INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD
MA5040 - Topology
Problem Sheet 4
Spring 2025

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1. Let X_n be a metric space with metric d_n, for $n \in \mathbb{Z}_+$.

   (a) Show that
   $$\rho(x, y) = \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\}$$
   is a metric for the product space $X_1 \times \cdots \times X_n$.

   (b) Let $\bar{d_i} = \min\{d_i, 1\}$. Show that
   $$D(x, y) = \sup \frac{\bar{d_i}(x_i, y_i)}{i}$$
   is a metric for the product space $\prod X_i$.

2. Show that $\mathbb{R}_\ell$ and the ordered square satisfy the first countability axiom. (This result does not, of course, imply that they are metrizable.)

3. Let X be a set, and let $f_n : X \to \mathbb{R}$ be a sequence of functions. Let $\bar{\rho}$ be the uniform metric on the space $\mathbb{R}^X$. Show that the sequence (f_n) converges uniformly to the function $f : X \to \mathbb{R}$ if and only if the sequence (f_n) converges to f as elements of the metric space $(\mathbb{R}^X, \bar{\rho})$.

4. Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\bigcup A_n$ is connected.

5. Let $\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connected subspace of X. Show that if $A \cap A_\alpha \neq \emptyset$ for all α, then $A \cup (\bigcup A_\alpha)$ is connected.

6. A space is *totally disconnected* if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

7. Is the space $\mathbb{R}_\ell$ connected? Justify your answer.

8. Determine whether or not $\mathbb{R}^\infty$ is connected in the uniform topology.

9. Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that
$$(X \times Y) - (A \times B)$$
is connected.

10. (a) Show that no two of the spaces $(0,1)$, $(0,1]$, and $[0,1]$ are homeomorphic.
    (b) Suppose that there exist imbeddings $f : X \to Y$ and $g : Y \to X$. Show by means of an example that X and Y need not be homeomorphic.
    (c) Show $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

11. Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show there exists a point x of S^1 such that $f(x) = f(-x)$.

12. Let $f : X \to X$ be continuous. Show that if $X = [0,1]$, there is a point x such that $f(x) = x$. The point x is called a *fixed point* of f. What happens if X equals $[0,1)$ or $(0,1)$?

13. Let X be an ordered set in the order topology. Show that if X is connected, then X is a linear continuum.

14. (a) Is a product of path-connected spaces necessarily path connected?
    (b) If $A \subset X$ and A is path connected, is $\bar{A}$ necessarily path connected?
    (c) If $f : X \to Y$ is continuous and X is path connected, is $f(X)$ necessarily path connected?
    (d) If $\{A_\alpha\}$ is a collection of path-connected subspaces of X and if $\bigcap A_\alpha \neq \emptyset$, is $\bigcup A_\alpha$ necessarily path connected?

15. Assume that $\mathbb{R}$ is uncountable. Show that if A is a countable subset of $\mathbb{R}^2$, then $\mathbb{R}^2 - A$ is path connected.

16. If A is a connected subspace of X, does it follow that Int A and Bd A are connected? Does the converse hold? Justify your answers.

17. What are the components and path components of $\mathbb{R}_\ell$? What are the continuous maps $f : \mathbb{R} \to \mathbb{R}_\ell$?

18. (a) What are the components and path components of $\mathbb{R}^\omega$ (in the product topology)?
    (b) Consider $\mathbb{R}^\omega$ in the uniform topology. Show that $\mathbf{x}$ and $\mathbf{y}$ lie in the same component of $\mathbb{R}^\omega$ if and only if the sequence
$$\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, \dots)$$
    is bounded.
    (c) Give $\mathbb{R}^\omega$ the box topology. Show that $\mathbf{x}$ and $\mathbf{y}$ lie in the same component of $\mathbb{R}^\omega$ if and only if the sequence $\mathbf{x} - \mathbf{y}$ is "eventually zero."