Laplace transform

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Outline

Uses of Laplace transform

- solving the problems that arise in engineering as well as in Mathematics
- The Ordinary differential Equations and partial differential equations describe certain quantities that vary with time
  - current in an electrical circuit
  - oscillations of a vibrating string
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(Example: Initial Value Problem)

$$y' + 4y = e^t,\ y(0) = 2.$$

POWER SERIES

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Continuous Analogue

$$F(x) = \int_0^{\infty} a(t) x^t dt$$

Substitute $s = -log(x), \ x > 0$.

Then $F(s) = \int_0^{\infty} e^{-st} f(t) dt$

Laplace transforms

Definition

Let $f : [0, \infty) \to \mathbb{R}$ be a function. The Laplace transform of f is defined by

$$\mathcal{L}(f) := \int_0^\infty e^{-st} f(t) dt$$

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Note: $\displaystyle\int_0^\infty e^{-st} f(t) dt = \lim_{b \to \infty} \int_0^b e^{-st} f(t) dt$

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Integration by parts

$$[u(x)v(x)]_a^b = \int_a^b u'(x)v(x)\,dx + \int_a^b u(x)v'(x)\,dx.$$

EXAMPLES

1. Let $a \in \mathbb{R}$ be fixed and $f(t) = e^{at},\ t \in [0, \infty)$. Then $\mathcal{L}(f) = \dfrac{1}{s-a},\ s > a$
2. Let $f(t) = e^{iwt},\ t \in [0, \infty),\ w \in \mathbb{R}$. Then $\mathcal{L}(f) = \dfrac{1}{s-iw},\ s > w$ or $Re(s) > w$
3. Let $a > -1$ and $f(t) = t^a,\ t \in [0, \infty)$. Then $\mathcal{L}(f) = \dfrac{\Gamma(a+1)}{s^{a+1}},\ s > 0$. Here $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$, the Gamma function

PIECEWISE CONTINUOUS

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A function $f : (0, b) \to \mathbb{R}$ is said to be piecewise continuous if f is continuous on $(0, b)$ except possibly at finite number of points $\{t_i : i = 1, 2, \ldots n\}$ at which f has a jump discontinuity.

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A function $g : [0, \infty) \to R$ is said to be piecewise continuous on $[0, \infty)$ if g is piecewise continuous on every finite sub interval of $[0, \infty)$.

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2. $f(t) = \begin{cases} 1, t \geq 0 \\ -1, t < 0. \end{cases}$ is piecewise continuous
3. $g(t) = \dfrac{1}{t^2},\ t \in (-1, 1) \setminus \{0\}$ is not piecewise continuous

Exponential order

A function $f : [0, \infty) \to \mathbb{R}$ is said to be of exponential order α, if there exists constants $M > 0$ and α such that for some $t_0 \geq 0$, we have

$$|f(t)| \leq Me^{\alpha t} \text{ for all } t \geq t_0$$

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Existence of Laplace transform

Let $f : [0, \infty) \to \mathbb{R}$ be a function satisfying

- f is piecewise continuous
- f is of exponential order with order α.

Then $\mathcal{L}(f)$ exists for all $s > \alpha$.

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2. $g(t) = 2te^{t^2}\cos(t^2),\ t \in [0, \infty)$. Then f is continuous but not of exponential order. But $\mathcal{L}(f)$ exists.

Properties

Let $D(\mathcal{L}) := \{f : [0, \infty) \to \mathbb{R} : f$ is piecewise continuous and exponenti

Then

- $\mathcal{L}$ is linear. That is $\mathcal{L}(af + bg) = a\mathcal{L}(f) + \mathcal{L}(g)$ for all $f, g \in D(\mathcal{L}),\ a, b \in \mathbb{R}$

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1. $\mathcal{L}(\sin(wt)) = \dfrac{w}{s^2 + w^2},\ s > w$
2. $\mathcal{L}(\sinh(at)) = \dfrac{a}{s^2 - a^2},\ s > a$

Vanishing property

Let $f : [0, \infty)$ be piecewise continuous and has exponential order α. Then $\mathcal{F}(s) \to 0$ as $s \to \infty$.

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The functions

1. $\mathcal{F}_1(s) = \frac{s-1}{s+1}, \quad s > -1$
2. $\mathcal{F}_2(s) = \frac{e^s}{s}, \quad s > 0$
3. $\mathcal{F}_3(s) = s^2, \quad s \in \mathbb{R}$

cannot be Laplace transform of any function.

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First Shift Theorem

Let $f : [0, \infty) \to \mathbb{R}$ be such that $\mathcal{L}(f)$ exists. Then

$$\mathcal{L}(e^{at}f(f)) = \mathcal{F}(s-a), \quad s > a.$$

HEAVISIDE STEP FUNCTION

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Hence $\mathcal{L}^{-1}(\frac{e^{-as}}{s}) = u_a(t)$.

Dirac delta Operator

Let $a \geq 0$. Then the Dirac delta operator is defined by

$$\delta(t-a) = \begin{cases} \infty, t = a \\ 0, \text{else}, \end{cases} \quad \text{and} \quad \int_0^\infty \delta(t-a)dt = 1.$$

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