1 - a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}.$$

Furthermore, $\|(a-\lambda)^{-1}\| \leq \dfrac{1}{|\lambda| - \|a\|}$.

(2) *Let $a \in \mathcal{A}$ be such that $\|1 - a\| < 1$, then $a \in G(\mathcal{A})$ and $a^{-1} = \displaystyle\sum_{n=0}^{\infty} (1-a)^n$.*

Proof. To prove (1), take $a_\lambda = \dfrac{a}{\lambda}$ and apply Lemma 2.1. To prove (2), replace a by $1 - a$ in Lemma 2.1. □

Proposition 2.3. *The set $G(\mathcal{A})$ is open in $\mathcal{A}$.*

Proof. Let $a \in G(\mathcal{A})$ and $D = \{b \in \mathcal{A} : \|a - b\| < \dfrac{1}{\|a^{-1}\|}\}$. Note that $a^{-1}(a - b) = 1 - a^{-1}b$. So $\|1 - a^{-1}b\| \leq \|a^{-1}\|\|a - b\| < 1$. Hence $a^{-1}b \in G(\mathcal{A})$. So $b = a(a^{-1}b) \in G(\mathcal{A})$. This shows that $D \subset G(\mathcal{A})$. □

Proposition 2.4. *The map $a \mapsto a^{-1}$ is continuous on $G(\mathcal{A})$.*

Proof. Let $(a_n) \subseteq G(\mathcal{A})$ be such that $a_n \to a \in G(\mathcal{A})$. Our aim is to show that $a_n^{-1} \to a^{-1}$. Consider,

$$\|a_n^{-1} - a^{-1}\| = \|a^{-1}(a_n - a)a_n^{-1}\| \leq \|a^{-1}\|\|a_n - a\|\|a_n^{-1}\|. \tag{2.1}$$

Hence, if we can show $\|a_n^{-1}\|$ is bounded by a fixed constant, we are done. As $a_n \to a$, there exists $n_0 \in \mathbb{N}$ such that $\|a_n - a\| < \dfrac{1}{2\,\|a^{-1}\|}$ for all $n \geq n_0$. Thus $\|a^{-1}a_n - 1\| < \dfrac{1}{2}$ for $n \geq n_0$. So for $n \geq n_0$, we have

$$\|aa_n^{-1}\| = \|(a^{-1}a_n)^{-1}\| = \sum_{k=0}^{\infty} \|(a^{-1}a_n - 1)^k\| < \sum_{k=0}^{\infty} \frac{1}{2^k} < 2.$$

Therefore $\|a_n^{-1}\| \leq \|a_n^{-1}a\|\,\|a^{-1}\| < 2\,\|a^{-1}\|$ for all $n \geq n_0$.

Choose $M = \max\{\|a_i^{-1}\|, 2\|a^{-1}\| : i = 1, 2, \ldots, n_0\}$. Now by Equation (2.1), it follows that $a_n^{-1} \to a^{-1}$. □

Remark 2.5. As the maps $(a, b) \to ab$ from $G(\mathcal{A}) \times G(\mathcal{A})$ into $G(\mathcal{A})$ and $a \to a^{-1}$ form $G(\mathcal{A})$ into $G(\mathcal{A})$ are continuous, we can conclude that $G(\mathcal{A})$ is a topological group.

Example 2.6.
(a) Let $\mathcal{A} = C(K)$, where K is compact, Hausdorff space. Then $G(\mathcal{A}) = \{f \in \mathcal{A} : f(t) \neq 0 \text{ for each } t \in K\}$.
(b) Let $\mathcal{A} = M_n(\mathbb{C})$. Then $G(\mathcal{A}) = \{A \in M_n(\mathbb{C}) : det(A) \neq 0\}$.

In this section we define the concept of spectrum of an element in a Banach algebra.

Definition 2.7. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. The resolvent $\rho_{\mathcal{A}}(a)$ of a with respect to $\mathcal{A}$ is defined by*

$$\rho_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C} : a - \lambda 1 \in G(\mathcal{A})\}.$$

The spectrum $\sigma_{\mathcal{A}}(a)$ of a with respect to $\mathcal{A}$ is defined by $\sigma_{\mathcal{A}}(a) = \mathbb{C} \setminus \rho_{\mathcal{A}}(a)$. That is

$$\sigma_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C} : a - \lambda 1 \textit{ is not invertible in } \mathcal{A}\}.$$

If $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$ such that $1 \in \mathcal{B}$. If $a \in \mathcal{B}$, then we can discuss the invertibilty of a in $\mathcal{B}$ as well as in $\mathcal{A}$. In such cases we write $\rho_{\mathcal{B}}(a)$ and $\rho_{\mathcal{A}}(a)$. Similar convention hold for the spectrum. But if we want to discuss the spectrum in one algebra, then we omit the suffix.

Example 2.8.
(1) Let $f \in C(K)$ for some compact Hausdorff space K. Then $\sigma(f) = \text{range}(f)$.

(2) Let $A \in M_n(\mathbb{C})$. Then $\sigma(A) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigen value of } A\}$.

(3) Let $f \in L^\infty(X,\mu)$ be real valued. Then $\sigma(f) = \text{essrange(f)}$, the essential range of f.

Recall that if $f \in L^\infty(X,\mu)$, then the essential range of f is defined by

$$\{w \in \mathbb{R} : \mu(\{x : |f(x) - w| < \epsilon\}) > 0 \text{ for every } \epsilon > 0\}.$$

First we show that the spectrum of an element in a Banach algebra is a non empty.

Theorem 2.9. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Then $\sigma(a) \neq \emptyset$.*

Proof. If a is not invertible, then $0 \in \sigma(a)$. Assume that a is invertible. Assume that $\sigma(a) = \emptyset$. Then $\rho(a) = \mathbb{C}$.

Let $\phi \in A^*$. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(\lambda) = \phi((a - \lambda 1)^{-1})$.

Let $\lambda_0 \in \mathbb{C}$. Then,

$$\begin{aligned} f(\lambda) - f(\lambda_0) &= \phi((a - \lambda 1)^{-1}) - \phi((a - \lambda_0 1)^{-1}) \\ &= \phi\Big((a - \lambda 1)^{-1} - (a - \lambda_0 1)^{-1}\Big) \\ &= -\phi\Big((a - \lambda 1)^{-1}(\lambda_0 - \lambda)(a - \lambda_0 1)^{-1}\Big). \end{aligned}$$

Hence $\lim_{\lambda \to \lambda_0} \dfrac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = -\phi\big((\lambda_0 1 - a)^2\big)$. As λ_0 is arbitrary, f is entire.

Note that $|f(\lambda)| \leq \|\phi\| \, \|(\lambda 1 - a)^{-1}\| \leq \|\phi\| \dfrac{1}{|\lambda| - \|a\|}$ for each $|\lambda| > \|a\|$. Hence $|f(\lambda)| \to 0$ as $\lambda \to \infty$. Hence f is bounded. By Liouville's theorem, f must be constant and hence $f = 0$. That is $\phi(\lambda 1 - a)^{-1} = 0$ for all $\phi \in \mathcal{A}^*$. Hence by the Hahn-Banach theorem, $(\lambda 1 - a)^{-1} = 0$, a contradiction. Therefore $\sigma(a)$ is non empty. □

Remark 2.10. (1) We know by Lemma 2.1 that if $\lambda \in \mathbb{C}$ such that $|\lambda| > \|a\|$, then $a - \lambda 1 \in G(\mathcal{A})$. Hence $\sigma(a) \subseteq \{z \in \mathbb{C} : |z| \leq \|a\|\}$. Hence $\sigma(a)$ is bounded subset of $\mathbb{C}$

(2) $\sigma(a)$ is closed, since the map $f : \mathbb{C} \to \mathcal{A}$ given by $f(\lambda) = a - \lambda 1$ is continuous and $\rho(a) = f^{-1}\big(G(\mathcal{A})\big)$.

Exercise 2.11. *Let $\phi \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Define $g : \mathbb{C} \setminus \sigma(a) \to \mathbb{C}$ by $g(\lambda) = \phi((\lambda.1 - a)^{-1})$. Show that g is analytic in $\rho(a)$.*

2.1. **Gelfand-Mazur theorem.**

Theorem 2.12 (Gelfand-Mazur theorem)**.** *Every Banach division algebra is isometrically isomorphic to $\mathbb{C}$.*

Proof. Let $a \in \mathcal{A}$. Then $\sigma(a) \neq \emptyset$. Let $\lambda \in \sigma(a)$. Then $a - \lambda 1 \notin G(\mathcal{A})$. As $\mathcal{A}$ is a divison algebra, $a - \lambda 1 = 0$. Hence $a = \lambda \cdot 1$. Now define a map $\eta : \mathbb{C} \to \mathcal{A}$ by $\eta(\lambda) = \lambda \cdot 1$. It can be checked that η is an isometric isomorphism. □

Proposition 2.13. *Let $\mathcal{A}$ be a unital Banach algebra and $a, b \in \mathcal{A}$. Then*

$$1 - ab \in G(\mathcal{A}) \Leftrightarrow 1 - ba \in G(\mathcal{A}).$$

Proof. Assume that $1 - ab \in G(\mathcal{A})$. Let $c := 1 + b(1 - ab)^{-1}a$. It can be checked easily that $c(1 - ba) = 1 = (1 - ba)c$. □

Corollary 2.14. *Let $\mathcal{A}$ be a unital Banach algebra and $a, b \in \mathcal{A}$. Then $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$.*

Exercise 2.15. *Let $\mathcal{A} = \mathcal{Q}(H)$, the Calkin algebra. Let $T \in \mathcal{B}(H)$, then $[T] \in \mathcal{Q}(H)$. The essential spectrum of T is defined by*

$$\sigma_{ess}(T) := \{\lambda \in \mathbb{C} : [T - \lambda I] \text{ is not invertible in } \mathcal{Q}(H)\}$$

Show that $[T - \lambda I]$ is invertible if and only if there exists $K_1, K_2 \in \mathcal{K}(H)$ and $S \in \mathcal{B}(H)$ such that $(T - \lambda I)S = K_1 + I$ and $S(T - \lambda I) = K_2 + I$.

Definition 2.16. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Then the spectral radius of a is defined by $r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}$.*

Note that from Remark 2.10, it follows that $0 \leq r(a) \leq \|a\|$.

Example 2.17.
(1) Let $\mathcal{A} = C(K)$ and $f \in \mathcal{A}$. Then

$$r(f) = \sup\{|\lambda| : \lambda \in \text{ range}(f)\} = \|f\|_\infty.$$

(2) Let $T \in \mathcal{B}(H)$ be normal. Then $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} = \|T\|$
(3) Let $\mathcal{A} = M_n(\mathbb{C})$. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Since A is nilpotent, $\sigma(A) = \{0\}$ and hence $r(A) = 0$. But $\|A\| = 1$
(4) Let $\mathcal{A} = C'[0,1]$ and $f \in \mathcal{A}$. Define $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Let $g : [0,1] \to \mathbb{C}$ be the inclusion map. Then $r(g) = 1$ and $\|g\| = 2$
(5) Let R be the right shift operator on ℓ^2. Then $\sigma(R) = \overline{\mathbb{D}}$.

Exercise 2.18. *Let $\mathcal{A}$ be a unital complex Banach algebra and $a \in \mathcal{A}$. Show that*
(1) *If a is invertible, then $\sigma(a^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(a)\}$*
(2) $\sigma(a+1) = \{\lambda + 1 : \lambda \in \sigma(a)\}$
(3) *Prove that $\sigma(a^2) = \sigma(a)^2 = \{\lambda^2 : \lambda \in \sigma(a)\}$*
(4) *Prove that if p is a polynomial with complex coefficients, then*

$$\sigma(p(a)) = p(\sigma(a)) = \{p(\lambda) : \lambda \in \sigma(a)\}$$

(5) *$r(a^n) = r(a)^n$ for all $n \in \mathbb{N}$*
(6) *If $b \in \mathcal{A}$, then $r(ab) = r(ba)$.*

2.2. **The spectral radius formula.** Here we give a proof of the spectral radius formula using the spectral radius formula for bounded operator on a a Banach space.

Theorem 2.19. *Let $\mathcal{A}$ be a complex unital Banach algebra and $a \in \mathcal{A}$. Then*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}.$$

Proof. Define $L_a : \mathcal{A} \to \mathcal{A}$ by $L_a(b) = ab$. We know that $\|L_a\| = \|a\|$ and $L_a^n = L_{a^n}$ for each $n \in \mathbb{N}$. It can be shown that $\sigma(a) = \sigma(L_a)$ and hence $r(a) = r(L_a)$. By the spectral radius formula for L_a, we have

$$\begin{aligned} r(a) = r(L_a) &= \lim_{n\to\infty} \|(L_a)^n\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \|(L_{a^n})\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}. \end{aligned}$$

□

Theorem 2.20. *Let $\mathcal{A}$ be a complex Banach algebra. The the following are equivalent;*

(1) *There exists a $c > 0$ such that $c\|a\|^2 \leq \|a^2\|$ for all $a \in \mathcal{A}$*
(2) *There exists a $d > 0$ such that $d\|a\| \leq r(a)$ for all $a \in \mathcal{A}$.*

Proof. Suppose there exists a $d > 0$ such that $d\|a\| \leq r(a)$ for all $a \in \mathcal{A}$. Then $d^2\|a^2\| \leq r(a^2) \leq \|a^2\|$. Taking $c = d^2$, we arrive to the conclusion.

On the other hand, assume that there exists a $c > 0$ such that

$$c\|a\|^2 \leq \|a^2\| \text{ for all } a \in \mathcal{A}, \tag{2.2}$$

then squaring both sides of the equation, we get

$$c^2\|a\|^4 \leq \|a^2\|^2 \leq \frac{1}{c}\|a^4\| \text{ (by Equation (2.2))}.$$

On simplification we get $c^3\|a\|^4 \leq \|a^4\|$. By induction, we get $c^{2k-1}\|a\|^{2^k} \leq \|a^{2^k}\|$ for all $k \in \mathbb{N}$. By the spectral radius formula, we have $c\|a\| \leq r(a)$. □

Let $a \in \mathcal{A}$. Write $a^0 := 1$ and $\exp(a) := \displaystyle\sum_{n=0}^{\infty} \frac{a^n}{n!}$. To show this series is convergent, consider the n^{th} partial sum $s_n = \displaystyle\sum_{k=0}^{n} \frac{a^k}{k!}$. Then $\|s_n\| \leq \displaystyle\sum_{k=0}^{n} \frac{\|a\|^k}{k!}$. Hence $\lim_{n\to\infty} \|s_n\| \leq \exp(\|a\|)$. Hence $\displaystyle\sum_{n=0}^{\infty} \frac{a^n}{n!}$ is convergent absolutely and since $\mathcal{A}$ is complete, it is convergent.

Also note that $\exp(a)\exp(-a) = 1 = \exp(-a)\exp(a)$. In general if $ab = ba$, then $\exp(a + b) = \exp(a)\exp(b)$.

Theorem 2.21. *Let $\mathcal{A}$ be a complex Banach algebra. Suppose there exists a $M > 0$ such that*

$$\|\exp(\lambda a) b \exp(-\lambda a)\| \leq M, \text{ for all } \lambda \in \mathbb{C},$$

then $ab = ba$.

Proof. Let $F(\lambda) = \exp(\lambda a)\, b \exp(-\lambda a)$ for each $\lambda \in \mathbb{C}$. By the series representation of the exponential function, we have

$$\begin{aligned}
F(\lambda) &= \left(1 + (\lambda a) + \frac{(\lambda a)^2}{2!} + \dots\right) b \left(1 - (\lambda a) + \frac{(\lambda a)^2}{2!} - \dots\right) \\
&= \left(b + (\lambda ab) + \frac{(\lambda a)^2 b}{2!} + \dots\right)\left(1 - (\lambda a) + \frac{(\lambda a)^2}{2!} - \dots\right) \\
&= \left(b - \lambda ba + \frac{b(\lambda a)^2}{2!} + \lambda ab - \lambda^2 aba + \frac{\lambda^2 a^2 b}{2!} \dots +\right) \\
&= b + \lambda(ab - ba) + \lambda^2\left(\frac{ba^2}{2!} - aba + \frac{a^2 b}{2!}\right) + \lambda^3(\dots) + \dots
\end{aligned}$$

This conclude that $F(\lambda)$ is entire. By the Hypothesis, $F(\lambda)$ is bounded. Hence by the Liouville's theorem, $F(\lambda)$ must be constatnt, so $F(\lambda) = F(0) = b$. Thus $ab = ba$. □

Theorem 2.22. *Let $\mathcal{A}$ be a complex Banach algebra. If there exists a $c > 0$ such that*

$$c\|a\|^2 \leq \|a^2\| \text{ for all } a \in \mathcal{A},$$

then $\mathcal{A}$ is commutative.

Proof. Let $a \in \mathcal{A}$ be arbitrary and $b \in \mathcal{A}$ be fixed. Let $z = \exp(\lambda a)\, b\, \exp(-\lambda a)$. Let $\mu \in \mathbb{C}$. Then $z - \mu 1 = \exp(\lambda a)\,(b - \mu 1)\, \exp(-\lambda a)$. Then $z - \mu 1$ is invertible if and only $b - \mu 1$ is invertible. Hence $\sigma(z) = \sigma(b)$ and $r(z) = r(b)$. Now by Theorem 2.20, there exists $d > 0$ such that $d\|z\| \leq r(z) = r(b)$. That is $\|z\| \leq \dfrac{r(b)}{d} < \infty$. Hence by Theorem 2.21, $ab = ba$ for all $a \in \mathcal{A}$. Hence $\mathcal{A}$ is commutative. □

2.2.1. *Topological Divisors of Zero.* In this section we discuss the topological divisors of zero in a Banach algebra and some properties of them.

Definition 2.23. *Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}$. Then a is called a left topological divisor of zero if there exists a sequence $(b_n) \subseteq \mathcal{A}$ such that $\|b_n\| = 1$ for all n and $ab_n \to 0$ as $n \to \infty$.*

The right topological divisor of zero can be defined in a similar way. An element in a Banach algebra is called a topological divisor of zero it is both left and right topological zero divisor. In particular, if $\mathcal{A}$ is commutative, both the left and the right topological divisors of zero divisors coincide.

Example 2.24. Let $\mathcal{A} = C[0,1]$ and $f(t) = t,\ t \in [0,1]$. Let

$$f_n(t) = \begin{cases} 1 - nt, \text{if } t \in [0, \frac{1}{n}], \\ 0 \text{ else.} \end{cases}$$

Then $f_n f = f f_n \to 0$ as $n \to \infty$.

Let $\mathcal{Z}$ denote the set of all topological zero divisors of $\mathcal{A}$ and S denote the set of singular elements in $\mathcal{A}$. The we have the following relations.

Proposition 2.25. *Let $\mathcal{A}$ be a unital Banach algebra. Then $\partial G(\mathcal{A}) \subseteq \mathcal{Z}$.*

Proof. Let $a \in \partial G(\mathcal{A})$. Choose $a_n \in G(\mathcal{A})$ such that $a_n \to a$ as $n \to \infty$. As $a \notin G(\mathcal{A})$, we have $a_n^{-1} a \notin G(\mathcal{A})$. By Lemma 2.1,

$$\begin{aligned} 1 \leq \|a_n^{-1} a - 1\| &= \|a_n^{-1}(a - a_n)\| \\ &\leq \|a_n^{-1}\| \|a - a_n\|. \end{aligned}$$

Therefore $\dfrac{1}{\|a_n^{-1}\|} \to 0$ as $n \to \infty$. Let $b_n = \dfrac{a_n^{-1}}{\|a_n^{-1}\|}$. Then $\|b_n\| = 1$ for each n and

$$ab_n = \frac{aa_n^{-1}}{\|a_n^{-1}\|} = \frac{(a - a_n)a_n^{-1}}{\|a_n^{-1}\|} + \frac{1}{\|a_n^{-1}\|} \to 0 \text{ as } n \to \infty.$$

With a similar argument, it can be shown that $b_n a \to 0$ as $n \to \infty$. □

Exercise 2.26. *Let $\mathcal{A}$ be a unital Banach algebra. Show that*

(1) $\mathcal{Z} \subseteq S$
(2) $\partial S \subseteq \mathcal{Z}$
(3) *If $\mathcal{Z} = \{0\}$, then $\mathcal{A} \simeq \mathbb{C}$*
(4) *If there exists a $k > 0$ such that $\|ab\| \geq k\|a\|\|b\|$ for all $a, b \in \mathcal{A}$, then $\mathcal{A} \simeq \mathbb{C}$.*

Let $\mathcal{A}$ be a Banach algebra with unit 1 and $\mathcal{B}$ be a closed subalgebra of $\mathcal{A}$ such that $1 \in \mathcal{B}$. If $a \in G(\mathcal{B})$, then $a \in G(\mathcal{A})$. If b is a topological zero divisor in $\mathcal{B}$, then it is also a topological zero divisor in $\mathcal{A}$.

If $a \in \mathcal{B}$, then we can discuss the invertibility of a with respect to both $\mathcal{A}$ and $\mathcal{B}$. But if $a \in G(\mathcal{A})$ need not imply that $a \in G(\mathcal{B})$.

Exercise 2.27. *Let $\mathcal{B}$ be maximal, commutative subalgebra of a unital Banach algebra $\mathcal{A}$ such that $1 \in \mathcal{B}$. Show that $\mathcal{B}$ is closed and if $a \in \mathcal{B}$, then $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$.*

3. Multiplicative Functionals

Proposition 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras with identities $1_{\mathcal{A}}, 1_{\mathcal{B}}$ respectively. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism such that $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. Then $\sigma_{\mathcal{B}}(\Phi(a)) \subset \sigma_{\mathcal{A}}(a)$. .*

Further more, if Φ is bijective, then $\sigma_{\mathcal{B}}(\Phi(a)) = \sigma_{\mathcal{A}}(a)$.

Proof. Let $\lambda \notin \sigma_{\mathcal{A}}(a)$. Then $a - \lambda 1_{\mathcal{A}} \in G(\mathcal{A})$. Since Φ preserves multiplication and the identity, it preserves the invertible elements. That is $\Phi(a) - \lambda 1_{\mathcal{B}} \in G(\mathcal{B})$. Hence $\lambda \notin \sigma_{\mathcal{B}}(\Phi(a))$.

If Φ is bijective, then Φ^{-1} is also a homomorphism and hence the equality holds. □

Definition 3.2. *Let $\mathcal{A}$ be a unital Banach algebra and $\phi : \mathcal{A} \to \mathbb{C}$ be a non zero linear map. Then ϕ is said to be multiplicative if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$.*

If ϕ is multiplicative, then $\phi(1) = 1$.

Example 3.3. Let $\mathcal{A} = C(K)$, where K is compact, Hausdorff space. Let $x \in K$. Define $\phi_x : \mathcal{A} \to \mathbb{C}$ by $\phi_x(f) = f(x)$ for all $f \in \mathcal{A}$. Then ϕ_x is multiplicative. In fact, we show later that all the multiplicative functionals on $\mathcal{A}$ are of this form.

Example 3.4. Let $\mathcal{A} = A(\mathbb{D})$ and $z \in \mathbb{D}$. Then the function defined by $\phi_z(f) = f(z)$ for all $f \in \mathcal{A}$, is a multiplicative functional on $\mathcal{A}$. Later we can witness that there are more multiplicative functionals on $\mathcal{A}$ other than these. In fact, we can define ϕ_z for each $z \in \overline{\mathbb{D}}$.

Exercise 3.5. *Let $\mathcal{A} = M_n(\mathbb{C})$. Then for $n \geq 2$, $\mathcal{A}$ have no multiplicative functionals.*

We denote the set of all multiplicative functionals on $\mathcal{A}$ by $\mathcal{M}_{\mathcal{A}}$. Next we show that a multiplicative functional is automatically continuous on a Banach algebra.

Proposition 3.6. *Let $\phi \in \mathcal{M}_{\mathcal{A}}$. Then $\|\phi\| = 1$.*

Proof. Since $\phi(1) = 1$, we have $\|\phi\| \geq |\phi(1)| = 1$. If $\|\phi\| > 1$, then there exists a $a_0 \in \mathcal{A}$ such that $\|a_0\| = 1$ and $|\phi(a_0)| > 1$. Let $a = a_0 - \phi(a_0)1$. Then $\phi(a) = 0$. Then we have,

$$\left\| \frac{a_0}{\phi(a_0)} \right\| = \left\| \frac{a + \phi(a_0)1}{\phi(a_0)} \right\| < 1.$$

Hence $\dfrac{a}{\phi(a_0)} \in G(\mathcal{A})$, consequently, $a \in G(\mathcal{A})$, which is not possible as $a \in ker(\phi)$. Thus $\|\phi\| = 1$. □

Proposition 3.7. *The set $\mathcal{M}_{\mathcal{A}}$ is Hausdorff and w^*-compact subset of $\mathcal{A}_1^*$.*

Proof. Note that $\mathcal{M}_{\mathcal{A}} \subset \mathcal{A}_1^*$, the unit ball of $\mathcal{A}^*$ and by Banach-Alaouglu's theorem $\mathcal{A}_1^*$ is w^* compact. To prove the result, it suffices to show that $\mathcal{M}_{\mathcal{A}}$ is closed in $\mathcal{A}_1^*$. Let $(\phi_\alpha) \subset \mathcal{M}_{\mathcal{A}}$ be such that $\phi_\alpha \xrightarrow{w^*} \phi \in \mathcal{A}_1^*$. That is $\phi_\alpha(a) \to \phi(a)$ for each $a \in \mathcal{A}$. Consequently,

$$\phi(ab) = \lim_\alpha \phi_\alpha(ab) = \lim_\alpha \phi_\alpha(a) \lim_\alpha \phi_\alpha(b) = \phi(a)\phi(b).$$

Hence $\phi \in \mathcal{M}_{\mathcal{A}}$. $\square$

3.1. **Multiplicative Functionals and Ideals.** Recall that if X is a Banach space and $f : X \to \mathbb{C}$ be linear, then $f \in X^*$ if and only if $ker(f)$ is closed . In this case, $X/ker(f) \simeq \mathbb{C}$. We have a similar kind of result for multiplicative functionals on a Banach algebra.

Proposition 3.8. *Let $\mathcal{A}$ be a commutative, unital Banach algebra. There exists a one-to-one correspondence between $\mathcal{M}_{\mathcal{A}}$ and the maximal ideals of $\mathcal{A}$.*

Proof. Let $\phi \in \mathcal{M}_{\mathcal{A}}$. Then $ker(\phi)$ is an ideal. We show that it is a maximal ideal. Let $a_0 \in \mathcal{A} \setminus ker(\phi)$. As $1 = \left(1 - \dfrac{a_0}{\phi(a_0)}\right) + \dfrac{a_0}{\phi(a_0)}$. Consider the algebra $L := \{ker(\phi), a_0\}$ generated by $ker(\phi)$ and a_0. Then L is an ideal containing 1 and hence it must be equal to $\mathcal{A}$. Thus $ker(\phi)$ is a maximal ideal.

Let I be a maximal ideal in $\mathcal{A}$. It can be verified easily that $\overline{I}$ is also an ideal. Since I is maximal, $I = \overline{I}$.

Note that for each $a \in I$, $\|1 - a\| \geq 1$, by Lemma 2.1.

Since $\mathcal{A}$ is commutative and I is maximal, there exists an isomorphism $\eta : \mathcal{A}/I \to \mathbb{C}$.

Now consider the quotient map $\pi : \mathcal{A} \to \mathcal{A}/I$. Define $\phi : \mathcal{A} \to \mathbb{C}$ by

$$\phi = \eta \circ \pi.$$

Being composition of multiplicative maps, ϕ is multiplicative and

$$\begin{aligned} ker(\phi) &= \{a \in \mathcal{A} : \phi(a) = 0\} \\ &= \{a \in \mathcal{A} : \eta(a + I) = 0\} \\ &= I. \end{aligned}$$

Let $\phi_1, \phi_2 \in \mathcal{M}_{\mathcal{A}}$ such that $ker(\phi_1) = I = ker(\phi_2)$. For $a \in \mathcal{A}$, write $b = (\phi_1(a) - \phi_2(a))1 = ([a - \phi_2(a)] - [a - \phi_1(a)])$. As $a - \phi_2(a), a - \phi_1(a) \in I$, it follows that $b \in I$. That is $\phi_1(b) = 0 = \phi_2(b)$, concluding $\phi_1(a) = \phi_2(a)$ for all $a \in \mathcal{A}$. $\square$

Exercise 3.9. *Prove the following;*

1. *Let $\mathcal{A}$ be a commutative unital Banach algebra. Show that $a \notin G(\mathcal{A})$ if and only if there exists a proper maximal ideal I of $\mathcal{A}$ such that $a \in I$.*
2. *Let $\mathcal{A}$ be a commutative unital Banach algebra. Show that every ideal is contained in a maximal ideal.*
3. *Let H be a separable Hilbert space. Show that $\mathcal{K}(H)$ is the only closed two sided ideal in $\mathcal{B}(H)$.*

Next, we prove the Gleason-Kahane-Zelazko theorem, which characterize all multiplicative functionals on complex Banach algebras. In order to do this, we need a result from Complex Analysis.

Lemma 3.10. *Let $f(z)$ is an entire function such that $f(0) = 1$, $f'(0) = 0$ and $0 < |f(z)| < e^{|z|}$ for all $z \in \mathbb{C}$. Then $f(z) = 1$ for all $z \in \mathbb{C}$.*

Proof. Let $f(z) = e^{g(z)}$, where $g(z)$ is an entire function. Let $u(z)$ and $v(z)$ be real and imaginary parts of $g(z)$ respectively. By the Hypothesis, we have $g(0) = 0 = g'(0)$. Also note that $e^{u(z)} = |f(z)| < e^{|z|}$, hence $u(z) < |z|$ for all $z \in \mathbb{C}$.

For $|z| \leq r$, $(r > 0)$, we have $|g(z)| = |\overline{g(z)}| = |2u(z) - g(z)| \leq |2r - g(z)|$.

Define $g_r(z) = \dfrac{r^2 g(z)}{z^2(2r - g(z))}$ for all $0 < |z| \leq r$.

Then $|g_r(z)| \leq 1$ for all $0 < z \leq r$. Fix z and letting $r \to \infty$, we get

$$|g(z)| \leq \frac{|z^2||2r - g(z)|}{r^2} \to 0.$$

But $g(0) = 0$. Hence $f = 1$. □

3.2. **G-K-Z theorem.** Now we prove the G-K-Z theorem.

Theorem 3.11. *Let $\mathcal{A}$ be a unital Banach algebra and $\phi : \mathcal{A} \to \mathbb{C}$ be linear. Then $\phi \in \mathcal{M}_{\mathcal{A}}$ if and only if $\phi(1) = 1$ and $\phi(a) \neq 0$ for all $a \in G(\mathcal{A})$.*

Proof. If ϕ is multiplicative, then $\phi(1) = 1$ and $\phi(a) \neq 0$ for all $a \in G(\mathcal{A})$ is clear.

On the other hand, assume that $\phi(1) = 1$ and $\phi(a) \neq 0$ for all $a \in G(\mathcal{A})$. We prove this in three steps.

Step 1: ϕ is bounded.

Let $\mathcal{N} = ker(\phi)$. Let $a \in \mathcal{A}$ with $\phi(a) \neq 0$. Then $\left(1 - \dfrac{a}{\phi(a)}\right) \in \mathcal{N}$. Hence $\|\dfrac{a}{\phi(a)}\| \geq 1$. That is $\|a\| \geq |\phi(a)|$. If $a \in \mathcal{N}$, then $\|a\| \geq |\phi(a)| = 0$. Thus $\|\phi\| = 1$.

Step 2: $a \in \mathcal{N} \Rightarrow a^2 \in \mathcal{N}$

Let $a \in \mathcal{N}$ and $\|a\| \leq 1$. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \sum_{n=0}^{\infty} \frac{\phi(a^n)z^n}{n!} \text{ for all } z \in \mathbb{C}.$$

As $|\phi(a^n)| \leq \|a^n\| \leq 1$ for all $n \geq 1$, f is entire and $|f(z)| \leq e^{|z|}$. Also $f(0) = 1$ and $f'(0) = 0$. The continuity of ϕ implies that $f(z) = \phi(\sum_{n=0}^{\infty} \frac{(za)^n}{n!}) = \phi(exp(za))$. Since $exp(za)$ is invertible $f(z) \neq 0$ for all $z \in \mathbb{C}$, hence $|f(z)| > 0$ for all $z \in \mathbb{C}$. Now by Lemma 3.10, $f(z) = 1$ for all $z \in \mathbb{C}$. This shows that

$$\frac{\phi(a^2)z^2}{2!} + \frac{\phi(a^3)z^3}{3!} + \cdots = 0 \text{ for all } z \in \mathbb{C}.$$

Hence $\phi(a^2) = 0 = \phi(a^3) = \ldots$, concluding $a^2 \in \mathcal{N}$.

Step 3: If $a \in \mathcal{N}$, then $ab + ba \in \mathcal{N}$ for all $b \in \mathcal{A}$

If $a, b \in \mathcal{A}$, then $a = a_1 + \phi(a)1,\ b = b_1 + \phi(b)1$, where $a_1, b_1 \in \mathcal{N}$. Then

$$\begin{aligned} ab &= a_1b_1 + a_1\phi(b) + b_1\phi(a) + \phi(a)\phi(b) \\ \phi(ab) &= \phi(a_1b_1) + \phi(a_1)\phi(b) + \phi(b_1)\phi(a) + \phi(a)\phi(b) \\ &= \phi(a_1b_1) + \phi(a)\phi(b). \qquad (*) \end{aligned}$$

It is enough to show that $\phi(a_1b_1) = 0$. Substituting $a = b$, in Equation (*), we get

$$\phi(a^2) = \phi(a_1^2) + \phi(a)^2 = \phi(a)^2 \text{ (since } a_1 \in \mathcal{N} \Rightarrow a_1^2 \in \mathcal{N}). \tag{3.1}$$

replacing a by $a+b$ in Equation 3.1, we have that $\phi((a+b)^2) = \phi(a+b)^2$. On simplication, we have

$$\phi(ab + ba) = 2\phi(a)\phi(b). \tag{3.2}$$

That is if $a \in \mathcal{N}$, then $ab + ba \in \mathcal{N}$ for all $b \in \mathcal{A}$. Hence $a(bab) + (bab)a \in \mathcal{N}$. We claim that $ab - ba \in \mathcal{N}$, whenever $a \in \mathcal{N}$ and $b \in \mathcal{A}$.

To this end, since

$$(ab - ba)^2 + (ab + ba)^2 = 2[a(bab) + (bab)a],$$

we have $0 = \phi(ab - ba)^2 = \phi(ab - ba)^2$, hence $ab - ba \in \mathcal{N}$.

Finally we have $2ab = (ab + ba) - (ab - ba) \in \mathcal{N}$. Similarly $ba \in \mathcal{N}$. Hence $\mathcal{N}$ is an ideal. In fact it is maximal, since $\mathcal{N} = ker(\phi)$. □

In general if $\phi : \mathcal{A} \to \mathcal{B}$ is linear and preserves invertible elements, then ϕ need not be multiplicative.

Example 3.12. Let $\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$ and $\mathcal{B} = \mathcal{M}_2(\mathbb{C})$.
Define $\phi : \mathcal{A} \to \mathcal{B}$ by

$$\phi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix}.$$

It clear that ϕ maps invertible elements into invertible elements but

$$\phi\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Here ϕ is not multiplicative, since

$$\phi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) = \phi\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix},$$

and

$$\phi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \phi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

Example 3.13. Let $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$. Then the map $\phi : \mathcal{A} \to \mathcal{A}$ given by

$$\phi(A) = A^t, \ A \in \mathcal{A},$$

maps I to I and invertible elements into invertible elements. But not multiplicative since $\phi(AB) = (AB)^t = B^t A^t = \phi(B)\phi(A)$.

If we assume an extra condition, namely semisimplicity on the algebra in codomain, then we can get a characterization of multiplicative linear maps.

Let $\mathcal{A}$ be an algebra. Then

$$J := \bigcap_{I \text{ is maximal ideal}} I$$

is called the **radical** of $\mathcal{A}$. If $J = \{0\}$, then $\mathcal{A}$ is called a **semi simple algebra**.

For example $C(K)$ is a semi simple Banach algebra.

Now we state a theorem without proof, on multiplicative linear maps between Banach algebras.

Theorem 3.14. *Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{B}$ be a semi-simple commutative unital Banach algebra and $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map. Then ϕ is multiplicative if and only $\phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ and ϕ maps invertible elements of $\mathcal{A}$ into invertible elements of $\mathcal{B}$.*

4. The Gelfand Map

Recall that if X is a complex normed linear space, then X can be embedded in $C(K)$, where $K = X_1^*$, the unit ball in X^* which is a w^*-compact, Hausdorff space. In this section, we prove a similar result for commutative Banach algebras.

Let $\mathcal{A}$ be a unital Banach algebra. Then the map $\Gamma : \mathcal{A} \to C(\mathcal{M}_{\mathcal{A}})$ defined by

$$\Gamma(a)(\phi) = \phi(a), \text{ for all } a \in \mathcal{A},\ \phi \in \mathcal{M}_{\mathcal{A}} \tag{4.1}$$

is called the **Gelfand map**.

Clearly Γ is linear. By definition, $\|\Gamma\| = \sup\{\|\Gamma a\|_\infty : a \in \mathcal{A},\ \|a\| \leq 1\}$. But

$$\begin{aligned}\|\Gamma a\|_\infty &= \sup\{|(\Gamma a)(\phi)| : \phi \in \mathcal{M}_{\mathcal{A}}\}\\ &= \sup\{|\phi(a)| : \phi \in \mathcal{M}_{\mathcal{A}}\}\\ &\leq \|a\|.\end{aligned}$$

Hence $\|\Gamma\| \leq 1$.

Let ϕ be multplicative. Then for all $a, b \in \mathcal{A}$,

$$\Gamma(ab)(\phi) = \phi(ab) = \phi(a)\phi(b) = (\Gamma a)(\phi)(\Gamma b)(\phi) = (\Gamma a)(\Gamma b)(\phi).$$

Hence $\Gamma(ab) = \Gamma(a)\Gamma(b)$. That is Γ is multiplicative.

Remark 4.1. If $\mathcal{A}$ is non commutative, then $\mathcal{M}_{\mathcal{A}}$ may be trivial. For example $\mathcal{A} = \mathcal{M}_n(\mathbb{C}),\ n \geq 2$ has no non trivial ideals and hence no multiplicative functionals on it. But if the Banach algebra is commutative, then it has non trivial maximal ideals and hence non zero multiplicative functionals.

Here we discuss some of the properties of the Gelfand map.

Proposition 4.2. *Let $\mathcal{A}$ be a commutative, unital Banach algebra. Then*

$$a \in G(\mathcal{A}) \Leftrightarrow \Gamma a \in G(C(\mathcal{M}_{\mathcal{A}})).$$

That is Γ maps invertible elements of $\mathcal{A}$ into the invertible elements of $C(\mathcal{M}_{\mathcal{A}})$.

Proof. If $a \in G(\mathcal{A})$, since Γ is multplicative, Γa is invertible in $C(\mathcal{M}_{\mathcal{A}})$. On the other hand, suppose that $a \notin G(\mathcal{A})$. Then there exists a maximal ideal I such that $a \in I$. That is there exists a $\phi_0 \in \mathcal{M}_{\mathcal{A}}$ such that $ker(\phi_0) = I$. Hence $0 = \phi_0(a) = (\Gamma a)(\phi_0)$. Therefore $\Gamma a \notin G(C(\mathcal{M}_{\mathcal{A}}))$. ☐

Corollary 4.3. (1) Γ *preserves spectrum. That is*

$$\begin{aligned}\sigma(a) = \sigma(\Gamma a) &= \textit{range of } \Gamma a\\ &= \{\Gamma a(\phi) : \phi \in \mathcal{M}_{\mathcal{A}}\}\\ &= \{\phi(a) : \phi \in \mathcal{M}_{\mathcal{A}}\}.\end{aligned}$$

(2) $r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} = \|\Gamma a\|_\infty$.

Exercise 4.4. *Show that Γ is an isometry if and only if $\|a\|^2 = \|a^2\|$ for all $a \in \mathcal{A}$.*

Here we illustrate the Gelfand map with examples.

4.1. **Examples.**

Theorem 4.5. *Let K be a compact, Hausdorff space. For each $x \in K$, the map $x \to \phi_x$ is a homeomorphism from K onto $\mathcal{M}_{C(K)}$.*

In this case the Gelfand map $\Gamma : C(K) \to C(K)$ is the identity map.

Proof. Let $\Phi : K \to \mathcal{M}_{C(K)}$ be given by

$$x \to \phi_x,$$

where ϕ_x is the evaluation map. Hence $\phi_x \in \mathcal{M}_{C(K)}$.

If $x_1 \neq x_2$, by the Uryson's lemma, there exists a $f \in C(K)$ such that $f(x_1) = 1$ and $f(x_2) = 0$. Hence $\Phi(x_1)(f) = f(x_1) \neq f(x_2) = \Phi(x_2)(f)$. That is Φ is one-to-one.

To show Φ is onto, let $\phi \in \mathcal{M}_{C(K)}$. Consider

$$I = \{f \in C(K) : \phi(f) = 0\} = ker(\phi).$$

We claim that there exists an $x_0 \in K$ such that $f(x_0) = 0$ for all $f \in I$. If this is not true, then for each $x \in K$, there exists a $f_x \in I$ such that $f_x(x) \neq 0$. Since f_x is continuous, there exists an open set U_x containing x such that $f_x(z) \neq 0$ for all $z \in U_x$. Note that $K \subseteq \bigcup\limits_{x \in K} U_x$. Since K is compact, there exists $x_1, x_2, \ldots, x_n \in K$ such that $K \subseteq \bigcup\limits_{k=1}^{n} U_{x_k}$.

Define

$$f(x) = \sum_{k=1}^{n} |f_{x_k}(x)|^2, \; x \in K.$$

By the definition of f_{x_k}, we can conclude that f is non vanishing on K and hence invertible in $C(K)$. But

$$\phi(f) = \sum_{k=1}^{n} \phi(f_{x_k}) \overline{\phi(f_{x_k})} = 0, \text{ since } f_{x_k} \in I,$$

contradicting the fact that f is invertible and ϕ is multiplicative. Thus $f(x_0) = 0$ for all $f \in I$.

Now if $g \in C(K)$, then $g - \phi(g) \in I$. Hence by the earlier observation, $(g - \phi(g))(x_0) = 0$. That is $\phi(g) = g(x_0) = \phi_{x_0}(g)$ for all $g \in C(K)$. Hence Φ is onto.

Let $x_\alpha \in K$ such that $x_\alpha \to x \in K$. Then $\Phi_{x_\alpha}(f) = f(x_\alpha) \to f(x) = \Phi(x)(f)$ for every $f \in C(K)$. That is $\Phi(x_\alpha) \to \Phi(x)$. Hence Φ is continuous.

We have seen that $\mathcal{M}_{C(K)}$ is Hausdorff and w^*-compact. By the Hypothesis, K is compact. Hence Φ is a homeomorphism.

If we identify $\mathcal{M}_{C(K)}$ with K, the Gelfand map $\Gamma : C(K) \to C(K)$ is the identity map. $\square$

As an application of the Gelfand map, we prove the spectral mapping theorem for analytic functions.

4.2. **The Spectral Mapping Theorem.** Let $\mathcal{A}$ be a unital Banach algebra, $x \in \mathcal{A}$ and $f(z)$ be an analytic function in a neighbourhood of the disc $D = \{z \in \mathbb{C} : |z| \leq \|x\|\}$. Then $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ for each $z \in D$. For each $z \in \mathbb{D}$, we have that $\sum_{n=0}^{\infty} |a_n||z|^n < \infty$. Thus $\sum_{n=0}^{\infty} a_n x^n$ converges to an element in $\mathcal{A}$, we denote this by $f(x)$.

Theorem 4.6. *Suppose $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$. If $f(z)$ is analytic in a neighbourhood of D, then we have*

$$\sigma(f(x)) = f(\sigma(x)) = \{f(\lambda) : \lambda \in \sigma(x)\}.$$

Proof. To prove the Theorem we use Proposition 2.27. Let $\lambda \in \rho(x)$ and $\mu \in \rho(f(x))$. Let $\mathcal{B}$ be the closed algebra generated by $1, x, (\lambda 1 - x)^{-1}, (\mu 1 - f(x))^{-1}$. Note if $ab = ba$ and b invertible, then $ab^{-1} = b^{-1}a$. This fact implies that $\mathcal{B}$ is commutative. It can be shown that

$$\sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{A}}(x) \text{ and } \sigma_{\mathcal{A}}(f(x)) = \sigma_{\mathcal{B}}(f(x)).$$

Let $\Gamma : \mathcal{B} \to C(\mathcal{M}_{\mathcal{B}})$ be the Gelfand transform. By Corollary 4.3,

$$\sigma_{\mathcal{B}}(f(x)) = \{\phi(f(x)) : \phi \in \mathcal{M}_{\mathcal{B}}\} = \sigma(\Gamma f(x)).$$

By the power series expansion, we have that for every $\phi \in \mathcal{M}_{\mathcal{B}}$,

$$\begin{aligned} \phi(f(x)) = \phi\left(\sum_{n=0}^{\infty} a_n x^n\right) &= \sum_{n=0}^{\infty} a_n \phi(x^n) \\ &= \sum_{n=0}^{\infty} a_n \phi(x)^n \\ &= f(\phi(x)). \end{aligned}$$

Therefore $\sigma_{\mathcal{B}}(f(x)) = f(\sigma_{\mathcal{B}}(x))$. By Proposition 2.27, we get the desired result. □

4.3. **Non unital Banach algebras.** Here we discuss the Gelfand map for non unital commutative Banach algebras. To do this, we consider the unitization of $\mathcal{A}$ and using the earlier case of unital Banach algebras we deduce the required results.

Let $\mathcal{A}$ be a non unital commutative Banach algebra. Then we denote the set of all non zero multiplicative functionals on $\mathcal{A}$ by $\mathcal{M}_{\mathcal{A}}$.

Proposition 4.7. *Let $\phi \in \mathcal{M}_{\mathcal{A}}$. Then $\|\phi(a)\| \leq \|a\|$ for each $a \in \mathcal{A}$.*

Proposition 4.8. *The set $\mathcal{M}_{\mathcal{A}}$ is a w^*-locally compact and Hausdorff.*

Theorem 4.9. *Let $\mathcal{A}$ be a non unital commutative Banach algebra. Then the image of the Gelfand map $\Gamma : \mathcal{A} \to C(\mathcal{A})$ is $C_0(\mathcal{M}_{\mathcal{A}})$.*

Example 4.10. Let Ω be a locally compact but non compact set and $\mathcal{A} = C_0(\Omega)$. Then $\mathcal{M}_{\mathcal{A}} \cong \Omega$ and the Gelfan map is the identity map.

Define a map $\Phi : \Omega \to \mathcal{M}_{\mathcal{A}}$ by

$$\Phi(x) = \phi_x,$$

where $\phi_x(f) = f(x)$ for every $f \in \mathcal{A}$. We have seen that $\{\phi_x : x \in \Omega\} \subseteq \mathcal{M}_{\mathcal{A}}$. Next, we show that Φ is one-to-one. Let $x_1 \neq x_2 \in \Omega$. Then by the Urysohn's lemma, there exists a $f_0 \in C(\Omega)$ such that $f_0(x_1) = 0$ and $f_0(x_2) = 1$. Hence

$$\Phi(x_1)(f_0) = \phi_{x_1}(f_0) = f_0(x_0) = 0 \neq 1 = f_0(x_2) = \phi_{x_2}(f_0) = \Phi(x_2).$$

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