INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD
MA5040 - Topology
Problem Sheet 5
Spring 2025

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1. Show that every compact subspace of a metric space is bounded in that metric and is closed. Find a metric space in which not every closed bounded subspace is compact.

2. Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B, respectively.

3. Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a closed map (that is, f carries closed sets to closed sets).

4. Show that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map.

5. Let $f : X \to Y$; let Y be compact Hausdorff. Then f is continuous if and only if the *graph* of f,
$$G_f = \{(x, f(x)) \mid x \in X\},$$
is closed in $X \times Y$.

6. Let A and B be subspaces of X and Y, respectively; let N be an open set in $X \times Y$ containing $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y, respectively, such that
$$A \times B \subset U \times V \subset N.$$

7. Prove that if X is an ordered set in which every closed interval is compact, then X has the least upper bound property.

8. Let X be a metric space with metric d; let $A \subset X$ be nonempty.
   (a) Show that $d(x, A) = 0$ if and only if $x \in \bar{A}$.
   (b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.
   (c) Define the ϵ-*neighborhood* of A in X to be the set
   $$U(A, \epsilon) = \{x \mid d(x, A) < \epsilon\}.$$
   Show that $U(A, \epsilon)$ equals the union of the open balls $B_d(a, \epsilon)$ for $a \in A$.

(d) Assume that A is compact; let U be an open set containing A. Show that some ϵ-neighborhood of A is contained in U.

(e) Show the result in (d) need not hold if A is closed but not compact.

9. Recall that $\mathbb{R}_K$ denotes $\mathbb{R}$ in the K-topology.

   (a) Show that $[0,1]$ is not compact as a subspace of $\mathbb{R}_K$.

   (b) Show that $\mathbb{R}_K$ is connected. *[Hint: $(-\infty, 0)$ and $(0, \infty)$ inherit their usual topologies as subspaces of $\mathbb{R}_K$.]*

   (c) Show that $\mathbb{R}_K$ is not path connected.

10. Show that a connected metric space having more than one point is uncountable.

11. Give $[0,1]^\omega$ the uniform topology. Find an infinite subset of this space that has no limit point.

12. Show that $[0,1]$ is not limit point compact as a subspace of $\mathbb{R}_\ell$.

13. Let X be limit point compact.

    (a) If $f : X \to Y$ is continuous, does it follow that $f(X)$ is limit point compact?

    (b) If A is a closed subset of X, does it follow that A is limit point compact?

14. A space X is said to be *countably compact* if every countable open covering of X contains a finite subcollection that covers X. Show that for a T_1 space X, countable compactness is equivalent to limit point compactness.

15. Show that X is countably compact if and only if every nested sequence $C_1 \supset C_2 \supset \cdots$ of closed nonempty sets of X has a nonempty intersection.

16. Show that the rationals $\mathbb{Q}$ are not locally compact.

17. Let X be a locally compact space. If $f : X \to Y$ is continuous, does it follow that $f(X)$ is locally compact? What if f is both continuous and open?

18. Show that $[0,1]^\omega$ is not locally compact in the uniform topology.
