Fourier Transforms

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Outline

1. Fourier Integrals

2. Fourier Transforms

Fourier Integral

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The representation

$$f(x) = \int_0^\infty \Big(A(w)\cos(wx) + B(w)\sin(wx) \Big) dw, \tag{1}$$

where

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v)\cos(wv)dv$$
$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v)\sin(wv)dv$$

is called the Fourier Integral representation of f.

Let $f : \mathbb{R} \to \mathbb{R}$ be such that

- piecewise cont in every finite interval
- absolutely integrable $\left(\int_{-\infty}^{\infty} |f(x)| dx < \infty \right)$
- f has left and right derivative at every point in the finite interval.

Then $f(x)$ can be represented by Fourier integral.
If f is discontinuous at x_0, then $f(x_0) = \frac{f(x_0+) + f(x_0-)}{2}$.

Find the Fourier integral representation of

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Sol: We have

$$A(w) = \frac{2\sin(w)}{\pi w}$$

$$B(w) = 0$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos(wx)\sin(w)}{w}\,dw$$

$$f(1) = \frac{f(1+) + f(1-)}{2} = \frac{1}{2}.$$

Fourier Cosine Integral

$$f(x) = \int_0^\infty A(w)\cos(wx)dw \tag{2}$$

where

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos(wv)dv \tag{3}$$

is called the Fourier cosine integral of f.

Fourier sine Integral

$$f(x) = \int_0^\infty B(w)\sin(wx)\,dw \tag{4}$$

where

$$B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin(wv)\,dv \tag{5}$$

is called the Fourier sine integral of f.

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$f(x) = e^{-kx},\ x > 0,\ k > 0.$

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$$\frac{\pi}{2k} e^{-kx} = \int_o^\infty \frac{\cos(wx)}{k^2 + w^2} dw \tag{6}$$

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$B(w) = \dfrac{2w}{\pi(k^2 + w^2)}$ and

$$\int_0^\infty \frac{w \sin(wx)}{k^2 + w^2} dw = \frac{\pi}{2} e^{-kx} \tag{7}$$

The integrals in Equations (6) and (7) are called as Laplace integrals.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then the cosine integral is given by Equations (2) and (3). Let $A(w) = \sqrt{\frac{2}{\pi}}\hat{f}_c(w)$, where

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(wx)dx$$

is called the Fourier cosine transform of f and

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w)\cos(wx)dw$$

is called the inverse Fourier Cosine transform of f.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then the sine integral is given by Equations (4) and (5). Let $B(w) = \sqrt{\frac{2}{\pi}}\hat{f}_s(w)$, where

$$\hat{f}_s(w) = \int_0^\infty f(x)\sin(wx)dx$$

is called the Fourier Sine transform of f and

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is called the inverse Fourier Sine transform of f.

Find the Fourier Cosine and sine transforms of the function

$$f(x = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}.$$

Ans: $\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \frac{k \sin(w)}{w}$.

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Find the Fourier cosine transform of $f(x) = e^{-x},\ x \in \mathbb{R}$.

Properties

Let $\mathcal{F}_c$ and $\mathcal{F}_s$ denote the Fourier Cosine and Sine transforms of f respectively. Then

1. $\mathcal{F}_c(af + bg) = a\mathcal{F}_c(w) + b\mathcal{F}_c(w)$
2. $F_s(af + bg) = a\mathcal{F}_s(w) + b\mathcal{F}_s(w)$

THEOREM

Let $f : \mathbb{R} \to \mathbb{R}$ be such that

1. f is continuous
2. f is absolutely integrable on $\mathbb{R}$
3. f' is piecewise continuous on each finite interval
4. $f(x) \to 0$ as $x \to \infty$.

Then

(A) $\mathcal{F}_c(f') = w\mathcal{F}_s(f) - \sqrt{\dfrac{2}{\pi}}f(0)$

(B) $\mathcal{F}_s(f') = -w\mathcal{F}_c(f)$.

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be such that

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Similarly we can prove the following:

(A) $\mathcal{F}_c(f'') = -w^2 \mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}} f'(0)$

(B) $\mathcal{F}_s(f'') = -w^2 \mathcal{F}_s(f) + \sqrt{\frac{2}{\pi}}\, w f'(0)$.

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Find the Fourier cosine transform of $f(x) = e^{-ax},\ (a > 0)$

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Ans: $\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2 + w^2}\right).$

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The Fourier transform of f is defined by

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx} f(x)dx. \tag{8}$$

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The inverse Fourier transform of $\hat{f}(w)$ is defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwx} \hat{f}(w) dw. \tag{9}$$

Existence of Fourier transform

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

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PROPERTIES

Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that $\mathcal{F}(f)$ and $\mathcal{F}(g)$ exists. Then

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$$\mathcal{F}(f') = iw\mathcal{F}(f).$$

If f'' is absolutely integrable, we can show that $\mathcal{F}(f'') = -w^2\mathcal{F}(f)$.

Let $f(x) = xe^{-x^2},\ x \in \mathbb{R}$. Find $\mathcal{F}(f)$.

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Let f and g are piecewise continuous and absolutely integrable. Then

$$\mathcal{F}(f * g) = \sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(g).$$

THANK YOU